Remarks on Non-Commutative Banach Function Spaces

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The purpose of this note is to outline an approach to the duality theory of non-commutative Banach function spaces which extends earlier work of Yeadon [Y1],[Y2]. The details will appear elsewhere.

Let $\mathcal{M}$ be a semifinite von Neumann algebra with a semifinite normal trace τ and let $\tilde{\mathcal{M}}$ be the $*$-algebra of τ-measurable operators (in the sense of Nelson [N]) affiliated with $\mathcal{M}$. For each $x \in \mathcal{M}$ and $0 < t \in \mathbb{R}$, the generalized singular value $\mu_t(x)$ is defined to be

$$\mu_t(x) = \inf\{\lambda \geq 0 : \tau(1 - e_\lambda) \leq t\}$$

where $\{e_\lambda\}$ denotes the spectral resolution of $|x|$. Our approach is based on the following result.

Proposition 1. *If $x, y \in \tilde{\mathcal{M}}$, then*

$$\sup\left\{ \int_E |\mu_t(x) - \mu_t(y)|dt : |E| \leq u \right\} \leq \int_0^u \mu_t(x-y)dt$$

for each $u \geq 0$.

The preceding result is a common generalization of the well known inequality of Markus ([M], Theorem 5.4) for compact operators and that of Lorentz and Shimogaki [LS] for the case that $\mathcal{M}$ is abelian. A similar inequality has been established by Hiai and Nakamura [HN] via the real interpolation method. Our present approach however is direct and is not based on interpolation methods.

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Suppose now that $L_\rho \subseteq L^0(\mathbb{R}^+, dm)$ is a rearrangement invariant Banach function space for which ρ is an invariant Fatou norm (see, for example [KPS], Chapter II). The non-commutative space $L_\rho(\mathcal{M})$ is defined by setting

$$L_\rho(\mathcal{M}) = \{x \in \tilde{\mathcal{M}} : \mu(x) \in L_\rho\}$$

and for $x \in L_\rho(\mathcal{M})$, $\|x\|_\rho$ is defined to be $\rho(\mu(x))$. The generalized Markus inequality given by Proposition 1 may be used to show that the spaces $L_\rho(\mathcal{M})$ are Banach spaces. We define the space

$$L_\rho(\mathcal{M})^\times = \{x \in \tilde{\mathcal{M}} : xy \in L^1(\mathcal{M}) \quad \text{for all} \quad y \in L_\rho(\mathcal{M})\}.$$

The space $L_\rho(\mathcal{M})^\times$ may be identified with a subspace of the Banach dual $L_\rho(\mathcal{M})^*$. If $L_\rho^\times$ denotes the (Köthe) associate space of L_ρ and if $L_\rho(\mathcal{M})^\times$ is equipped with the norm induced by $L_\rho(\mathcal{M})^*$, then we have the following identification.

Proposition 2.

$$L_\rho(\mathcal{M})^\times = L_\rho^\times(\mathcal{M})$$

In turn, the non-commutative associate space $L_\rho^\times(\mathcal{M})$ may be identified via a Radon-Nikodym type theorem as that subspace of the Banach dual $L_\rho(\mathcal{M})^*$ consisting of normal linear functionals.

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