On some C*-dynamical systems

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The problems addressed in this talk were partly motivated by attempts to classify the *-derivations defined on a class of smooth elements for an automorphic action α of a locally compact group G on a C*-algebra A. It has been proved that such a derivation δ has the form $\delta = d\alpha(X) + \tilde{\delta}$, where X is an element of the Lie algebra g of G and $d\alpha$ is a bounded derivation, under various circumstances, for example the following four:

(i) G is compact and there exists a faithful covariant representation π of A with $\pi(A^{\alpha})' \cap \pi(A)'' = \mathbb{C}1$. (Bratteli - Goodman and Longo, see [1, Theorem 2.9.2] and [8, Corollary 4.3]).

(ii) G is abelian, there exists sufficiently many G - invariant pure states and $\Gamma(\alpha) = \hat{G}$ (Batty - Ikunishi - Kishimoto, see [1, Theorem 2.9.10 and Corollary 2.9.17]).

(iii) G is abelian or compact, A is simple separable and there exists a sequence τ_n of automorphisms of A such that $\tau_n\alpha_g = \alpha_g\tau_n$ for all $n \in \mathbb{N}$ and $g \in G$, and $\lim_{n=\infty} || \tau_n(x)y - y\tau_n(x) || = 0$ for all $x, y \in A$. (Bratteli - Kishimoto, see [1, Theorem 2.9.31])

Actually, using (iii) one proves.

(iv) G is abelian or compact and there exists an irreducible representation π of A on a Hilbert space $\mathcal{H}$ which is strongly non-covariant in the sense that the center of the direct integral representation

$$\int_G^{\otimes} dg\ \pi\circ\alpha_g \quad \text{on } \mathcal{H}\otimes L^2(G) \quad \text{is } 1\otimes L^{\infty}(G),$$

and this is used in proving the decomposition of a derivation defined on the smooth elements.

Note that these conditions are typically fulfilled for product type actions of G on a UHF - algebra. Recently it har been realized that these conditions actually are equivalent under general circumstances, and they are also equivalent to the existence of an embedding (in Glimm's sense) of a product type action in (A, G, α). Results of this sort has been proved for compact abelian groups G in [4], [5], [2], for abelian groups in [6], and for compact groups in [4] [5], [3]. As a sample we cite part of the main result in [3]:

<u>Theorem</u> Let A be a separable C*-algebra, G a compact group with $G \neq \{ e \}$ and α a faithful action of G on A. The following 6 conditions are equivalent

(1) There exists a $\delta>0$ such that $\sup \{ \|xay\| \mid a\in A^{\alpha}, \|a\| = 1\} \geq \delta\|x\| \|y\|$ for all $x, y\in A$, where A^{α} is the fixed point algebra under α.

(2) Condition (1) with $\delta = 1$.

(3) There exists a faithful irreducible representation π of A such that $\pi|A^{\alpha}$ is irreducible.
(This is equivalent to condition (iv) above).

(4) There exists a pure invariant state ω of A such that $\pi_{\omega}|_{A^{\alpha}}$ is faithful (and thus π_{ω} is faithful and A^{α} is prime).

(5) If (ξ_n) is a sequence of finite dimensional representations of

G, $d_n = \dim(\xi_n)$, $\beta = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(\xi_n)$ is the corresponding product type representation of G on the UHF algebra $C = \bigotimes_{n=1}^{\infty} M_{d_n}$, then there exists a globally α - invariant projection q in the bidual A^{**} of A such that

(5a) $q \in B'$.

(5b) $qAq = Bq$.

(5c) $q \in J^{**} \subseteq A^{**}$ for any nonzero closed ideal J of A.

(5d) $(Bq, G, \alpha^{**}|Bq)$ is isomorpfic to (C, G, β) as C*- dynamical systems.

(6) For each irreducible representation γ of G of dimension d there exists a $\delta_\gamma > 0$ such that for each concrete matrix representative $\gamma_{ij}(g)$ of γ there exists a sequence $y_n = (y_{n_1}, \ldots, y_{n_d})$ of d-tuples in A such that $\alpha_g(y_n) = y_n\, [\gamma_{ij}(g)]$, $n \to y_{n1}$ is a central sequence, and $\lim_{n=\infty} \sup ||a\, y_{n1}|| \geq \delta_\gamma ||a||$ for any $a \in A$.

For a more detailed survey of these results, see [7].

References

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