IRREDUCIBLE REPRESENTATIONS THAT CANNOT BE SEPARATED FROM THE TRIVIAL REPRESENTATION

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Let G be a locally compact group and $\hat{G}$ the dual space of G, i.e. the set of equivalence classes of irreducible unitary representations of G equipped with the usual topology. In general, $\hat{G}$ is far away from being a Hausdorff space. In fact, a connected group G has a Hausdorff dual iff G is an extension of a compact group by a vector group [2], and if G is discrete, then $\hat{G}$ is Hausdorff iff the centre of G has finite index in G. Therefore, it is reasonable to study the set of all those $\pi \in \hat{G}$ that cannot be separated from the trivial 1-dimensional representation 1_G, the so-called cortex cor(G) of G.

Interest in this closed subset of the dual also arose from the fact that the topology in the neighbourhood of 1_G is related to the group structur of G and to the cohomology of G in unitary representation spaces. It is well known (see [1]) that G has the Kazhdan property (T), i.e. $\{1_G\}$ is open in $\hat{G}$, iff $H^1(G,\pi) = 0$ for every unitary representation π of G. The following remarkable result has independently been obtained by Vershik and Karpushev [8] and by Larsen [7]: If G is second countable and $\pi \in \hat{G}$, then $H^1(G,\pi) \neq 0$ implies $\pi \in \mathrm{cor}(G)$.

Clearly, for $n \geq 3$, $SL(n,\mathbb{C})$, $SL(n,\mathbb{R})$ and $SL(n,\mathbb{Z})$ have a trivial cortex since they are groups with property (T). The cortex of $SL(2,\mathbb{C})$ consists of 1_G and the principal series representation which is usually denoted by $\pi_{2,0}$. $\mathrm{cor}(SL(2,\mathbb{R}))$ contains, except 1_G, two discrete series representations. It turns out that for every connected semi-simple Lie group G, cor(G) is finite [3]. To show this one observes that, for any connected Lie group G and $\pi \in \mathrm{cor}(G)$, the infinitesimal character χ_π is trivial, i.e. $\chi_\pi = \chi_{1_G}$, and then uses the fact that if G is semi-simple and has a finite centre, then $\chi_\rho = \chi_{1_G}$ for at most finitely many $\rho \in \hat{G}$.

In [3] Bekka and Kaniuth investigate the structure of amenable Lie groups having a compact, in particular a finite, cortex. One of the main tools is the following explicit description of the cortex for amenable discrete G .

THEOREM 1 [3]. *Let* G *be an amenable discrete group, and denote by* G_F *the normal subgroup of* G *consisting of all elements with finite conjugacy classes. Then* $\mathrm{cor}(G) = \widehat{G/G_F}$.

A locally compact group G is called $[FC]^-$ group if every element of G has a relatively compact conjugacy class. For the structure of these groups compare [5]. Every $[FC]^-$ group G has a Hausdorff primitive ideal space [6], so that $\mathrm{cor}(G) = \{1_G\}$. It turns out that, conversely, G having a finite cortex amounts to that G is almost $[FC]^-$. Let us mention two of the results obtained in [3]:

Theorem 2 [3]. *For an almost connected amenable group* G *the following conditions are equivalent:*

(i) cor(G) *is finite;*

(ii) cor(G) *is compact;*

(iii) G *contains a compact normal subgroup* C *such that* G/C *is a finite extension of a vector group.*

THEOREM 3 [3]. *Let* G *be an amenable Lie group. Then* *(i)* $\mathrm{cor}(G) = \{1_G\}$ *if and only if* G *is an* $[FC]^-$ *group.*

(ii) *If* G *is* σ*-compact, then* $\mathrm{cor}(G)$ *is finite iff* G *is a finite extension of an* $[FC]^-$ *group.*

One of the typical arguments used to prove Theorems 2 and 3 is the following. Let G be an amenable group, and suppose that V is a normal vector subgroup of G. Let $\lambda \in \hat{V}$, and denote by H the stability subgroup of λ in G. Then the support $\mathrm{supp}(\mathrm{ind}_H^G 1_H)$ of the induced representation $\mathrm{ind}_H^G 1_H$ is contained in $\mathrm{cor}(G)$. This can be seen as follows. Define $\lambda_n \in \hat{V}$ by $\lambda_n(V) = \lambda(\frac{1}{n} v)$, $n \in \mathbb{N}$. Then $G_{\lambda_n} = H$, and by the amenability of H, 1_H is weakly contained in the set of all $\tau \in \hat{H}$ such that $\tau|V$ is a multiple of λ_n for some n. We conclude that there exists a net $\pi_\iota = \mathrm{ind}_H^G \tau_\iota$, $\tau_\iota \in \hat{H}$, $\iota \in I$, such that $\tau_\iota \to 1_H$. Hence $\pi_\iota \to \mathrm{ind}_H^G 1_H$, and since G is amenable this shows that, for any $\rho \in \mathrm{supp}(\mathrm{ind}_H^G 1_H)$, ρ and 1_G cannot be separated in $\hat{G}$.

Let now G be a simply connected nilpotent Lie group and $\mathfrak{g}$ its Lie algebra. Denote by $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$ and by Ad^* the coadjoint representation of G on $\mathfrak{g}^*$. Then, by Kirillov's theory, every $f \in \mathfrak{g}^*$ defines an irreducible representation π_f of G, and this gives rise to a bijection between the orbit space $\mathfrak{g}^*/\mathrm{Ad}^*$ and $\hat{G}$. Moreover, this correspondence is known to be a homeomorphism. It is clear that $\pi_f \in \mathrm{cor}(G)$ iff there exist $f_n \in \mathfrak{g}^*$ and $x_n \in G$, $n \in \mathbb{N}$, such that $f_n \to 0$ and $\mathrm{Ad}^*(x_n)f_n \to f$ in $\mathfrak{g}^*$. There was some hope that this theory should enable to clarify the structure of $\mathrm{cor}(G)$ up to a certain extent.

Looking at all the simply connected nilpotent Lie groups G of dimension ≤ 6, one observes that

(I) $\mathrm{cor}(G)$ coincides with the set of irreducible representations of G

having a trivial infinitesimal character;

(II) cor(G) is a subdual of $\hat{G}$, i.e. cor(G) = $\widehat{G/N}$ for some closed normal subgroup N of G .

The expectation that (I) and (II) may hold in general was also supported by the following

THEOREM 4 . *Suppose that* G *is a nilpotent group of the form* $G = \mathbb{R} \ltimes \mathbb{R}^n$. *Then (I) holds for* G[*4*]. *Furthermore,* cor(G) = $\widehat{G/N}$ *for some* N *, and* N *can be described explicitly* [*3,4*]. *In particular, let* $\mathfrak{g}$ *be the threadlike algebra of dimension* n *, i.e.* $\mathfrak{g}$ *has a basis* $X_1,\ldots,X_n$ *with non-trivial products* $[X_n,X_{j+1}] = X_j$, $1 \leq j \leq n-2$. *If* $G = \exp \mathfrak{g}$ *and* $N = \exp(\mathbb{R}X_1+\ldots+\mathbb{R}X_{[n/2]})$, *then* cor(G) = $\widehat{G/N}$.

Surprisingly, it turned out that (I) as well as (II) may fail even for 2-step nilpotent simply connected nilpotent Lie groups. In [3] an 8-dimensional example G is given that neither satisfies (I) nor (II). The Lie algebra of G has a basis $X_1,\ldots,X_6,Z_1,Z_2$ with non-trivial commutators $[X_1,X_5] = [X_2,X_3] = Z_1$ and $[X_1,X_6] = [X_2,X_4] = Z_2$. It is possible to calculate the algebra $I(\mathfrak{g}^*)$ of all $Ad^*(G)$-invariant complex valued polynomial functions on $\mathfrak{g}^*$. Since π_f has a trivial infinitesimal character iff $P(f) = 0$ for all $P \in I(\mathfrak{g}^*)$ with $P(0) = 0$, this gives the set of all $\pi \in \hat{G}$ with $\chi_\pi = \chi_{1_G}$. On the other hand, cor(G) can be computed by using the fact hat if G is 2-step nilpotent and $G = \exp \mathfrak{g}$, then $\pi_f \in$ cor(G) iff f belongs to the closure in $\mathfrak{g}^*$ of the set of all $ad^*(X)g$, $g \in \mathfrak{g}^*$, $X \in \mathfrak{g}$.

Finally we consider motion groups, i.e. semi-direct products $G = K \ltimes V$, where K is a compact connected Lie group and V a vector group.

Then there exists a unique conjugacy class $\mathcal{K}$ of subgroups of K with the following property: the set D of all $\lambda \in \hat{V}$ for which the stability subgroup belongs to $\mathcal{K}$ is open and dense in $\hat{V}$. It is not hard to show

THEOREM 5 [3]. *Let* G *and* $\mathcal{K}$ *be as above, and fix* $H \in \mathcal{K}$. *Then* $\mathrm{cor}(G)$ *consists exactly of all irreducible subrepresentations of* $\mathrm{ind}_H^K 1_H$.

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