3. INTEGRALS

Besides an integrating gauge, ρ, on a family of functions, $\mathcal{K}$, we consider a functional, μ, on $\mathcal{K}$ which can be extended to a continuous linear functional, μ_ρ, defined on the whole of $\mathcal{L} = \mathcal{L}(\rho,\mathcal{K})$. The continuity is understood with respect to the seminorm, q_ρ, induced by ρ on $\mathcal{L}$, defined in the previous chapter. More generally, we consider a map, μ, from $\mathcal{K}$ into an arbitrary Banach space, E, and a continuous linear map, μ_ρ, from $\mathcal{L}$ into E, generated by μ. Given a function $f \in \mathcal{L}$, the number, or vector, $\mu_\rho(f)$ is looked upon as the integral of f with respect to μ.

The classical case of integration with respect to a (positive) measure, ι, is obtained by taking for $\mathcal{K}$ a sufficiently rich family of (characteristic functions of) sets of finite measure and putting both ρ and μ equal to (the restriction to $\mathcal{K}$ of) ι. If μ is an additive set function having finite and σ-additive variation, then integration with respect to μ can be introduced by choosing ρ equal to the variation of μ. Of course, this choice is not available in general, and so, given an additive set function, μ, the problem of integration with respect to μ is reduced to that of finding a suitable ρ. This problem will be treated more systematically in Chapter 4.

Here we show how the integration with respect to Banach space valued measures, due to R.G. Bartle, N. Dunford and J.T. Schwartz, [2], fits into the presented scheme. Also in this chapter, the definitions of the Orlicz, the Sobolev and the Hardy spaces are shown to be special cases of the construction of the space $\mathcal{L}(\rho,\mathcal{K})$ for suitable choices of $\mathcal{K}$ and ρ.

A. Let $\mathcal{K}$ be a nontrivial family of functions on a space Ω. Let E be a Banach space. Let $\mu : \mathcal{K} \to E$ be a linear map. Recall that the domain of a linear map, or a linear functional, is not necessarily a vector space. (See Section 1E.)

We shall say that a gauge, ρ, on $\mathcal{K}$ integrates for the map μ if it is integrating (see Section 2D) and $|\mu(f)| \le c q_\rho(f)$, for some number $c \ge 0$ and every function $f \in \text{sim}(\mathcal{K})$.

If the gauge ρ integrates for the linear map $\mu : \mathcal{K} \to E$, then there exists a unique linear map $\mu_\rho : \mathcal{L}(\rho,\mathcal{K}) \to E$ such that $\mu_\rho(f) = \mu(f)$, for every $f \in \mathcal{K}$, and $|\mu_\rho(f)| \le cq_\rho(f)$, for some number $c \ge 0$ and every $f \in \mathcal{L}(\rho,\mathcal{K})$. In fact, μ has a unique linear extension on $\text{sim}(\mathcal{K})$. In fact, μ has a unique linear extension on $\text{sim}(\mathcal{K})$ (see Section 1B) which, by the assumption, is continuous with respect to q_ρ and $\text{sim}(\mathcal{K})$ is q_ρ-dense in $\mathcal{L}(\rho,\mathcal{K})$.

We shall also use the conventional notation

$$\text{(A.1)} \qquad \int_\Omega f \mathrm{d}_\rho\mu = \int_\Omega f(\omega)\mu(\mathrm{d}_\rho\omega) = \mu_\rho(f)$$

for every $f \in \mathcal{L}(\rho,\mathcal{K})$. The subscript is omitted when ρ is understood or immaterial.

If $\mathcal{K}$ happens to be a vector space, then an integrating gauge ρ on $\mathcal{K}$ integrates for the additive map $\mu : \mathcal{K} \to E$ if and only if there exists a constant $c \ge 0$ such that $|\mu(f)| \le c\rho(f)$, for every $f \in \mathcal{K}$. In fact, in this case, $\text{sim}(\mathcal{K}) = \mathcal{K}$ and $q_\rho(f) = \rho(f)$ for every $f \in \mathcal{K}$. For an arbitrary nontrivial family of functions $\mathcal{K}$, we have the following

PROPOSITION 3.1. *An integrating gauge ρ on $\mathcal{K}$ integrates for the additive map $\mu : \mathcal{K} \to E$ if and only if there exists a constant $c \ge 0$ such that $|\mu(f)| \le c\rho(f)$, for every $f \in \mathcal{K}$, and*

$$\text{(A.2)} \qquad \lim_{n\to\infty} \left| \sum_{j=1}^{n} c_j\mu(f) \right| = 0$$

for any numbers c_j and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots$, such that

$$\text{(A.3)} \qquad \sum_{j=1}^{\infty} |c_j|\rho(f_j) < \infty$$

and

$$\text{(A.4)} \qquad \sum_{j=1}^{\infty} c_jf_j(\omega) = 0$$

for every $\omega \in \Omega$ for which

$$\text{(A.5)} \qquad \sum_{j=1}^{\infty} |c_jf_j(\omega)| < \infty .$$

Proof. Let the gauge ρ integrate for μ. Let c_j be numbers and $f_j \in \mathcal{K}$ functions, $j = 1,2,\ldots,$ satisfying condition (A.3), such that (A.4) holds for every $\omega \in \Omega$ for which (A.5) does. Then, by Proposition 2.1,

$$\lim_{n\to\infty} q_\rho\left[\sum_{j=1}^{n} c_j f_j\right] = 0 .$$

Because, by the assumption,

$$\left|\sum_{j=1}^{n} c_j\, \mu(f_j)\right| \le c q_\rho\left[\sum_{j=1}^{n} c_j f_j\right],$$

for some $c \ge 0$ and every $n = 1,2,\ldots,$ the equality (A.2) follows.

Conversely, assume that ρ is an integrating gauge on $\mathcal{K}$, that there exists a number $c \ge 0$ such that $|\mu(f)| \le c\rho(f)$, for every $f \in \mathcal{K}$, and that (A.2) holds for any numbers c_j and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ satisfying (A.3), such that (A.4) holds for every $\omega \in \Omega$ for which (A.5) does. Then, for any function $f \in \mathcal{L}(\gamma,\mathcal{K})$, let $\tilde{\mu}(f)$ be the element of the space E such that

$$\tilde{\mu}(f) = \sum_{j=1}^{\infty} c_j \mu(f_j) ,$$

where the c_j are some numbers and the f_j some functions from $\mathcal{K}$, $j = 1,2,\ldots,$ satisfying condition (A.3), such that

$$f(\omega) = \sum_{j=1}^{\infty} c_j f_j(\omega)$$

for every $\omega \in \Omega$ for which the inequality (A.5) holds. By the assumption, the vector $\tilde{\mu}(f)$ depends on the function f alone and not on a particular choice of the numbers c_j and the functions f_j, $j = 1,2,\ldots$. Consequently, $\tilde{\mu}(f) = \mu(f)$ for every $f \in \mathrm{sim}(\mathcal{K})$. Furthermore, for every $\epsilon > 0$, we can choose these numbers and functions so that

$$\sum_{j=1}^{\infty} |c_j| \rho(f_j) < q_\rho(f) + \epsilon .$$

Hence, $|\tilde{\mu}(f)| \le cq_\rho(f) + \epsilon$, because $|\mu(f_j)| \le c\rho(f_j)$ for every $j = 1,2,\ldots$. So, $|\tilde{\mu}(f)| \le cq_\rho(f)$, for every $f \in \mathcal{L}(\rho,\mathcal{K})$.

Whenever applicable the following proposition is of course easier to use. By Proposition 2.27, it can be used, in particular, when $\mathcal{K}$ is a quasiring of sets.

PROPOSITION 3.2. *Let ρ be an integrating gauge on $\mathcal{K}$ such that, for every function $f \in \mathrm{sim}(\mathcal{K})$,*

$$q_\rho(f) = \inf \sum_{j=1}^{n} |c_j| \rho(f_j) ,$$

where the infimum is taken over all expressions of f in the form

$$f = \sum_{j=1}^{n} c_j f_j ,$$

with arbitrary $n = 1,2,\ldots,$ numbers c_j and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,n$. Let $\mu : \mathcal{K} \to E$ be an additive map such that $|\mu(f)| \le c\rho(f)$, for some $c \ge 0$ and every $f \in \mathcal{K}$.

Then the gauge ρ integrates for the map μ.

Proof. The assumptions imply that $|\mu(f)| \le cq_\rho(f)$, for every $f \in \mathrm{sim}(\mathcal{K})$.

B. Let $\mathcal{Q}$ be a quasiring of sets in a space Ω. Let ι be a σ-additive non-negative real valued set function on $\mathcal{Q}$. (See Sections 1D and 1F.)

Because $|\iota(f)| \le \iota(|f|)$, for every $f \in \mathrm{sim}(\mathcal{Q})$, by Proposition 2.13, ι is a gauge which integrates for itself. So, there exists a unique linear functional, ι_ι, on $\mathcal{L}(\iota,\mathcal{Q})$ such that $\iota_\iota(X) = \iota(X)$ for every $X \in \mathcal{Q}$, and the inequality $|\iota_\iota(f)| \le q_\iota(f)$ holds for every function $f \in \mathcal{L}(\iota,\mathcal{Q})$. Conforming to standard notation, we shall of course write

$$\iota(f) = \int_\Omega f d\iota = \int_\Omega f(\omega)\iota(d\omega) = \iota_\iota(f)$$

for every function $f \in \mathcal{L}(\iota,\mathcal{Q})$.

PROPOSITION 3.3. *If* $f \in \mathcal{L}(\iota,\mathcal{Q})$, *then also* $|f| \in \mathcal{L}(\iota,\mathcal{Q})$ *and* $q_\iota(f) = \iota(|f|)$ *for every function* $f \in \mathcal{L}(\iota,\mathcal{Q})$.

Proof. Let $\rho(f) = \iota(|f|)$, for every $f \in \mathrm{sim}(\mathcal{Q})$. Then the seminorm ρ is monotonic and, by Proposition 2.13, $\mathcal{L}(\iota,\mathcal{Q}) = \mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q}))$. Hence, by Proposition 2.20, if $f \in \mathcal{L}(\iota,\mathcal{Q})$, then also $|f| \in \mathcal{L}(\iota,\mathcal{Q})$. Now, the seminorms $f \mapsto \iota(|f|)$ and $f \mapsto q_\iota(f)$, $f \in \mathcal{L}(\iota,\mathcal{Q})$, are both q_ι-continuous and they agree on a q_ι-dense subspace, $\mathrm{sim}(\mathcal{Q})$, of $\mathcal{L}(\iota,\mathcal{Q})$. Therefore, they agree on the whole of $\mathcal{L}(\iota,\mathcal{Q})$.

The Beppo Levi monotone convergence theorem and the Lebesgue dominated convergence theorem are now special cases of the two respective statements of Proposition 2.21. The Fatou lemma can then be deduced in the well-known manner. (See e.g. [59], no. 20.)

Let $\mathcal{R}(\iota)$ be the family of all ι-integrable sets, that is, sets with characteristic function belonging to $\mathcal{L}(\iota,\mathcal{Q})$. Then $\mathcal{R}(\iota)$ is a δ-ring of sets in Ω . The existence of a (finite) non-negative σ-additive extension of ι onto the whole of $\mathcal{R}(\iota)$ is now obvious. Moreover, by Proposition 2.7, $\mathcal{L}(\iota,\mathcal{R}(\iota)) = \mathcal{L}(\iota,\mathcal{Q})$. Therefore, we may suppress the domain, $\mathcal{Q}$, of ι in the symbol for the space of ι-integrable functions and write simply $\mathcal{L}(\iota) = \mathcal{L}(\iota,\mathcal{Q})$.

There are now several possibilities of defining ι-measurable sets and functions. We may call a set ι-measurable if it belongs to the σ-algebra or just the σ-ring of sets generated by $\mathcal{R}(\iota)$. A larger family of ι-measurable sets is obtained if we call ι-measurable any set $X \subset \Omega$ such that $X \cap \Omega$ such that $X \cap Z \in \mathcal{R}(\iota)$ for every $Z \in \mathcal{R}(\iota)$. The choice of the definition depends of course on the purpose to which it is to be used. But in either case, it is customary to put $\iota(X) = \infty$ for every ι-measurable set X which is not ι-integrable.

So, the set function ι determines a measure in the space Ω which is of course denoted still by ι .

It should be noted perhaps that the term "measure" is not used in the same fashion throughout the literature. It often designates a non-negative extended real valued (∞ is allowed as a value) set function on a σ-ring of sets covering the whole space or a σ-algebra. Other authors designate by this term the corresponding integral, that is, the linear functional whose value at an integrable function, f, is equal to the integral of f with respect to the measure in question, or even its restriction to a linear subspace dense in the L^1-seminorm in the space of all integrable functions.

This lack of uniformity will not cause any inconvenience in the sequel, because, however the term "measure" is interpreted, specifying a measure, ι, entails the specification of the following objects: a vector lattice, $\mathcal{L}(\iota)$, of functions on Ω and a positive linear functional, ι, on $\mathcal{L}(\iota)$ such that $\mathcal{L}(\rho,\mathcal{L}(\iota)) = \mathcal{L}(\iota)$, where $\rho(f) = \iota(|f|)$ for every $f \in \mathcal{L}(\iota)$, and if $\mathcal{R}(\iota)$ is the family of sets (with characteristic functions) belonging to $\mathcal{L}(\iota)$, then $\mathcal{L}(\rho,\mathcal{R}(\iota)) = \mathcal{L}(\iota)$. The functions belonging to $\mathcal{L}(\iota)$ and sets belonging to $\mathcal{R}(\iota)$ are then called integrable with respect to the measure ι or ι-integrable.

Now, returning to the the measure, ι, determined by its values on the quasiring $\mathcal{Q}$, let us note that, in view of Proposition 2.13 and Proposition 3.1, the definitions adopted in Sections 2A, 2D and 3A, give us a direct and economical representation of integrable functions circumventing the Carathéodory theory of extension of ι onto all measurable sets. Namely, a function f is ι-integrable if and only if there exist numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots$, such that

$$\sum_{j=1}^{\infty} |c_j|\iota(X_j) < \infty$$

and

$$f(\omega) = \sum_{j=1}^{\infty} c_j X_j(\omega)$$

for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |c_j| X_j(\omega) < \infty .$$

The integral of such a function f is then given by the formula

$$\int_{\Omega} f \mathrm{d}\iota = \sum_{j=1}^{\infty} c_j \iota(X_j) .$$

It is striking how close this characterization of integrability and integral is to the ideas of Archimedes, especially to one of his calculations of the area of a parabolic section; see e.g. [21]. As noted by J. Mikusiński in the Preface to his book [50], it makes the presentation of the Lebesgue integral at elementary level more viable than that of the Riemann integral. For further elementary comments, see [33].

An approach to integration along similar lines was suggested by J.L. Kelley and T.P. Srinivasan, [28]; see also [29].

As suggested, a measure in the space Ω is sometimes specified by specifying the values of the corresponding integral on a sufficiently rich vector subspace of the space of all integrable functions. It is done by invoking a theory of the Daniell integral or its generalization. Such theories too are instances of the general scheme presented in Section A. To describe the main points, let us recall some notation.

For a real valued function, f, on Ω, we write $f^+ = \frac{1}{2}(|f|+f)$, $f^- = \frac{1}{2}(|f|-f)$ and $f\wedge 1 = g$, where $g(\omega) = \frac{1}{2}(f(\omega)+1-|f(\omega)-1|)$ for every $\omega \in \Omega$. For a nontrivial family, $\mathcal{K}$, of real valued functions on Ω, we write $\mathcal{K}^+ = \{f \in \mathcal{K} : f \geq 0\}$ and $\mathcal{K}^+ - \mathcal{K}^+ = \{f-g : f \in \mathcal{K}^+, g \in \mathcal{K}^+\}$.

Let $\mathcal{K}$ be a vector lattice of real valued functions on the space Ω. A positive linear functional, ι, on $\mathcal{K}$ (see Section 2E) is called a Daniell integral, if $\iota(f_n) \to 0$, as $n \to \infty$, for any functions $f_n \in \mathcal{K}$ such that $f_n(\omega) \geq f_{n+1}(\omega)$, $n = 1,2,\ldots$, and $f_n(\omega) \to 0$, for every $\omega \in \Omega$, as $n \to \infty$.

It is easy to show using Proposition 2.12, say, that a positive linear functional, ι, on $\mathcal{K}$ is a Daniell integral, if and only if, the seminorm, ρ, defined by $\rho(f) = \iota(|f|)$, for every $f \in \mathcal{K}$, is integrating.

Assume now that ι is a Daniell integral on $\mathcal{K}$. It is then obvious that the seminorm ρ integrates for the functional ι. Let us write $\mathcal{L}(\iota) = \mathcal{L}(\rho,\mathcal{K})$ and denote by $\mathcal{R}(\iota)$ the family of sets (with characteristic functions) belonging to $\mathcal{L}(\iota)$.

We say that the Daniell integral ι satisfies the Stone condition if the function $f\wedge 1$ belongs to $\mathcal{L}(\iota)$ whenever the function f does. It is well-known that, if $f\wedge 1$ belongs to $\mathcal{K}$ whenever f does, then ι satisfies the Stone condition.

Let ι_ρ be the unique continuous linear functional on $\mathcal{L}(\iota)$ that extends ι. (See Section A.) Its restriction to $\mathcal{R}(\iota)$ is a non-negative σ-additive set function. M.H. Stone has shown, [62], that $\mathcal{L}(\iota) = \mathcal{L}(\iota_\rho, \mathcal{R}(\iota))$ if and only if ι satisfies the Stone condition.

M. Leinert, [40], and H. König, [36], have generalized the notion of a Daniell integral by requiring that $\mathcal{K}$ be merely a vector space and not necessarily a vector lattice. Such generalization is interesting because it represents the abstract core of situations not infrequently occurring in analysis; see [37], [41].

So, let $\mathcal{K}$ be a vector space of real valued functions on Ω and let ι be a positive linear functional on $\mathcal{K}$. For any real valued function f on Ω, let

$$\iota^+(f) = \inf \sum_{j=1}^{\infty} \iota(f_j) ,$$

where the infimum is taken over all choices of functions $f_j \in \mathcal{K}^+$, $j = 1,2,\ldots$, such that

$$f(\omega) \leq \sum_{j=1}^{\infty} f_j(\omega) \tag{B.1}$$

for every $\omega \in \Omega$. The possibility that $\rho(f) = \infty$ is of course admitted. Let, further,

$$\bar{\iota}(f) = \inf \sum_{j=1}^{\infty} \iota(f_j) ,$$

where the infimum is taken over all choices of functions $f_1 \in \mathcal{K}$ and $f_j \in \mathcal{K}^+$, $j = 2,3,\ldots$, such that (B.1) holds for every $\omega \in \Omega$.

We say that the functional ι satisfies the König continuity condition, if $\iota^+(f^+) = \iota(f) + \iota^+(f^-)$, for every function $f \in \mathcal{K}$.

We say that the functional ι satisfies the Leinert continuity condition, if $\iota^+(f^+) \geq \iota(f)$, for every $f \in \mathcal{K}$.

Clearly, if ι satisfies the König continuity condition then it satisfies the Leinert continuity condition. Moreover, if $\mathcal{K}$ happens to be a vector lattice, then ι satisfies the König continuity condition if and only if it is a Daniell integral, and also it satisfies the Leinert continuity condition if and only if it is a Daniell integral.

Now, assume that $\mathcal{K} = \mathcal{K}^+ - \mathcal{K}^+$. Then we can define gauges, ρ_1 and ρ_2, on $\mathcal{K}$ by letting $\rho_1(f) = \bar{\iota}(|f|)$ and $\rho_2(f) = \iota^+(|f|)$, respectively, for every $f \in \mathcal{K}$. Let, further, ρ_3 be the gauge on $\mathcal{K}^+$ such that $\rho_3(f) = \iota(f)$, for every $f \in \mathcal{K}^+$.

If ι satisfies the Leinert continuity condition, then the gauge ρ_3 is integrating. The gauge ρ_2, which is a seminorm, is automatically integrating. If ι satisfies the König continuity condition, then $\rho_1 = \rho_2$, the gauge ρ_1 is integrating and $\mathcal{L}(\rho_1,\mathcal{K}) = \mathcal{L}(\rho_3,\mathcal{K}^+)$. We may note that, while the König condition is sufficient, it is not necessary for the gauge ρ_1 to be integrating. However, the König condition is convenient to use without loss in the context of uniform algebras.

For a more complete consolidated exposition we refer to [37].

C. Natural seminorms in the classical function spaces defined in terms of a measure usually turn out to be integrating.

Let ι be a measure in a space Ω and p a real number such that $1 \leq p < \infty$. The family of all functions f on Ω such that $f|f|^{p-1} \in \mathcal{L}(\iota)$ is denoted by $\mathcal{L}^p(\iota)$. So, in particular, $\mathcal{L}^1(\iota) = \mathcal{L}(\iota)$. It is well-known that $\mathcal{L}^p(\iota)$ is a vector space. Moreover, if

$$\|f\|_{p,\iota} = \left[\int_\Omega |f|^p d\iota\right]^{1/p},$$

for every $f \in \mathcal{L}^p(\iota)$, then $\|\cdot\|_{p,\iota}$ is an integrating seminorm on $\mathcal{L}^p(\iota)$ such that $\mathcal{L}(\|\cdot\|_{p,\iota},\mathcal{L}^p(\iota)) = \mathcal{L}^p(\iota)$. This fact is implicit in the standard proof of the completeness of $\mathcal{L}^p(\iota)$ which avoids the notion of convergence in measure. The induced normed space is of course denoted by $L^p(\iota)$. M.H. Stone, [62], introduced the L^p-spaces along these lines in the context of Daniell integrals instead of measures.

These spaces (based on a measure rather than a Daniell integral) are special cases of the general Banach function spaces studied systematically by W.A.J. Luxemburg and A.C. Zaanen in a series of papers of which the first one, [47], contains an introduction to the subject with the relevant historical background. See also [72], §§63-64.

Let $\mathcal{S}$ be a σ-algebra of sets in the space Ω and let $\mathcal{Z}$ be a σ-ideal in the space Ω (see Section 1D) such that $\mathcal{Z} \subset \mathcal{S}$. Let $\mathcal{M} = \mathcal{M}(\mathcal{S})$ be the family of all complex valued $\mathcal{S}$-measurable functions and $\mathcal{M}^+$ the family of non-negative real-valued functions belonging to $\mathcal{M}$. Let $\mathcal{N} = \mathcal{M}(\mathcal{Z})$ be the family of all functions f on Ω such that the set $\{\omega : f(\omega) \neq 0\}$ belongs to $\mathcal{Z}$. Clearly, $\mathcal{N} \subset \mathcal{M}$ because $\mathcal{Z} \subset \mathcal{S}$.

Following [48], Definition 3.1, a functional, ρ, from $\mathcal{M}$ into $[0,\infty]$ (the value ∞ is allowed) will be called a function norm (with respect to $\mathcal{S}$ and $\mathcal{Z}$) if it has the following properties:

(i) $\rho(f) = 0$ if and only if $f \in \mathcal{N}$;

(ii) $\rho(f) = \rho(|f|)$ for every $f \in \mathcal{M}$;

(iii) $\rho(\alpha f) = |\alpha|\rho(f)$ for every number α and every function $f \in \mathcal{M}$;

(iv) $\rho(f + g) \le \rho(f) + \rho(g)$ for every $f \in \mathcal{M}^+$ and $g \in \mathcal{M}^+$; and

(v) if $f \in \mathcal{M}^+$, $g \in \mathcal{M}^+$ and $f \le g$, then $\rho(f) \le \rho(g)$.

Given a function norm, ρ, let $\mathcal{K}_\rho = \{f \in \mathcal{M} : \rho(f) < \infty\}$. Then the restriction of ρ to $\mathcal{K}_\rho$ is a seminorm; it will be called the seminorm induced by the function norm ρ and still denoted by ρ. Our aim is to characterize those function norms which induce in this manner integrating seminorms such that $\mathcal{K}_\rho = \mathcal{L}(\rho,\mathcal{K}_\rho)$ and the family of ρ-null functions coincides with $\mathcal{N}$.

The function norm, ρ, is said to have the Riesz-Fischer property, see [48], Definition 4.1, if, for any functions $f_j \in \mathcal{M}^+$, $j = 1,2,\ldots,$ such that

(C.1)
$$\sum_{j=1}^{\infty} \rho(f_j) < \infty ,$$

the set, Y, of all points $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} f_j(\omega) = \infty$$

belongs to $\mathcal{Z}$, and, if f is a function on Ω such that

(C.2) $$f(\omega) = \sum_{j=1}^{\infty} f_j(\omega)$$

for every $\omega \in \Omega$ not belonging to Y, then $f \in \mathcal{K}_\rho$.

For the sake of unity of style, our formulation differs from that of W.A.J. Luxemburg and A.C. Zaanen, but the difference is merely technical. Luxemburg and Zaanen achieve some simplicity of the formulation by admitting into $\mathcal{M}$ also functions with infinite values. However, the resulting theories are equivalent because then, for every function $f \in \mathcal{M}$ such that $\rho(f) < \infty$, the set $Y = \{\omega : |f(\omega)| = \infty\}$ belongs to $\mathcal{Z}$. Indeed, $Y \le n^{-1}|f|$, and, by (iii) and (v), $\rho(Y) \le n^{-1}\rho(f)$, for every $n = 1,2,\ldots$. So, $\rho(Y) = 0$, and, by (i), $Y \in \mathcal{Z}$.

The following lemma and proposition are due to I. Halperin and W.A.J. Luxemburg, [20].

LEMMA 3.4. *If ρ has the Riesz-Fischer property, then*

$$\rho(f) \le \sum_{j=1}^{\infty} \rho(f_j),$$

whenever $f_j \in \mathcal{M}$, $j = 1,2,\ldots,$ are functions satisfying condition (C.1) *and $f \in \mathcal{M}$ is a function such that*

$$|f(\omega)| \le \sum_{j=1}^{\infty} |f_j(\omega)|$$

for every $\omega \in \Omega$.

Proof. If not, there exist such f_j, $j = 1,2,\ldots,$ and f as in the statement of the lemma, but

$$\rho(f) > \alpha + \sum_{j=1}^{\infty} \rho(f_j)$$

with some $\alpha > 0$. Consequently, for each $k = 1,2,\ldots,$ there exist functions $f_{kj} \in \mathcal{M}^+$, $j = 1,2,\ldots,$ and a function $f_k \in \mathcal{M}^+$ such that

$$f_k(\omega) = \sum_{j=1}^{\infty} f_{kj}(\omega)$$

for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} f_{kj}(\omega) < \infty$$

and

(C.3) $$\infty > \rho(f_k) > k + \sum_{j=1}^{\infty} \rho(f_{kj}) .$$

Because

$$\rho\left[\sum_{j=1}^{r} f_{kj}\right] \le \sum_{j=1}^{r} \rho(f_{kj})$$

for every $r = 1,2,...,$ we can assume that, besides (C.3),

$$\sum_{j=1}^{\infty} \rho(f_{kj}) < k^{-2}$$

for every $k = 1,2,...$. Let us arrange the functions f_{kj} , $j = 1,2,...,$ $k = 1,2,...,$ into a single sequence q_n , $n = 1,2,...$. Then

$$\sum_{n=1}^{\infty} \rho(g_n) < \sum_{k=1}^{\infty} k^{-2} < \infty .$$

Let g be a function such that

$$g(\omega) = \sum_{n=1}^{\infty} g_n(\omega)$$

for every $\omega \in \Omega$ for which

$$\sum_{n=1}^{\infty} g_n(\omega) < \infty .$$

Then, for every $k = 1,2,...,$ there is a function $h_k \in \mathcal{N}$ such that $g(\omega) \ge f_k(\omega) + h_k(\omega)$, for every $\omega \in \Omega$, and, hence,

$$\rho(g) \ge \rho(f_k) \ge k .$$

So, $\rho(g) = \infty$, contrary to ρ having the Riesz-Fischer property.

PROPOSITION 3.5. *The function norm* ρ *has the Riesz-Fischer property if and only if the induced seminorm on* $\mathcal{K}_\rho$ *is integrating and* $\mathcal{L}(\rho,\mathcal{K}_\rho) = \mathcal{K}_\rho$.

Proof. If ρ has the Riesz-Fischer property, let $f_j \in \mathcal{K}_\rho$, $j = 1,2,\ldots,$ be functions satisfying (3.2) and let f be a function on Ω such that (C.2) holds for every $\omega \in \Omega$ such that

$$\text{(C.4)} \qquad \sum_{j=1}^{\infty} |f_j(\omega)| < \infty .$$

Then $f \in \mathcal{M}$ and $\rho(f) < \infty$, that is, $f \in \mathcal{K}_\rho$, because $\mathcal{Z} \subset \mathcal{S}$ and ρ has the Riesz-Fischer property. Furthermore,

$$f(\omega) - \sum_{j=1}^{n} f_j(\omega) = \sum_{j=n+1}^{\infty} f_j(\omega)$$

for every $\omega \in \Omega$ for which (C.4) holds, and so, by Lemma 3.4,

$$\rho\left[f - \sum_{j=1}^{n} f_j\right] \le \sum_{j=n+1}^{\infty} \rho(f_j) ,$$

for every $n = 1,2,\ldots$. Therefore, ρ is integrating and $\mathcal{L}(\rho,\mathcal{K}_\rho) = \mathcal{K}_\rho$.

Conversely, if the seminorm induced by ρ is integrating and $\mathcal{L}(\rho,\mathcal{K}_\rho) = \mathcal{K}_\rho$, then, obviously, ρ has the Riesz-Fischer property.

Besides the L^p-spaces, the classical spaces which are covered by this proposition include notably the Orlicz spaces.

Let ι be a σ-finite measure in the space Ω ; that is, the space Ω is equal to the union of a sequence of ι-integrable sets. Let $\mathcal{M}$ be the family of all ι-measurable functions; the assumption of σ-finiteness implies that the definitions of measurability mentioned in the previous section are equivalent. $\mathcal{N}$ is the family of ι-null functions. Let Φ be a Young function. (See Section 1G.)

For any function $f \in \mathcal{M}$, let

$$M_\Phi(f) = \int_\Omega \Phi(|f|)\mathrm{d}\iota .$$

We are using the convention that, if the function $\omega \mapsto \Phi(|f(\omega)|)$, $\omega \in \Omega$, is not ι-integrable, then $M_\Phi(f) = \infty$. Let, further,

$$\rho_\Phi(f) = \inf\{k : k > 0 \, , \, M_\Phi(k^{-1}f) \le 1\}$$

for every $f \in \mathcal{M}$. Now we are using the convention that the infimum of the empty set is ∞.

PROPOSITION 3.6. *The functional* ρ_Φ *is a function norm having the Riesz-Fischer property.*

Proof. For brevity, we write $\rho = \rho_\Phi$.

If the set $\{\omega : |f(\omega)| > 0\}$ has non-zero measure, then, for some $\epsilon > 0$, the set $X = \{\omega : |f(\omega)| \ge \epsilon\}$ has non-zero measure and, hence, $M_\Phi(f) \ge \Phi(\epsilon)\iota(X) > 0$. Consequently, $0 < \rho(f) \le \infty$. Conversely, if $\iota(\{\omega : |f(\omega)| > 0\}) = 0$, then $M_\Phi(\alpha f) = 0$, for every $\alpha > 0$, and, hence, $\rho(f) = 0$.

Assume that $0 < \rho(g) < \infty$. Choosing a decreasing sequence of numbers k_n, $n = 1,2,\ldots,$ tending to $\rho(g)$ and applying the sequential form of the Beppo Levi theorem on the functions $k_n^{-1}|g|$, $n = 1,2,\ldots,$ tending point-wise monotonically to $(\rho(g))^{-1}|g|$, we deduce that $M_\Phi((\rho(g))^{-1}g) \le 1$. From this observation we deduce further that, if $|f| \le |g|$, then $\rho(f) \le \rho(g)$. For, if $|f| \le |g|$, then $M_\Phi((\rho(g))^{-1}f) \le M_\Phi((\rho(g))^{-1}g) \le 1$.

Now, assuming that $f \ge 0$, $g \ge 0$, $\rho(f) + \rho(g) = \gamma > 0$, let $\rho(f) = \alpha\gamma$ and $\rho(g) = \beta\gamma$, so that $\alpha + \beta = 1$. Then, by the Jensen inequality,

$$M_\Phi((f+g)/\gamma) = M_\Phi(\alpha f/\alpha\gamma + \beta g/\beta\gamma) \le \alpha M_\Phi(f/\rho(f)) + \beta M_\Phi(g/\rho(g)) \le \alpha + \beta = 1 \, ,$$

and so, $\rho(f+g) \le \gamma$.

From these remarks and from the definition of ρ, it follows easily that ρ satisfies all the requirements (i) - (v), which means that it is a function norm on $\mathcal{M}$.

To prove that ρ has the Riesz-Fischer property, let $g_n \in \mathcal{M}^+$, $n = 1,2,\ldots,$ be functions forming a non-decreasing sequence such that $0 < \alpha = \sup\{\rho(g_n) : n = 1,2,\ldots\} < \infty$. Because $M_\Phi(\alpha^{-1}g_n) \le 1$, for every $n = 1,2,\ldots,$ by the Beppo Levi theorem, $M_\Phi(\alpha^{-1}g) \le 1$. Hence, $\rho(g) \le \alpha$. In particular $g \in \mathcal{K}_\rho$. It is now evident that ρ has the Riesz-Fischer property.

The proof of Proposition 3.6 gives slightly more than the Riesz-Fischer property of the function norm ρ_Φ . For more details we refer to [72].

Let us note that the space $\mathcal{K}_\rho$, for $\rho = \rho_\Phi$, consists of all functions $f \in \mathcal{M}$ for which there exists a number $k > 0$ such that $M_\Phi(kf) < \infty$. Furthermore, if the Young function Φ satisfies condition (Δ_2) , then $f \in \mathcal{K}_\rho$ if and only if $M_\Phi(f) < \infty$. If $\iota(\Omega) < \infty$ and Φ satisfies condition (Δ_2) for large values of the argument, then $f \in \mathcal{K}_\rho$ if and only if $M_\Phi(f) < \infty$.

The space $\mathcal{K}_\rho$ is conventionally denoted by $\mathcal{L}^\Phi(\iota)$ and the corresponding normed space by $L^\Phi(\iota)$. These spaces are known as Orlicz spaces. One writes $\|f\|_{\Phi,\iota} = \rho_\Phi(f)$, that is,

$$\|f\|_{\Phi,\iota} = \inf\{k>0 : \int_\Omega \Phi(k^{-1}|f(\omega)|)\iota(d\omega) \le 1\} ,$$

for any function $f \in \mathcal{L}^\Phi(\iota)$. The seminorm $\|\cdot\|_{\Phi,\iota}$ and the induced norm on $L^\Phi(\iota)$ are called the Luxemburg seminorm and the Luxemburg norm, respectively. Another seminorm on $\mathcal{L}^\Phi(\iota)$ is defined by the formula

$$\|f\|^0_{\Phi,\iota} = \sup\{ | \int_\Omega fg \, d\iota | : \int_\Omega \Psi(|g|)d\iota \le 1\} ,$$

for every $f \in \mathcal{L}^\Phi(\iota)$, where Ψ is the Young function complementary to Φ . (See Section 1G.) The so-defined seminorm and the corresponding norm on $L^\Phi(\iota)$ are called the Orlicz seminorm and the Orlicz norm, respectively. The inequalities

$$\|f\|_{\Phi,\iota} \le \|f\|^0_{\Phi,\iota} \le 2\|f\|_{\Phi,\iota}$$

hold for every $f \in \mathcal{L}^\Phi(\iota)$, so that the Luxemburg and the Orlicz norms are equivalent.

The classical reference about Orlicz spaces is [38]. Useful information can also be found in [39], especially Sections 3.1 - 3.9, and of course elsewhere.

For the definition of the class $\mathcal{L}^\Phi(\iota)$, the assumption that the measure ι be σ-finite is of course not necessary. Explicitly, $\mathcal{L}^\Phi(\iota)$ consists of the ι-measurable functions f on Ω for which there exists a number $k > 0$ (depending on f) such that

$$\int_{\Omega} \Phi(k^{-1}|f(\omega)|)\iota(d\omega) < \infty .$$

D. Another class of important and extensively studied integrating seminorms is constituted by the seminorms inducing the natural norms of the Sobolev spaces. Following A. Kufner, O. John and S. Fučík, [39], Section 5.1, we present a general scheme for introducing these spaces which may be useful also in other contexts.

Let $\mathcal{K}$ be a vector space of functions on a space Ω. Let ρ_0 be an integrating seminorm on $\mathcal{K}$. Let J be an index set. For every $\alpha \in J$, let ρ_α be an integrating seminorm on a vector space, $\mathcal{L}_\alpha$, of functions on a space Ω_α such that $\mathcal{L}_\alpha = \mathcal{L}(\rho_\alpha, \mathcal{L}_\alpha)$ and let $S_\alpha : \mathcal{K} \to \mathcal{L}_\alpha$ be a linear map.

The maps S_α, $\alpha \in J$, will be called collectively closable if $\rho_\alpha(h_\alpha) = 0$ for any functions $h_\alpha \in \mathcal{L}_\alpha$ for which there exist functions $g_n \in \mathcal{K}$, $n = 1,2,\ldots,$ such that

(D.1) $$\lim_{n\to\infty} \rho_0(g_n) = 0 \text{ and } \lim_{n\to\infty} \rho_\alpha(S_\alpha g_n - h_\alpha) = 0 ,$$

for every $\alpha \in J$.

PROPOSITION 3.7. *If the set J is finite and the maps S_α, $\alpha \in J$, are collectively closable, then the functional, ρ, defined by*

(D.2) $$\rho(f) = \rho_0(f) + \sum_{\alpha\in J} \rho_\alpha(S_\alpha f) ,$$

for every $f \in \mathcal{K}$, is an integrating seminorm on $\mathcal{K}$.

Proof. Clearly, ρ is a seminorm. Let $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ be functions such that

$$\sum_{j=1}^{\infty} \rho(f_j) < \infty$$

and

$$\sum_{j=1}^{\infty} f_j(\omega) = 0$$

for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |f_j(\omega)| < \infty .$$

Then

$$\sum_{j=1}^{\infty} \rho_0(f_j) < \infty \text{ and } \sum_{j=1}^{\infty} \rho_\alpha(f_j) < \infty$$

for every $\alpha \in J$. Let

$$g_n = \sum_{j=1}^{n} f_j ,$$

for $n = 1,2,\dots$. Then, by Proposition 2.8, $\rho_0(g_n) \to 0$ as $n \to \infty$, because the seminorm ρ_0 is integrating. Furthermore, by Theorem 2.4, for every $\alpha \in J$, there exists a function $h_\alpha \in \mathcal{L}_\alpha$ such that $\rho_\alpha(S_\alpha g_n - h_\alpha) \to 0$ as $n \to \infty$. Then $\rho_\alpha(h_\alpha) = 0$, for every $\alpha \in J$, because the maps S_α, $\alpha \in J$, are collectively closable. By Proposition 2.1, $\rho_\alpha(S_\alpha g_n) \to 0$, for every $\alpha \in J$, and, hence, $\rho(g_n) \to 0$ as $n \to \infty$, because the set J is finite. By Proposition 2.8, the seminorm ρ is integrating.

To describe the most important particular cases, let $n \geq 1$ be an integer. Let ι be the Lebesgue measure in $\mathbb{R}^n$. Let Ω be a non-empty bounded open set in $\mathbb{R}^n$. Let $k \geq 0$ be an integer and $1 \leq p \leq \infty$. For J, we take the set of all n-tuples $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of integers $\alpha_1 \geq 0$, $\alpha_2 \geq 0, \dots, \alpha_n \geq 0$ such that

$$0 < |\alpha| = \sum_{j=1}^{n} \alpha_j \leq k .$$

For any such $\alpha \in J$, let $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n}$, where $D_1, D_2, \dots, D_n$ are the operators of partial differentiation on $\mathbb{R}^n$ with respect to the first, second, ..., n-th variable, respectively.

Now, for $\mathcal{K}$, we take the space of all restrictions to Ω of C^∞-functions on $\mathbb{R}^n$ and let

$$(\rho_0(f))^p = \int_\Omega |f(\omega)|^p \iota(\mathrm{d}\omega) ,$$

for every $f \in \mathcal{K}$. For every $\alpha \in J$, we take $\mathcal{L}_\alpha$ to be the space of all ι-measurable functions on Ω such that $(\rho_0(f))^p < \infty$ and put $\rho_\alpha(f) = \rho_0(f)$, for every $f \in \mathcal{L}_\alpha$. Finally, we put $S_\alpha f = \mathrm{D}^\alpha f$, for every $f \in \mathcal{K}$ and every $\alpha \in J$.

LEMMA 3.8. *For every $\alpha \in J$, the map $S_\alpha : \mathcal{K} \to \mathcal{L}_\alpha$ is closable.*

Proof. Let $g_n \in \mathcal{K}$, $n = 1,2,\ldots$, and $h_\alpha \in \mathcal{L}_\alpha$, be functions such that (D.1) holds. Then, by the Green formula and the Hölder inequality,

$$\int_\Omega h_\alpha \varphi \mathrm{d}\iota = \lim_{n\to\infty} \int_\Omega \mathrm{D}^\alpha g_n \varphi \mathrm{d}\iota = (-1)^{|\alpha|} \lim_{n\to\infty} \int_\Omega g_n \mathrm{D}^\alpha \varphi \mathrm{d}\iota = 0 ,$$

for every C^∞-function φ whose support is contained in Ω. Consequently, $\rho_\alpha(h_\alpha) = 0$.

This lemma obviously implies that the maps S_α, $\alpha \in J$, are collectively closable. So, by Proposition 3.7, the seminorm, ρ, defined by (D.2) for every $f \in \mathcal{K}$, is integrating. The corresponding Banach space $L(\rho,\mathcal{L})$ is usually denoted by $W^{k,p}(\Omega)$.

Let $\mathcal{K}_0$ be the space of all C^∞ functions with supports contained in Ω. Then $\mathcal{K}_0 \subset \mathcal{K}$, but the restriction of ρ to $\mathcal{K}_0$ is still denoted by ρ. The corresponding space $L(\rho,\mathcal{K}_0)$ is denoted by $W_0^{k,p}(\Omega)$. The spaces $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ do not coincide, in general.

For further discussion, examples and ramification along these lines we refer to [39], Chapters 5, 7 and 8. The literature on the Sobolev space is of course very large.

E. Both the classical and the real-variable definitions of the Hardy spaces can be viewed as the special cases of the construction of the space $L(\rho,\mathcal{K})$ with a suitably chosen integrating gauge ρ on a family of functions $\mathcal{K}$. Let us start with the classical definition.

Let $1 \le p \le \infty$. Let $\mathcal{K} = \mathcal{K}_\delta$ be the family of complex functions on the closed unit disc $\bar{D}_1 = \{z \in \mathbb{C} : |z| \le 1\}$ for which there exists a δ such that $0 < \delta \le \infty$ and $f \in \mathcal{K}$ if and only if f has an analytic continuation on the disc $D_{1+\delta} = \{z \in \mathbb{C} : |z| < 1 + \delta\}$. In particular, $\mathcal{K}_\infty$ consists of the restrictions to $\bar{D}_1$ of the entire functions. Given an r such that $0 < r < 1$, let

$$\rho_r(f) = \left[\frac{1}{2\pi}\int_0^{2\pi} |f(r\exp(i\theta))|^p d\theta\right]^{1/p}$$

for every $f \in \mathcal{K}$. Finally, let

$$\rho(f) = \sup\{\rho_r(f) : 0 < r < 1\}$$

for every $f \in \mathcal{K}$.

PROPOSITION 3.9. *The functional ρ is an integrating seminorm on $\mathcal{K}$.*

Proof. We have $\rho(f) < \infty$ for every $f \in \mathcal{K}$ because every function belonging to $\mathcal{K}$ is bounded on $\bar{D}_1$. Using the analyticity of the functions in $\mathcal{K}$, it is easy to deduce that each seminorm ρ_r, $0 < r < 1$, is integrating. Then, by Proposition 2.14, the seminorm ρ is integrating too.

The space $L(\rho,\mathcal{K})$ is usually denoted by H^p.

It may be noted that, for $\mathcal{K}$, the space of all complex polynomials could be taken, which is even smaller than $\mathcal{K}_\infty$, or, on the other hand, the space of all functions continuous in $\bar{D}_1$ and analytic in the open disc D_1, which is larger than all the spaces $\mathcal{K}_\delta$, $\delta > 0$.

The given definition of the space H^p can of course be adapted to the case of the space $\mathbb{R}^2_+ = \{(x,y) : x \in \mathbb{R}, y \geq 0\}$, or even $\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n, y \geq 0\}$ for any $n = 1,2,\ldots$, replacing the disc $\bar{D}_1$.

Let us turn now to the real variable definition. We will consider only the H^1 spaces on $\mathbb{R}^n$. That will suffice for our purposes; any attempt to treat systematically the Hardy spaces, or even just their connection with the theory of integrating gauges, is out of place here anyway. We may refer, however, to the survey [6] in which the history and the richness of the subject are elegantly presented.

Let $n \geq 1$ be an integer. Let ι be the Lebesgue measure in $\mathbb{R}^n$.

By an H^1-atom in $\Omega = \mathbb{R}^n$ is understood any function, f, for which there exists a (solid) ball B such that $|f(\omega)| \leq (\iota(B))^{-1}B(\omega)$, for ι-almost every $\omega \in \Omega$, and $\iota(f) = 0$. We say that the atom f is supported by the ball B. Let $\mathcal{K} = \mathcal{K}_{\mathrm{a}}(H^1(\mathbb{R}^n))$ be the family of all H^1-atoms in Ω.

For every $f \in \mathcal{K}$, let

$$\rho_{\mathrm{a}}(f) = \inf \sum_{j=1}^{\infty} |c_j| ,$$

where the infimum is taken over all choices of the numbers c_j, $j = 1,2,\ldots,$ such that

(E.1)
$$\sum_{j=1}^{\infty} |c_j| < \infty ,$$

and there exist functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ such that

(E.2)
$$f(\omega) = \sum_{j=1}^{\infty} c_j f_j(\omega) ,$$

for every $\omega \in \Omega$ for which

(E.3)
$$\sum_{j=1}^{\infty} |c_j| \, |f_j(\omega)| < \infty .$$

If c is a number and $f \in \mathcal{K}$, then, clearly, $\iota(|cf|) \leq |c|$. Therefore, condition (E.1) implies that the inequality (E.3) is satisfied ι-almost everywhere. So, by the Beppo Levi theorem, $\iota(|f|) \leq \rho_{\mathrm{a}}(f) \leq 1$, for every $f \in \mathcal{K}$.

PROPOSITION 3.10. *The functional* ρ_{a} *is an integrating gauge on the family of functions* $\mathcal{K} = \mathcal{K}_{\mathrm{a}}(H^1(\mathbb{R}^n))$. *A function* f *belongs to the space* $\mathcal{L}(\rho_{\mathrm{a}},\mathcal{K})$ *if and only if there exist numbers* c_j, $j = 1,2,\ldots,$ *satisfying condition* (E.1), *and functions* $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ *such that the equality* (E.2) *holds for every* $\omega \in \Omega$ *for which the inequality* (E.3) *does.*

Proof. Let $\sigma(f) = 1$, for every function $f \in \mathcal{K}$ which does not vanish ι-almost everywhere, and $\sigma(f) = 0$, if $f(\omega) = 0$ for almost every $\omega \in \Omega$. Then $q_\sigma(f) = \rho_{\mathrm{a}}(f)$, for every $f \in \mathcal{K}$, by the definition of q_σ (Section 2A) and that of ρ_{a}. Because, by Proposition 2.7, q_σ is an integrating seminorm on $\mathcal{L}(\sigma,\mathcal{K})$, its restriction, ρ_{a}, to the family $\mathcal{K}$ too is integrating.

Now, by Proposition 2.7, $\mathcal{L}(q_\sigma,\mathcal{L}(\sigma,\mathcal{K})) = \mathcal{L}(\sigma,\mathcal{K})$ and, by Corollary 2.5, $\mathcal{L}(\rho_{\mathrm{a}},\mathcal{K}) = \mathcal{L}(q_\sigma,\mathcal{L}(\sigma,\mathcal{K}))$. So, $\mathcal{L}(\rho_{\mathrm{a}},\mathcal{K}) = \mathcal{L}(\sigma,\mathcal{K})$ which means that the space $\mathcal{L}(\rho_{\mathrm{a}},\mathcal{K})$ consists precisely of those functions f for which there exist numbers c_j and atoms $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ such that (E.1) holds and the equality (E.2) holds for all $\omega \in \Omega$ for which the inequality (E.3) does.

In view of the atomic representation of $H^1(\mathbb{R}^n)$, this proposition says that the spaces $L(\rho_{\mathrm{a}},\mathcal{K})$ and $H^1(\mathbb{R}^n)$ are identical and their respective norms are equivalent. This fact can also be deduced from the consideration of their duals; cf. the discussion in [6]. We will only identify the dual of the space $L(\rho_{\mathrm{a}},\mathcal{K})$ by showing that the continuous linear functionals on it are generated by functions of bounded mean oscillation. Let us recall the definition.

A function F on $\Omega = \mathbb{R}^n$ is said to have bounded mean oscillation if it is locally integrable and there is a constant M such that

$$(\iota(B))^{-1}\int_B \left|F(\omega) - (\iota(B))^{-1}\int_B F\mathrm{d}\iota\right|\iota(\mathrm{d}\omega) \le M, \tag{E.4}$$

for every ball $B \subset \Omega$. The infimum of all the constants M for which (E.4) holds is denoted by $\|F\|_{\mathrm{BMO}}$. Let us note that $\|F\|_{\mathrm{BMO}} = 0$ if and only if the function F is ι-almost everywhere equal to a constant.

PROPOSITION 3.11. *If F is a function of bounded mean oscillation, then there exists a unique continuous linear functional, ℓ, on the space $L(\rho,\mathcal{K})$, $\mathcal{K} = \mathcal{K}_{\mathrm{a}}(H^1(\mathbb{R}^n))$, $\rho = \rho_{\mathrm{a}}$, such that*

$$\ell([f]_\rho) = \int_\Omega f(\omega)F(\omega)\iota(\mathrm{d}\omega), \tag{E.5}$$

for every $f \in \mathcal{K}$; the norm of ℓ is equal to $\|F\|_{\mathrm{BMO}}$. Conversely, for every continuous linear functional, ℓ, on $L(\rho,\mathcal{K})$, there is a function F of bounded mean oscillation such that (E.5) *holds for every $f \in \mathcal{K}$.*

Proof. Let the function F have bounded mean oscillation. Then the formula (E.5) determines the number $\ell([f]_\rho)$ unambiguously for every atom f. Moreover, if $f \in \mathcal{K}$, let c_j be numbers satisfying (E.1) and $f_j \in \mathcal{K}$ atoms, $j = 1,2,\ldots$, such that (E.2) holds for every $\omega \in \Omega$ for which (E.3) does; let the atom f_j be supported by the ball B_j, $j = 1,2,\ldots$. Then, by definitions of atoms and of $\|F\|_{\mathrm{BMO}}$,

$$\left|\int_\Omega f_j F \mathrm{d}\iota\right| = \left|\int_{B_j} f_j(\omega)\left[F(\omega) - (\iota(B_j))^{-1}\int_{B_j} F\mathrm{d}\iota\right]\iota(\mathrm{d}\omega)\right| \le \|F\|_{\mathrm{BMO}},$$

for every $j = 1,2,\ldots$. So, by the series version of the Beppo Levi theorem,

$$\ell([f]_\rho) = \int_\Omega fF\mathrm{d}\iota = \sum_{j=1}^{\infty} c_j \int_\Omega f_j F\mathrm{d}\iota .$$

Consequently, $|\ell([f]_\rho)| \le \rho(f)\|F\|_{\mathrm{BMO}}$, because the numbers c_j, $j = 1,2,\ldots$, can be chosen so that the sum of their absolute values is arbitrarily close to $\rho(f)$. This argument can obviously be applied to any function $f \in \mathcal{L}(\rho,\mathcal{K})$. Alternatively, Proposition 3.2 implies that there is a unique continuous linear functional ℓ on $L(\rho,\mathcal{K})$, satisfying (E.5) for every $f \in \mathcal{K}$, whose norm is not larger than $\|F\|_{\mathrm{BMO}}$. Because, however, there are atoms, f, such that $\rho(f) = 1$ and $\iota(fF)$ is as close to $\|F\|_{\mathrm{BMO}}$ as we please, the norm of ℓ is actually equal to $\|F\|_{\mathrm{BMO}}$.

Conversely, assume that ℓ is a continuous linear functional on $L(\rho,\mathcal{K})$. For $n = 1,2,\ldots$, let $\mathcal{L}_n$ be the subspace of $\mathcal{L}(\rho,\mathcal{K})$ consisting of the (equivalence classes of) functions, f, represented in the form (E.2), where the numbers c_j satisfy (E.1) and the atoms f_j are supported by balls wholly contained in $B_n = \{\omega : |\omega| \le n\}$, $j = 1,2,\ldots$. Then $\mathcal{L}_n$ contains all essentially bounded functions supported by B_n with integral equal to 0. Because the dual space of L^∞ (on a space of finite measure) is equal to L^1, there is a function F_n, determined uniquely ι-almost everywhere on B_n, such that $\ell([f]_\rho) = \iota(fF_n)$, for every $f \in \mathcal{L}_n$, and

$$\int_{B_n} F_n \mathrm{d}\iota = 0 ,$$

$n = 1,2,\ldots$. Consequently, there is a locally integrable function F which coincides ι-almost everywhere on B_n with the function F_n, for every $n = 1,2,\ldots$. Then, using a similar argument as in the first part of the proof, it is straightforward to deduce that

$\ell([f])_\rho) = \iota(fF)$, for every $f \in \mathcal{L}(\rho,\mathcal{K})$, or just for every $f \in \mathcal{K}$. By what we have proved already, the function F has bounded mean oscillation.

F. Let E be a Banach space. Let $\mathcal{Q}$ be a quasiring of sets in a space Ω and $\mu : \mathcal{Q} \to E$ an additive set function. (See Sections 1D and 1E.)

For every set $X \in \mathcal{Q}$, let

$$\iota(X) = \sup \sum_{j=1}^{n} |\mu(X_j)| ,$$

where the supremum is taken over all integers $n = 1,2,...$ and all choices of pair-wise disjoint sets $X_j \in \mathcal{Q}$, $j = 1,2,...,n$, whose union is equal to X . Then ι is an extended real valued additive set function on $\mathcal{Q}$ such that

(i) $|m(X)| \le \iota(X)$, for every $X \in \mathcal{Q}$; and

(ii) if κ is any extended real valued additive set function on $\mathcal{Q}$ such that $|\mu(X)| \le \kappa(X)$, for every $X \in \mathcal{Q}$, then $\iota(X) \le \kappa(X)$, for every $X \in \mathcal{Q}$.

The set function ι is called the variation of μ . We write $v(\mu) = \iota$, $v(\mu,X) = \iota(X)$ for $X \in \mathcal{Q}$ and even $v(\mu,f) = \iota(f)$ for any function f such that $\iota(f)$ is defined. Alternatively, we write $|\mu| = \iota$.

The set function μ is said to have finite variation if $v(\mu,X) < \infty$ for every $X \in \mathcal{Q}$.

It is well-known that the variation of a σ-additive set function is σ-additive. Also, if the space E is finite-dimensional, $\mathcal{Q}$ is a δ-ring and the set function $\mu : \mathcal{Q} \to E$ is σ-additive, then μ has finite variation.

The conventions about the integration 'with respect to μ ' are not fixed even if the set function $\mu : \mathcal{Q} \to E$ is σ-additive. The reason being that there may exist several gauges on $\mathcal{Q}$, or sim$(\mathcal{Q})$, integrating for μ but generating different spaces of integrable functions, all considered 'natural' from alternative points of view.

If μ has finite variation which is σ-additive then we can let $v(\mu)$ integrate for μ . That is to say, we let $\iota = v(\mu)$ and note that there exists a unique linear map $\mu_\iota : \mathcal{L}(\iota) \to E$ such that

(i) $\mu_\iota(f) = \mu(X)$, whenever f is the characteristic function of a set $X \in \mathcal{Q}$;

and

(ii) $|\mu_\iota(f)| \le \iota(|f|)$, for every $f \in \mathcal{L}(\iota)$.

Then we write $\mathcal{L}(\mu) = \mathcal{L}(\iota)$ and

$$\mu(f) = \int_\Omega f \, d\mu = \int_\Omega f(\omega)\mu(d\omega) = \mu_\iota(f)$$

for every $f \in \mathcal{L}(\mu)$. It is often assumed that $\mathcal{Q}$ is a σ-algebra, or at least a δ-ring, [10], but this assumption has no significant bearing on the theory.

Another possibility arises when, for every $x' \in E'$, the set function $x' \circ \mu$ has finite and σ-additive variation and

(F.1) $$\sup\{v(x' \circ \mu, X) : x' \in E' , |x'| \le 1\} < \infty$$

for every $X \in \mathcal{Q}$. In that case, let

(F.2) $$\rho(f) = \sup\{v(x' \circ \mu, |f|) : x' \in E' , |x'| \le 1\} ,$$

for every $f \in \mathrm{sim}(\mathcal{Q})$. By Proposition 3.3 and Proposition 2.14, ρ is an integrating seminorm on $\mathrm{sim}(\mathcal{Q})$. Obviously, the seminorm ρ integrates for μ . So, one can define $\mathcal{L}(\mu) = \mathcal{L}(\rho, \mathrm{sim}(\mathcal{Q}))$ and

$$\mu(f) = \int_\Omega f \, d\mu = \int_\Omega f \, d_\rho \mu$$

for every $f \in \mathcal{L}(\mu)$.

Condition (F.1) is surely satisfied and the seminorm (F.2) integrates for μ if μ has finite σ-additive variation. In that case, $\mathcal{L}(v(\mu)) \subset \mathcal{L}(\rho, \mathrm{sim}(\mathcal{Q}))$ and the inclusion may be proper even when $\mathcal{Q}$ is a σ-algebra.

EXAMPLE 3.12. Let $\Omega = \{1,2,\ldots\}$ be the set of all positive integers and let $\mathcal{Q}$ be the family of all subsets of Ω . Let $\{x_j\}_{j=1}^\infty$ be an absolutely summable sequence of elements of the space E . Let

$$\mu(X) = \sum_{\omega \in X} x_\omega$$

for every $X \in \mathcal{Q}$.

Then $\mathcal{L}(v(\mu))$ consists of all functions f on Ω such that the sequence $\{f(j)x_j\}_{j=1}^{\infty}$ is absolutely summable. The space $\mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q}))$ consists of all functions f on Ω such that the sequence $\{f(j)x_j\}_{j=1}^{\infty}$ is unconditionally summable.

So, if μ has finite and σ-additive variation, then the symbol " $\mathcal{L}(\mu)$ " is ambiguous and would remain so even if the domain of μ were indicated. Though, if the space E is finite-dimensional, then $\mathcal{L}(\iota) = \mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q}))$, with $\iota = v(\mu)$ and ρ defined by (F.2), and the respective seminorms are equivalent.

Of course, it might be possible to form the space $\mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q}))$, with ρ defined by (F.2), also when μ does not have finite variation. By the following proposition, this space surely can be formed when $\mathcal{Q}$ is a δ-ring and μ is σ-additive.

PROPOSITION 3.13. *Let $\mathcal{Q}$ be a δ-ring of sets in the space Ω. Let $\mathcal{S}$ be the σ-algebra of all sets $X \subset \Omega$ such that $X \cap Z \in \mathcal{Q}$ for every $Z \in \mathcal{Q}$. Let $\mu : \mathcal{Q} \to E$ be a σ-additive set function.*

Then, for every $x' \in E'$, the set function $x' \circ \mu$ has finite variation and the inequality (F.1) *holds for every $X \in \mathcal{Q}$.*

Let the seminorm ρ be defined by (F.2) *for every $f \in \mathrm{sim}(\mathcal{Q})$. Then the seminorm ρ integrates for μ. The seminorm ρ is monotonic and the space $\mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q}))$ is a vector lattice. A function on Ω is ρ-null if and only if it is $v(x' \circ \mu)$-null for every $x' \in E'$. The seminorm ρ is equivalent to the seminorm σ defined by*

$$\sigma(f) = \sup\{|\mu(Xf)| : X \in \mathcal{Q}\}$$

for every $f \in \mathrm{sim}(\mathcal{Q})$.

Let f be a function on Ω. Then the following statements are equivalent:

(i) $f \in \mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q}))$.

(ii) *There exist $\mathcal{Q}$-simple functions f_n, $n = 1,2,\ldots,$ such that*

(F.3) $$f(\omega) = \lim_{n\to\infty} f_n(\omega)$$

for ρ-almost every $\omega \in \Omega$ and the sequence $\{\mu(Xf_n)\}_{n=1}^{\infty}$ converges to an element of the space E, for every $X \in \mathcal{S}$.

(iii) *For every $x' \in E'$, the function f is $v(x' \circ \mu)$-integrable and, for every $X \in \mathcal{S}$, there exists an element $\nu(X)$ of E such that*

$$x'(\nu(X)) = \int_\Omega Xf\,\mathrm{d}(x' \circ \mu)$$

for every $x' \in E'$.

Proof. Some of the statements were already proved. The equivalence of the seminorms ρ and σ was noted by R.G. Bartle, N. Dunford and J.T. Schwartz in [2]; see also [14], Lemma IV.10.4(b). They also noted that a set is ρ-null if and only if it is $v(x' \circ \mu)$-null for every $x' \in E'$. Hence, by Proposition 2.2, a function is ρ-null if and only if it is $v(x' \circ \mu)$-null for every $x' \in E'$.

Given a function f on Ω, the equivalence of the statements (i), (ii) and (iii) was essentially proved by D.R. Lewis in [44]. In fact, (i) obviously implies (ii) and (ii) implies (iii). Now, let $\mathcal{K}$ be the family of all functions f for which the statement (iii) holds. Define $\rho(f)$ by (F.2) for every $f \in \mathcal{K}$. Then $\rho(f) < \infty$, for every $f \in \mathcal{K}$ because, by the Orlicz-Pettis lemma, the set function ν is σ-additive and

$$\rho(f) = \sup\{v(x' \circ \nu, \Omega) : x' \in E' \,|\, \le 1\}.$$

By Theorem 3.5 of [44], for every $f \in \mathcal{K}$ and $\epsilon > 0$, there exists a function $g \in \mathrm{sim}(\mathcal{Q})$ such that $\rho(f-g) < \epsilon$. Hence, by Theorem 2.4, for every $f \in \mathcal{K}$, one can produce a sequence, $\{f_n\}_{n=1}^{\infty}$, of $\mathcal{Q}$-simple functions such that

$$\lim_{n\to\infty} \rho(f-f_n) = 0$$

and (F.3) holds for ρ-almost every $\omega \in \Omega$. So, $\mathcal{K} \subset \mathcal{L}(\rho, \mathrm{sim}(\mathcal{Q}))$.

This proposition summarizes the main approaches to integration 'with respect to Banach valued measures' of not necessarily finite variation which appeared in the literature. R.G. Bartle, N. Dunford and J.T. Schwartz, [2], used condition (ii) to define integrability in the case when $\mathcal{Q}$ is a σ-algebra; see also [14], Section IV.10. Property (iii) was used by D.R. Lewis in [44] and [45]. Different approaches, leading to different spaces of integrable functions are of course possible. One of them will be described in Example 4.27 (Section 4F).

G. The structure described in Section 3A represents a possibility for defining, in a reasonably systematic manner, integrals of the form

$$\int_a^b f \mathrm{d}w ,$$

where w is an arbitrary continuous function in the interval $[a,b]$. In this section, we present a way of doing so sketched in [32]. We shall return to this theme again in Sections 4C and 4D, where we impose on w some additional conditions, similar to but still much weaker than the finiteness of variation, and, on the other hand, extend the generality of the whole set-up.

Let w be a bounded continuous real or complex valued function on the real-line, $\Omega = (-\infty,\infty)$.

Let $C_0((-\infty,\infty])$ be the Banach space of all functions continuous on the two-point compactification, $[-\infty,\infty]$, of the space Ω and vanishing at $-\infty$, under the usual sup-norm, $\|\cdot\|_\infty$. Let E be the space of all bounded sequences of elements of $C_0((-\infty,\infty])$ equipped with the norm defined by

$$\|\varphi\| = \sup\{\|\varphi_n\|_\infty : n = 1,2,\ldots\} ,$$

for every element, $\varphi = \{\varphi_n\}_{n=1}^\infty$, of E. Let F be the subspace of E consisting of those sequences of elements of $C_0((-\infty,\infty])$ which are convergent in $C_0((-\infty,\infty])$.

Let ι be the Lebesgue measure in the space Ω. As usual, this measure is not shown in integrals written down using a dummy variable. For the functions f and g

on Ω, we denote

$$(f*g)(t) = \int_\Omega f(t-s)g(s)\mathrm{d}s\,,$$

for every $t \in \Omega$ for which this integral exists (in the sense described in Section B).

Now, let k_n, $n = 1,2,\ldots$, be continuously differentiable functions on Ω, ι-integrable together with their derivatives such that $k_n*\varphi \to \varphi$, as $n \to \infty$, uniformly on Ω, for every continuous function φ on Ω with compact support, and $k'_n*\varphi \to \varphi'$, as $n \to \infty$, uniformly on Ω, for every continuously differentiable function φ on Ω with compact support (the dash denotes the derivative). For example, we can take

$$k_n(t) = \frac{n}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}nt^2)\,,$$

for every $t \in \Omega$ and $n = 1,2,\ldots$.

Given a function $f \in \mathcal{L}(\iota)$, let

$$\nu_n(f)(t) = \int_{-\infty}^{t} f(s)(k'_n*w)(s)\mathrm{d}s\,,$$

for every $t \in [-\infty,\infty]$ and $n = 1,2,\ldots$.

Let $\mathcal{K}$ be the vector space of all functions $f \in \mathcal{L}(\iota)$ such that the sequence $\nu(f) = \{\nu_n(f)\}_{n=1}^{\infty}$ belongs to E, and let

$$\rho(f) = \iota(|f|) + \|\nu(f)\|\,,$$

for every $f \in \mathcal{K}$. Let $\mathcal{J}$ be the subspace of $\mathcal{K}$ consisting of the functions $f \in \mathcal{K}$ such that $\nu(f)$ belongs to F.

PROPOSITION 3.14. *The functional ρ is an integrating seminorm on $\mathcal{K}$ such that $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$ and $\mathcal{L}(\rho,\mathcal{J}) = \mathcal{J}$.*

Proof. It is obvious that ρ is a seminorm. So, let $f_j \in \mathcal{K}$, $j = 1,2,\ldots$, be functions such that

$$\sum_{j=1}^{\infty} \rho(f_j) < \infty$$

and let f be a function on Ω such that

$$f(t) = \sum_{j=1}^{\infty} f_j(t) ,$$

for every $t \in \Omega$ for which

$$\sum_{j=1}^{\infty} |f_j(t)| < \infty .$$

Then

$$\sum_{j=1}^{\infty} \iota(|f_j|) < \infty ,$$

and, by the Beppo Levi theorem (or Proposition 2.1 applicable by Proposition 2.13), $f \in \mathcal{L}(\iota)$. Also

$$\sum_{j=1}^{\infty} \|\nu(f_j)\| < \infty ,$$

and, because the space E is complete, there exists an element, $\varphi = \{\varphi_n\}_{n=1}^{\infty}$, of E such that

$$\sum_{j=1}^{\infty} \nu(f_j) = \varphi$$

in the sense of convergence in the space E. It follows that $\nu_n(f) = \varphi_n$, because the continuity of the map $\nu_n : \mathcal{L}(\iota) \to C_0((0,\infty])$ implies that

$$\nu_n(f) = \sum_{j=1}^{\infty} \nu_n(f_j) ,$$

for every $n = 1,2,...$. So, by the definition of $\mathcal{K}$, the function f belongs to it, by Proposition 2.8, the seminorm ρ is integrating and, by the definition of the space $\mathcal{L}(\rho,\mathcal{K})$, the equality $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$ holds. Then also the restriction of ρ to $\mathcal{J}$ is integrating and the same argument shows that $\mathcal{L}(\rho,\mathcal{J}) = \mathcal{J}$.

Let LIM be a Banach limit. That is, LIM is a continuous linear functional on the space of all bounded sequences of scalars equipped with the sup-norm, independent on any finite number of coordinates, such that

$$\mathrm{LIM}\{\alpha_n\}_{n=1}^{\infty} = \lim_{n\to\infty} \alpha_n$$

whenever the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is convergent.

Given a function $f \in \mathcal{K} = \mathcal{L}(\rho,\mathcal{K})$, we define

$$\int_{-\infty}^{t} f\,\mathrm{d}w = \mathrm{LIM}\ \{\nu_n(f)(t)\}_{n=1}^{\infty}$$

for every $t \in [-\infty,\infty]$. Then

$$f \mapsto \mu_{\mathrm{LIM}}(f) = \int_{-\infty}^{\infty} f\,\mathrm{d}w\,,\ f \in \mathcal{K}\,,$$

is a continuous linear functional on the complete space $\mathcal{L}(\rho,\mathcal{K})$ such that

$$\int_{-\infty}^{\infty} f\,\mathrm{d}w = -\int_{-\infty}^{\infty} f'(t)w(t)\mathrm{d}t\,,$$

for every continuously differentiable function f on Ω with compact support. This functional depends of course on the choice of the Banach limit LIM. However, its values on the functions belonging to $\mathcal{J}$ are determined uniquely.