2. INTEGRATING GAUGES

An integration theory involves two constructions, namely that of the space of integrable function and that of the integral. These two constructions are often carried out simultaneously. However, having in mind the generalizations pursued here, it is desirable to keep them at least conceptually separated. In this chapter, spaces of integrable functions are introduced; integrals will be dealt with in the next one.

We start with a family of functions, $\mathcal{K}$, defined on a space Ω , which contains the zero-function but is not necessarily a vector space, and a non-negative real valued functional, ρ , on $\mathcal{K}$, called a gauge, such that $\rho(0) = 0$. Then we introduce the vector space $\mathcal{L} = \mathcal{L}(\rho,\mathcal{K})$ of functions, f , on Ω which can be expressed in the form

$$(*) \qquad f(\omega) = \sum_{j=1}^{\infty} c_j f_j(\omega) ,$$

for all $\omega \in \Omega$ subject to certain exceptions, where c_j are numbers and f_j functions belonging to $\mathcal{K}$, $j = 1,2,\ldots,$ such that

$$(\ddagger) \qquad \sum_{j=1}^{\infty} |c_j| \rho(f_j) < \infty .$$

The equality (*) is not required to hold for those points $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |c_j f_j(\omega)| = \infty ,$$

even if the sum on the right in (*) exists as the limit of the sequence of partial sums; the values of f at such points are arbitrary. For the seminorm, $q(f)$, of such a function f we take the infimum of the numbers ($\ddagger$) . The space $\mathcal{L}$ is complete in this seminorm and the linear hull of $\mathcal{K}$ is dense in it. Of course, to avoid the obvious pathology that the seminorm of some functions $f \in \mathcal{K}$ with $\rho(f) > 0$ collapses to 0 , some conditions have to be imposed on the gauge ρ . Accordingly, the gauge ρ is called integrating if $q(f) = \rho(f)$, for every function $f \in \mathcal{K}$.

If $\mathcal{K}$ is the family of characteristic functions of sets from a σ-algebra, say, and ρ is a measure on it, then this construction gives us precisely the family of functions integrable with respect to ρ and the corresponding seminorm of convergence in mean. Similarly, if $\mathcal{K}$ is a vector lattice and $\rho(f) = \iota(|f|)$, for every $f \in \mathcal{K}$, where ι is a Daniell integral on $\mathcal{K}$, then $\mathcal{L}$ is the family of all ι-integrable functions. Other choices of $\mathcal{K}$ and ρ lead to other classical and less classical spaces some of which will be described in the next chapter.

A. Let $\mathcal{K}$ be a nontrivial family of functions on a space Ω. (See Section 1D.) A non-negative real valued functional ρ on $\mathcal{K}$ such that $\rho(0) = 0$ will be called a gauge on $\mathcal{K}$. Good examples of gauges to keep in mind, in what follows, are seminorms on vector spaces of functions and (finite) non-negative additive, or just sub-additive, set functions on quasirings of sets. (Recall that we identify sets with their characteristic functions.)

The following definition can be viewed as the abstract core of the construction of the space of integrable functions and its L^1-seminorm from a given elementary measure or content.

Let ρ be a gauge on the family of functions $\mathcal{K}$. A function f on Ω will be called integrable with respect to ρ, or, briefly, ρ-integrable, if there exist numbers c_j and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots$, such that

(A.1) $$\sum_{j=1}^{\infty} |c_j| \rho(f_j) < \infty$$

and

(A.2) $$f(\omega) = \sum_{j=1}^{\infty} c_j f_j(\omega)$$

for every $\omega \in \Omega$ for which

(A.3) $$\sum_{j=1}^{\infty} |c_j f_j(\omega)| < \infty .$$

The family of all (individual) functions integrable with respect to ρ is denoted by $\mathcal{L}(\rho,\mathcal{K})$.

For any function $f \in \mathcal{L}(\rho,\mathcal{K})$, let

$$q_\rho(f) = \inf \sum_{j=1}^{\infty} |c_j| \rho(f_j) ,$$

where the infimum is taken over all choices of the numbers c_j and the functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ satisfying condition (A.1), such that the equality (A.2) holds for every $\omega \in \Omega$ for which the inequality (A.3) does.

Clearly, $\mathcal{L}(\rho,\mathcal{K})$ is a vector space such that $\mathrm{sim}(\mathcal{K}) \subset \mathcal{L}(\rho,\mathcal{K})$. (See Section 1D.) Also, it is not difficult to see that q_ρ is a seminorm on $\mathcal{L}(\rho,\mathcal{K})$; it is called the seminorm generated by the gauge ρ. Consequently, we can speak of q_ρ-Cauchy and q_ρ-convergent sequences of functions from $\mathcal{L}(\rho,\mathcal{K})$.

The ρ-equivalence class of a function $f \in \mathcal{L}(\rho,\mathcal{K})$, consisting of all functions $g \in \mathcal{L}(\rho,\mathcal{K})$ such that $q_\rho(f-g) = 0$, is denoted by $[f]_\rho$. The set $\{[f]_\rho : f \in \mathcal{L}(\rho,\mathcal{K})\}$ of all ρ-equivalence classes of functions from $\mathcal{L}(\rho,\mathcal{K})$ is denoted by $L(\rho,\mathcal{K})$. Then $L(\rho,\mathcal{K})$ is a normed space with respect to the linear operations induced by those of $\mathcal{L}(\rho,\mathcal{K})$ and the norm induced by the seminorm q_ρ. This norm is still denoted by q_ρ.

It is sometimes useful, even necessary, to indicate the domain, $\mathcal{K}$, of the gauge ρ not only in the symbol of the space $\mathcal{L}(\rho,\mathcal{K})$ but also in the symbol for its seminorm. Then, instead of q_ρ, we write more precisely $q_{\rho,\mathcal{K}}$. In fact, it is customary not to distinguish in the notation between a gauge ρ on $\mathcal{K}$ and its restriction to a nontrivial subfamily, $\mathcal{J}$, of $\mathcal{K}$. But then $\mathcal{L}(\rho,\mathcal{J}) \subset \mathcal{L}(\rho,\mathcal{K})$ and $q_{\rho,\mathcal{K}}(f) \le q_{\rho,\mathcal{J}}(f)$ for every $f \in \mathcal{L}(\rho,\mathcal{J})$. What is more, the inclusion may be strict and, for some functions $f \in \mathcal{L}(\rho,\mathcal{J})$, the inequality may be strict too.

PROPOSITION 2.1. *Let $f_j \in \mathcal{L}(\rho,\mathcal{K})$, $j = 1,2,\ldots,$ be functions such that*

(A.4) $$\sum_{j=1}^{\infty} q_\rho(f_j) < \infty$$

and let f be a function on Ω such that

(A.5) $$f(\omega) = \sum_{j=1}^{\infty} f_j(\omega)$$

for every $\omega \in \Omega$ for which

(A.6) $$\sum_{j=1}^{\infty} |f_j(\omega)| < \infty .$$

Then $f \in \mathcal{L}(\rho,\mathcal{K})$ *and*

(A.7) $$\lim_{n\to\infty} q_\rho\left[f - \sum_{j=1}^{n} f_j\right] = 0 .$$

Proof. For every $j = 1,2,\ldots,$ let c_{jk} be numbers and $f_{jk} \in \mathcal{K}$ functions, $k = 1,2,\ldots,$ such that

$$\sum_{k=1}^{\infty} |c_{jk}|\rho(f_{jk}) < q_\rho(f_j) + 2^{-j}$$

and

$$f_j(\omega) = \sum_{k=1}^{\infty} c_{jk}f_{jk}(\omega)$$

for every $\omega \in \Omega$ such that

$$\sum_{k=1}^{\infty} |c_{jk}f_{jk}(\omega)| < \infty .$$

Then, for any $n = 0,1,2,\ldots,$

$$\sum_{j=n+1}^{\infty} \sum_{k=1}^{\infty} |c_{jk}|\rho(f_{jk}) < \sum_{j=n+1}^{\infty} q_\rho(f_j) + 2^{-n} < \infty$$

and

$$f(\omega) - \sum_{j=1}^{n} f_j(\omega) = \sum_{j=n+1}^{\infty} \sum_{k=1}^{\infty} c_{jk}f_{jk}(\omega)$$

for every $\omega \in \Omega$ for which

$$\sum_{j=n+1}^{\infty} \sum_{k=1}^{\infty} |c_{jk}f_{jk}(\omega)| < \infty .$$

Therefore, the function

$$f - \sum_{j=1}^{n} f_j$$

belongs to $\mathcal{L}(\rho,\mathcal{K})$ and

$$q_\rho\left[f - \sum_{j=1}^{n} f_j\right] < \sum_{j=n+1}^{\infty} q_\rho(f_j) + 2^{-n}$$

for every $n = 0,1,2,\ldots$.

The most important implication of this proposition is, of course, that the space $\mathcal{L}(\rho,\mathcal{K})$ is q_ρ-complete so that $L(\rho,\mathcal{K})$ is a Banach space.

B. Let $\mathcal{K}$ be a nontrivial family of functions on a space Ω and let ρ be a gauge on $\mathcal{K}$.

A function f on Ω is said to be ρ-null if $f \in \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) = 0$. A set $X \subset \Omega$ is said to be ρ-null if its characteristic function is ρ-null. The family of all ρ-null sets is denoted by $\mathcal{Z}_\rho$. We shall use the customary jargon related to null sets. So, for example, we refer to a ρ-null set by saying that ρ-almost all points of Ω belong to its complement.

The next proposition says, among other things, that $\mathcal{Z}_\rho$ is a σ-ideal in the space Ω. (See Section 1D.)

PROPOSITION 2.2. *A function f is ρ-null if and only if the set $\{\omega \in \Omega : f(\omega) \neq 0\}$ is ρ-null.*

If the function f is ρ-null, then there exist numbers c_j and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots$, satisfying condition (A.1), *such that*

$$\sum_{j=1}^{\infty} |c_j f_j(\omega)| = \infty \tag{B.1}$$

for every $\omega \in \Omega$ for which $f(\omega) \neq 0$.

Conversely, if there exist functions $f_j \in \mathcal{L}(\rho,\mathcal{K})$, $j = 1,2,\ldots$, satisfying condition (A.4), *such that*

$$\sum_{j=1}^{\infty} |f_j(\omega)| = \infty \tag{B.2}$$

for every $\omega \in \Omega$ for which $f(\omega) \neq 0$, then the function f is ρ-null.

If X_j, $j = 1,2,\ldots$, are ρ-null sets and

$$X \subset \bigcup_{j=1}^{\infty} X_j ,$$

then the set X too is ρ-null.

Proof. Let X be a ρ-null set. Then, by the definition of $\mathcal{L}(\rho,\mathcal{K})$ and q_ρ, for every $k = 1,2,\ldots$, there exist numbers c_{kn} and functions $f_{kn} \in \mathcal{K}$, $n = 1,2,\ldots$, such that

$$\sum_{n=1}^{\infty} |c_{kn}|\rho(f_{kn}) < 2^{-k}$$

and

$$\sum_{n=1}^{\infty} |c_{kn}f_{kn}(\omega)| \geq 1 ,$$

for every $\omega \in X$. Then

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} |c_{kn}|\rho(f_{kn}) < \infty$$

and

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} |c_{kn}f_{kn}(\omega)| = \infty$$

for every $\omega \in X$. So there exist numbers c_j and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ satisfying condition (A.1), such that (B.1) holds for every $\omega \in X$.

Let function g be ρ-null. Let f be the characteristic function of the set $\{\omega : g(\omega) \neq 0\}$. Then the function $f_j = jg$ is ρ-null and $q_\rho(f_j) = 0$, for every $j = 1,2,\ldots$. Hence, condition (A.4) is satisfied and the equality (A.5) holds for every $\omega \in \Omega$ for which the inequality (A.6) does. Therefore, by Proposition 2.1, $f \in \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) = 0$.

Let f be a function such that the set $\{\omega : f(\omega) \neq 0\}$ is ρ-null. Let f_1 be the characteristic function of this set and let $f_j = jf_1$, for every $j = 1,2,3,\ldots$. Then $q_\rho(f_j) = 0$, for every $j = 1,2,\ldots,$ and so, condition (A.4) is satisfied. Furthermore, the equality (A.5) holds for every $\omega \in \Omega$ for which the inequality (A.6) does. So, by Proposition 2.1, $f \in \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) = 0$.

Now, let f be a function on Ω and let $f_j \in \mathcal{L}(\rho,\mathcal{K})$, $j = 1,2,\ldots,$ be functions satisfying condition (A.4), such that the equality (B.2) holds for every $\omega \in \Omega$ for which $f(\omega) \neq 0$. Then, for every $n = 1,2,\ldots,$

$$\sum_{j=n}^{\infty} q_\rho(f_j) + \sum_{j=n}^{\infty} q_\rho(-f_j) < \infty$$

and

$$f(\omega) = \sum_{j=n}^{\infty} f_j(\omega) + \sum_{j=n}^{\infty} (-f_j(\omega)) = 0$$

for every $\omega \in \Omega$ for which

$$\sum_{j=n}^{\infty} |f_j(\omega)| + \sum_{j=n}^{\infty} |-f_j(\omega)| < \infty .$$

Then, by Proposition 2.1, $f \in \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) = 0$ so that f is ρ-null.

C. The following theorem is the Beppo Levi theorem stated in terms of absolute summability rather than monotone convergence.

THEOREM 2.3. *A function* f *on* Ω *belongs to* $\mathcal{L}(\rho,\mathcal{K})$ *if and only if there exist numbers* c_j *and functions* $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ *satisfying condition* (A.1), *such that the equality* (A.2) *holds for* ρ-*almost every* $\omega \in \Omega$.

Let $f_j \in \mathcal{L}(\rho,\mathcal{K})$, $j = 1,2,\ldots,$ *be functions satisfying condition* (A.4). *Then the inequality* (A.6) *holds for* ρ-*almost every* $\omega \in \Omega$. *If, moreover,* f *is a function on* Ω *such that the equality* (A.5) *holds for* ρ-*almost every* $\omega \in \Omega$, *then* $f \in \mathcal{L}(\rho,\mathcal{K})$ *and the equality* (A.7) *holds.*

Proof. It is a direct consequence of Proposition 2.1 and Proposition 2.2.

In the terminology of N. Aronszajn and K.T. Smith, [1], the following theorem says that $\mathcal{L}(\rho,\mathcal{K})$ is a complete normed functional space, in fact, it is a functional completion of $\mathrm{sim}(\mathcal{K})$.

THEOREM 2.4. *A function* f *on* Ω *belongs to* $\mathcal{L}(\rho,\mathcal{K})$ *if and only if there exists a* q_ρ-*Cauchy sequence of functions* $h_n \in \mathrm{sim}(\mathcal{K})$, $n = 1,2,\ldots,$ *such that*

(C.1) $$f(\omega) = \lim_{n\to\infty} h_n(\omega)$$

for ρ-*almost every* $\omega \in \Omega$.

Every q_ρ-*Cauchy sequence of functions* $g_n \in \mathcal{L}(\rho,\mathcal{K})$, $n = 1,2,\ldots,$ *has a subsequence,* $\{h_n\}_{n=1}^{\infty}$, *such that the sequence of numbers* $\{h_n(\omega)\}_{n=1}^{\infty}$ *is convergent for* ρ-*almost every* $\omega \in \Omega$. *Moreover, if* $\{h_n\}_{n=1}^{\infty}$ *is such a subsequence of* $\{g_n\}_{n=1}^{\infty}$ *and* f *is a function on* Ω *such that the equality* (C.1) *holds for* ρ-*almost every* $\omega \in \Omega$, *then* $f \in \mathcal{L}(\rho,\mathcal{K})$ *and*

(C.2) $$\lim_{n\to\infty} q_\rho(f-g_n) = 0 .$$

Proof. If the sequence, $\{g_n\}_{n=1}^{\infty}$, of functions from $\mathcal{L}(\rho,\mathcal{K})$ is q_ρ-Cauchy, we can select a subsequence $\{h_n\}_{n=1}^{\infty}$ such that

(C.3)
$$\sum_{j=1}^{\infty} q_\rho(h_{j+1}-h_j) < \infty .$$

Then Theorem 2.3, applied to the functions f_j such that $f_1 = h_1$ and $f_{j+1} = h_{j+1} - h_j$ for $j = 1,2,\ldots$, implies that the sequence $\{h_n(\omega)\}_{n=1}^{\infty}$ is convergent for ρ-almost every $\omega \in \Omega$.

Now, if $\{h_n\}_{n=1}^{\infty}$ is a subsequence of $\{g_n\}_{n=1}^{\infty}$ such that the sequence $\{h_n\}_{n=1}^{\infty}$ is convergent for ρ-almost every $\omega \in \Omega$, we can achieve, by passing to a subsequence of $\{h_n\}_{n=1}^{\infty}$, if necessary, that (C.3) holds. Then, if (C.1) holds for ρ-almost every $\omega \in \Omega$, by Theorem 2.3, $f \in \mathcal{L}(\rho,\mathcal{K})$ and

$$\lim_{n\to\infty} q_\rho(f-h_n) = 0 .$$

Because $\{h_n\}_{n=1}^{\infty}$ is a subsequence of the q_ρ-Cauchy sequence $\{g_n\}_{n=1}^{\infty}$, (C.2) holds.

COROLLARY 2.5. *Let $\mathcal{J}$ be a q_ρ-complete vector space, containing every ρ-null function, such that $\mathcal{K} \subset \mathcal{J} \subset \mathcal{L}(\rho,\mathcal{K})$. Then $\mathcal{J} = \mathcal{L}(\rho,\mathcal{K})$.*

D. Theorems 2.3 and 2.4 demonstrate the usefulness of the space $\mathcal{L}(\rho,\mathcal{K})$ and its seminorm q_ρ. But this usefulness could be limited by the fact that, in general, we can only say that $q_\rho(f) \le \rho(f)$, for every $f \in \mathcal{K}$, and the inequality may be sharp for some f even if $\mathcal{K}$ is a vector space and ρ is a seminorm on it.

EXAMPLE 2.6. Let $\Omega = (0,1]$, $\mathcal{Q} = \{(u,v] : 0 \le u \le v \le 1\}$, $\mathcal{K} = \mathrm{sim}(\mathcal{Q})$ and

$$\rho(f) = \lim_{t\to 0+} |f(t)| ,$$

for every $f \in \mathcal{K}$. Then every function on Ω belongs to $\mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) = 0$ for every $f \in \mathcal{L}(\rho,\mathcal{K})$.

So, of particular interest are the gauges singled out in the following definition.

We shall call the gauge ρ integrating if $q_\rho(f) = \rho(f)$ for every function f belonging to its domain, $\mathcal{K}$.

Obviously, if a gauge on a vector space is integrating, then it is a seminorm. A seminorm which is an integrating gauge will of course be called an integrating seminorm.

PROPOSITION 2.7. *The gauge* ρ *is integrating if and only if*

$$\rho(f) \leq \sum_{j=1}^{\infty} |c_j| \rho(f_j)$$

for any function $f \in \mathcal{K}$, *numbers* c_j *and functions* $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ *such that the equality* (A.2) *holds for every* $\omega \in \Omega$ *for which the inequality* (A.3) *does.*

Let ρ *be an integrating gauge and let* $\mathcal{J}$ *be a nontrivial subfamily of its domain,* $\mathcal{K}$. *Then the restriction,* σ, *of* ρ *to* $\mathcal{J}$ *is an integrating gauge,* $\mathcal{L}(\sigma,\mathcal{J}) \subset \mathcal{L}(\rho,\mathcal{K})$ *and* $q_\rho(f) \leq q_\sigma(f)$, *for every* $f \in \mathcal{L}(\sigma,\mathcal{J})$.

If ρ *is any gauge on a nontrivial family of functions,* $\mathcal{K}$, *then the functional* q_ρ *is an integrating seminorm on* $\mathcal{L}(\rho,\mathcal{K})$ *such that* $\mathcal{L}(q_\rho,\mathcal{L}(\rho,\mathcal{K})) = \mathcal{L}(\rho,\mathcal{K})$ *and* $q_{q_\rho}(f) = q_\rho(f)$, *for every* $f \in \mathcal{L}(\rho,\mathcal{K})$.

Proof. The first statement is a direct consequence of the definitions and the second one follows from it. The third statement is a corollary to Proposition 2.1.

PROPOSITION 2.8. *Let* $\mathcal{K}$ *be a vector space of functions on* Ω *and let* ρ *be a seminorm on* $\mathcal{K}$. *Then* ρ *is integrating if and only if*

(D.1) $$\lim_{n\to\infty} \rho\left[\sum_{j=1}^{n} f_j\right] = 0$$

for any functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ *satisfying the inequality*

(D.2) $$\sum_{j=1}^{\infty} \rho(f_j) < \infty,$$

such that

(D.2)
$$\sum_{j=1}^{\infty} f_j(\omega) = 0$$

for every $\omega \in \Omega$ *for which the inequality* (A.6) *holds.*

Proof. By Proposition 2.1, the stated condition is necessary for ρ to be integrating. Conversely, assume that (D.1) holds for any functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ satisfying (D.2) such that (D.3) holds for every $\omega \in \Omega$ for which (A.6) does. Let $f \in \mathcal{K}$ and let $\epsilon > 0$. Then there exist functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ such that (A.5) holds for every $\omega \in \Omega$ for which (A.6) does and

$$\sum_{j=1}^{\infty} \rho(f_j) < q_\rho(f) + \epsilon .$$

Then, by the assumption,

$$\lim_{n\to\infty} \rho\left[f - \sum_{j=1}^{n} f_j\right] = 0 .$$

Hence

$$\rho(f) \le \rho\left[\sum_{j=1}^{n} f_j\right] + \epsilon \le \sum_{j=1}^{n} \rho(f_j) + \epsilon$$

for a sufficiently large n. Consequently, $\rho(f) \le q_\rho(f) + 2\epsilon$.

PROPOSITION 2.9. *The seminorm* ρ *on a vector space* $\mathcal{K}$ *is integrating if and only if*

(D.4)
$$\lim_{n\to\infty} \rho(g_n) = 0$$

for every ρ*-Cauchy sequence* $\{g_n\}_{n=1}^{\infty}$ *of functions from* $\mathcal{K}$ *such that*

(D.5)
$$\lim_{n\to\infty} g_n(\omega) = 0$$

for ρ*-almost every* $\omega \in \Omega$.

Proof. Assume that (D.4) holds for every ρ-Cauchy sequence $\{g_n\}_{n=1}^{\infty}$ of functions from $\mathcal{K}$ which converges ρ-almost everywhere to 0. Let $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ be functions satisfying condition (D.2) such that the equality (D.3) holds for every $\omega \in \Omega$ for which the inequality (A.6) does. Let

(D.6) $$g_n = \sum_{j=1}^{n} f_j ,$$

for every $n = 1,2,\dots$. Then, by Proposition 2.2, the equality (D.5) holds for ρ-almost every $\omega \in \Omega$. Hence, by the assumption, (D.4) holds, which means that (D.3) does. So, by Proposition 2.8, the seminorm ρ is integrating.

Conversely, assume that the seminorm ρ is integrating. Assume that $\{g_n\}_{n=1}^{\infty}$ is a ρ-Cauchy sequence of functions from $\mathcal{K}$ such that (D.5) holds for ρ-almost every $\omega \in \Omega$. To prove (D.4), it suffices to show that $\rho(h_n) \to 0$, as $n \to \infty$, for a subsequence $\{h_n\}_{n=1}^{\infty}$ of the sequence $\{g_n\}_{n=1}^{\infty}$. Therefore, assume that, if $f_1 = g_1$ and $f_j = g_j - g_{j-1}$, for $j = 2,3,\dots$, then (D.2) holds. Because (D.5) holds for ρ-almost every $\omega \in \Omega$, we have (D.3) for ρ-almost every $\omega \in \Omega$. Then, by Theorem 2.3,

$$\lim_{n\to\infty} \rho(g_n) = \lim_{n\to\infty} \rho\left[\sum_{j=1}^{n} f_j\right] = 0 .$$

PROPOSITION 2.10. *Let ρ be a gauge on a nontrivial family of functions, $\mathcal{K}$. For every $f \in \mathrm{sim}(\mathcal{K})$, let*

$$\sigma(f) = \inf \sum_{j=1}^{n} |c_j| \rho(f_j) ,$$

where the infimum is taken over all expressions of the function f in the form

$$f = \sum_{j=1}^{n} c_j f_j$$

with arbitrary $n = 1,2,\dots,$ numbers c_j and functions $f_j \in \mathcal{K}$, $j = 1,2,\dots,n$.

Then $\mathcal{L}(\sigma,\mathrm{sim}(\mathcal{K})) = \mathcal{L}(\rho,\mathcal{K})$ and $q_\sigma(f) = q_\rho(f)$, for every $f \in \mathcal{L})\sigma,\mathrm{sim}(\mathcal{K}))$. The equality $\sigma(f) = q_\rho(f)$ holds for every $f \in \mathrm{sim}(\mathcal{K})$ if and only if the seminorm σ is integrating.

Proof. Obviously, $\mathcal{L}(\rho,\mathcal{K}) \subset \mathcal{L}(\sigma,\mathrm{sim}(\mathcal{K}))$ and $q_\sigma(f) \le q_\rho(f)$, for every $f \in \mathcal{L}(\rho,\mathcal{K})$. On the other hand, $q_\rho(f) \le \sigma(f)$, for every $f \in \mathrm{sim}(\mathcal{K})$ and, therefore, $\mathcal{L}(\sigma,\mathrm{sim}(\mathcal{K})) \subset \mathcal{L}(q_\rho,\mathrm{sim}(\mathcal{K})) \subset \mathcal{L}(q_\rho,\mathcal{L}(\rho,\mathcal{K}))$. Because, by Proposition 2.7,

$\mathcal{L}(q_\rho,\mathcal{L}(\rho,\mathcal{K})) = \mathcal{L}(\rho,\mathcal{K})$, we have $\mathcal{L}(\sigma,\text{sim}(\mathcal{K})) \subset \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) \le q_\sigma(f)$, for every $f \in \mathcal{L}(\sigma,\text{sim}(\mathcal{K}))$.

Now, because $q_\sigma = q_\rho$, if $\sigma(f) = q_\rho(f)$, for every $f \in \text{sim}(\mathcal{K})$, then the seminorm σ is integrating. Conversely, if σ is integrating, then $q_\sigma(f) = \sigma(f)$, for every $f \in \text{sim}(\mathcal{K})$ and, hence, $\sigma(f) = q_\rho(f)$, for every $f \in \text{sim}(\mathcal{K})$.

EXAMPLES 2.11. (i) Let $\mathcal{K}$ be a vector space of bounded functions on a space Ω . Let $\rho(f) = \sup\{|f(\omega)| : \omega \in \Omega\}$, for every $f \in \mathcal{K}$. Then ρ is an integrating seminorm on $\mathcal{K}$.

(ii) Let $-\infty < a < b < \infty$ and $\Omega = [a,b]$. Let $\mathcal{K}$ be the space of all functions on $[a,b]$ of bounded variation and let $\rho(f) = |f(a)| + \text{var}(f)$, for every $f \in \mathcal{K}$. Then ρ is an integrating seminorm on $\mathcal{K}$.

E. It can be easily deduced from the general theory of measure and integral that (positive) measures are integrating gauges. However, we wish to show that the classical measure and integration theory is an instance of the theory presented here. Therefore, we prove first that a measure is an integrating gauge. Actually, we prove two slightly more general results. It is convenient to start with a re-statement of Stone's condition, [62], for a positive linear functional to be a Daniell integral.

Let $\mathcal{K}$ be a vector space consisting of real valued functions on a space Ω . A real valued linear functional, ι , on $\mathcal{K}$ is said to be positive if $\iota(f) \ge 0$, for every function $f \in \mathcal{K}$ such that $f(\omega) \ge 0$ for every $\omega \in \Omega$.

In this definition, it is not assumed that $\mathcal{K}$ is a vector lattice (see Section 1D), but, in the following proposition, such an assumption is made.

PROPOSITION 2.12. *Let $\mathcal{K}$ be a vector lattice of real valued functions on Ω and let ι be a positive linear functional on $\mathcal{K}$. Let $\rho(f) = \iota(|f|)$, for every $f \in \mathcal{K}$.*

Then ρ is a seminorm on $\mathcal{K}$ which is integrating if and only if

(E.1) $$\iota(|f|) \le \sum_{j=1}^{\infty} \iota(|f_j|)$$

for any functions $f \in \mathcal{K}$ *and* $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ *such that*

$$\text{(E.2)} \qquad |f(\omega)| \leq \sum_{j=1}^{\infty} |f_j(\omega)|$$

for every $\omega \in \Omega$.

Proof. If this condition is satisfied then, by Proposition 2.7, the seminorm ρ is integrating.

Conversely, let us assume that the seminorm ρ is integrating. Let $f \in \mathcal{K}$ and $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ be functions such that the inequality (E.2) holds for every $\omega \in \Omega$. Using the fact that $\mathcal{K}$ is a vector lattice, we construct inductively functions $g_j \in \mathcal{K}$ such that $|g_j| \leq |f_j|$, $j = 1,2,\ldots,$ and

$$f(\omega) = \sum_{j=1}^{\infty} g_j(\omega)$$

for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |g_j(\omega)| < \infty .$$

Then

$$\iota(|f|) = \rho(f) \leq \sum_{j=1}^{\infty} \rho(g_j) = \sum_{j=1}^{\infty} \iota(|g_j|) \leq \sum_{j=1}^{\infty} \iota(|f_j|) ,$$

because ρ is integrating.

The following proposition says slightly more than that a non-negative σ-additive set function is an integrating gauge, even if we do not assume that its domain is rich. If we wanted to prove merely that a non-negative σ-additive set function on a quasiring (see Section 1D) is an integrating gauge, then the proof could be slightly simplified. (See Example 4.28(i) in Section 4G.)

PROPOSITION 2.13. *Let* ι *be a nonnegative real valued additive set function on a quasiring of sets* $\mathcal{Q}$ *in a space* Ω . *Then* ι *is an integrating gauge on* $\mathcal{Q}$ *if and only if it is* σ*-additive. Moreover, if* ι *is* σ*-additive and* $\rho(f) = \iota(|f|)$, *for every* $f \in \mathrm{sim}(\mathcal{Q})$, *then* ρ *is an integrating seminorm on* $\mathrm{sim}(\mathcal{Q})$, $\mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q})) = \mathcal{L}(\iota,\mathcal{Q})$ *and* $q_\rho(f) = q_\iota(f)$ *for every* $f \in \mathcal{L}(\iota,\mathcal{Q})$.

Proof. If ι is not σ-additive, then, obviously, ι is not an integrating gauge. So, let us assume that ι is σ-additive. Let $\mathcal{K} = \mathrm{sim}(\mathcal{Q})$ and let $\rho(f) = \iota(|f|)$ for every $f \in \mathcal{K}$. If we show that the seminorm ρ is integrating, it will follow, by Proposition 2.7, that ι is an integrating gauge. To do that, by Proposition 2.12, it suffices to show that (E.1) holds for any functions $f \in \mathcal{K}$ and $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ such that (E.2) holds for every $\omega \in \Omega$. But this follows from a result of F. Riesz ([59], Lemma A and Lemma B in no. 16). For completeness we include the proof.

Let m be a positive integer, $d_j \geq 0$ numbers and $Y_j \in \mathcal{Q}$ pair-wise disjoint sets, $j = 1,2,\ldots,m,$ such that

$$|f| = \sum_{j=1}^{m} d_j Y_j .$$

Let Y be the union of the sets Y_j and d the largest of the numbers d_j, $j = 1,2,\ldots,m$. Let $\epsilon > 0$ and,

$$g_n = \sum_{j=1}^{n} |f_j| ,$$

$$Z_n = \{\omega \in Y : g_n(\omega) > |f(\omega)| - \epsilon\} ,$$

for every $n = 1,2,\ldots$. Then $\iota(Y \setminus Z_n) \to 0$, as $n \to \infty$, because the sets $Y \setminus Z_n$ decrease monotonically to $\emptyset$, they belong to the ring generated by $\mathcal{Q}$ and the extension of ι to this ring is σ-additive. Furthermore,

$$\iota(g_n) \geq \iota(Yg_n) = \iota(Z_n g_n) + \iota((Y\setminus Z_n)g_n) \geq \iota(Z_n(|f| - \epsilon)) + \iota((Y\setminus Z_n)(g_n - |f|)) =$$

$$= \iota(Z_n|f|) - \epsilon\iota(Z_n) + \iota((Y\setminus Z_n)g_n) - \iota((Y/Z_n)|f|) \geq$$

$$\geq \iota(|f|) - \epsilon\iota(Z_n) - 2\iota((Y\setminus Z_n)|f|) \geq \iota(|f|) - \epsilon\iota(Y) - 2d\iota(Y\setminus Z_n) ,$$

for every $n = 1,2,\ldots$. Therefore,

$$\sum_{j=1}^{\infty} \iota(|f_j|) = \lim_{n\to\infty} \iota(g_n) \geq \iota(|f|) - \epsilon\iota(Y) ,$$

and (E.1) follows.

Now, by Proposition 2.7, $\mathcal{L}(\iota,\mathcal{Q}) \subset \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) \leq q_\iota(f)$. On the other hand, $\mathcal{K} \subset \mathcal{L}(\iota,\mathcal{Q})$ and $\rho(f) = q_\iota(f)$, for every $f \in \mathcal{K}$, because, obviously, $q_\iota(f) \leq \rho(f)$

and, since the seminorm ρ is integrating, $\rho(f) = q_\rho(f) \le q_\iota(f)$. So, if $f \in \mathcal{L}(\rho,\mathcal{K})$, let $f_j \in \mathcal{K}$, $j = 1,2,\ldots$, be functions, satisfying condition (D.2), such that the equality (A.5) holds for every $\omega \in \Omega$ for which the inequality (A.6) does. Then, by Proposition 2.1, $f \in \mathcal{L}(\iota,\mathcal{Q})$ and

$$q_\iota(f) \le \sum_{j=1}^{\infty} q_\iota(f_j) = \sum_{j=1}^{\infty} \rho(f_j) .$$

Consequently, $q_\iota(f) \le q_\rho(f)$.

F. In this section, we present some methods of producing new integrating gauges if some are already given.

PROPOSITION 2.14. *Let $\mathcal{K}$ be a nontrivial family of functions on a space Ω . Let P be a collection of integrating gauges on $\mathcal{K}$ such that*

$$\sigma(f) = \sup\{\rho(f) : \rho \in \mathrm{P}\} < \infty ,$$

for every $f \in \mathcal{K}$. Then σ is an integrating gauge on $\mathcal{K}$.

Proof. Let $f \in \mathcal{K}$ and $\epsilon > 0$. Let $\rho \in \mathrm{P}$ be a gauge such that $\sigma(f) - \epsilon < \rho(f)$. Then

$$\sigma(f) - \epsilon < \rho(f) = q_\rho(f) \le \sum_{j=1}^{\infty} |c_j| \rho(f_j) \le \sum_{j=1}^{\infty} |c_j| \sigma(f_j) ,$$

for any numbers c_j and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots$, such that

$$\sum_{j=1}^{\infty} \sigma(f_j) < \infty ,$$

and (A.5) holds for every $\omega \in \Omega$ for which (A.6) does. Hence, $\sigma(f) \le q_\sigma(f)$, which means that σ is integrating.

EXAMPLE 2.15. Let Ω be any space. Let W be a real valued function on Ω such that $W(\omega) > 0$, for every $\omega \in \Omega$. Let $1 \le p \le \infty$. If $J \subset \Omega$ is a finite set, let

$$\rho_J(f) = \left[\sum_{\omega \in J} W(\omega) |f(\omega)|^p \right]^{1/p} ,$$

if $1 \leq p < \infty$, and

$$\rho_J(f) = \max\{ W(\omega)|f(\omega)| : \omega \in J\} ,$$

if $p = \infty$, for any scalar valued function f on Ω. Let $\mathcal{K}$ be the family of all functions f on Ω such that

$$\rho(f) = \sup \rho_J(f) < \infty ,$$

where the supremum is taken over all finite subsets, J, of Ω. By Proposition 2.14, ρ is an integrating gauge on $\mathcal{K}$.

It is straightforward that $\mathcal{K}$ is a vector space and that ρ is a seminorm on $\mathcal{K}$. Actually, ρ is a norm because the only ρ-null set is the empty set. Then it is not difficult to ascertain that $\mathcal{K}$ is ρ-complete. Hence, by Corollary 2.5, $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$ and $q_\rho = \rho$. Of, course, $\mathcal{K}$ is the classical weighted l^p space on Ω with the weight W and ρ in its norm;

$$\rho(f) = \left[\sum_{\omega\in\Omega} W(\omega)|f(\omega)|^p \right]^{1/p} ,$$

for $1 \leq p < \infty$, and $\rho(f) = \sup\{ W(\omega)|f(\omega)| : \omega \in \Omega\}$ for $p = \infty$, $f \in \mathcal{K}$.

PROPOSITION 2.16 *Let $\mathcal{L}$ be a vector space of scalar valued functions on a space Ω and let σ be an integrating seminorm on $\mathcal{L}$. Let $\mathcal{K}$ be a vector subspace of $\mathcal{L}$ and let ρ be a seminorm on $\mathcal{K}$ such that*

(i) *$\sigma(f) \leq \rho(f)$ for every $f \in \mathcal{K}$:*

(ii) *every σ-null function f is ρ-null, belongs to $\mathcal{K}$ and $\rho(f) = 0$; and*

(iii) *the space $\mathcal{K}$ is ρ-complete.*

Then $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$ and $q_\rho = \rho$, so that the seminorm ρ is integrating.

Proof. Let $\{g_n\}_{n=1}^{\infty}$ be a ρ-Cauchy sequence of functions from $\mathcal{K}$ such that (D.5) holds for ρ-almost every $\omega \in \Omega$. Let $f \in \mathcal{K}$ be a function, existing by (iii), such that

$$\lim_{n\to\infty} \rho(f-g_n) = 0 .$$

The requirement (i) implies that every ρ-null function is σ-null. Hence, (D.5) holds for σ-almost every $\omega \in \Omega$. Furthermore, by (i) the sequence $\{g_n\}_{n=1}^{\infty}$ is σ-Cauchy. Hence, by Theorem 2.4, the function f is σ-null. Therefore, by (ii), $\rho(f) = 0$. Consequently, the equality (D.4) holds and, by Proposition 2.9, the seminorm ρ is integrating. By Corollary 2.5, $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$ and so, $q_\rho = \rho$.

PROPOSITION 2.17. *Let $\mathcal{L}$ be a vector space of scalar valued functions on a space Ω and let σ be an integrating seminorm on $\mathcal{L}$ such that $\mathcal{L}(\sigma,\mathcal{L}) = \mathcal{L}$. Let $\mathcal{K}$ be a vector subspace of $\mathcal{L}$, let E be a Banach space and let $\mu : \mathcal{K} \to E$ be a closed linear map. Let*

$$\rho(f) = \sigma(f) + |\mu(f)|$$

for every $f \in \mathcal{K}$.

Then ρ is an integrating seminorm on $\mathcal{K}$ and $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$.

Proof. Let $f_j \in \mathcal{K}$, $j = 1,2,\ldots$, be functions satisfying condition (D.2) and let f be a function on Ω such that the equality (A.5) holds for each $\omega \in \Omega$ for which the inequality (A.6) does. Then

$$\sum_{j=1}^{\infty} \sigma(f_j) < \infty \text{ and } \sum_{j=1}^{\infty} |\mu(f_j)| < \infty .$$

Let the functions g_n be given by (D.6) for every $n = 1,2,\ldots$. Then, by Proposition 2.1, $f \in \mathcal{L}$ and $\sigma(g_n - f) \to 0$, as $n \to \infty$. Furthermore, there exists an element x of E such that $|\mu(g_n) - x| \to 0$. Therefore, $f \in \mathcal{K}$ and $\mu(f) = x$, because the map μ is closed. But then $\rho(g_n - f) \to 0$, as $n \to \infty$. Consequently, by Proposition 2.8, the seminorm ρ is integrating and, by the definition of ρ-integrable functions, $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$.

G. The space $\mathcal{L}(\rho,\mathcal{K})$ is not necessarily a vector lattice. (See Section 1D.)

EXAMPLE 2.18. Let $\mathcal{K}$ be the family of all functions continuous in the closed unit disc, $\Omega = \{(x,y) : x^2 + y^2 \le 1\}$, and harmonic in its interior. Let $\rho(f) = \sup\{|f(\omega)| : \omega \in \partial\Omega\}$, for every $f \in \mathcal{K}$. Then the seminorm ρ is integrating and $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$, but the space $\mathcal{K}$ is of course not a vector lattice.

We are going to give a sufficient condition for $\mathcal{L}(\rho,\mathcal{K})$ to be a vector lattice. The formulation of the following definition and propositions is slightly more general; it allows also for complex valued functions to belong to $\mathcal{K}$ and $\mathcal{L}(\rho,\mathcal{K})$.

A gauge, ρ , on a nontrivial family of functions, $\mathcal{K}$, will be called monotonic if $\rho(f) \leq \rho(g)$ for any functions $f \in \mathcal{K}$ and $g \in \mathcal{K}$ such that $|f| \leq |g|$.

The seminorm ρ in Example 2.18 is obviously monotonic.

PROPOSITION 2.19. *Let $\mathcal{K}$ be a vector space of scalar valued functions on a space Ω . Let ρ be an integrating seminorm on $\mathcal{K}$. Assume that $|f| \in \mathcal{L}(\rho,\mathcal{K})$, for every $f \in \mathcal{K}$, and that $q_\rho(|f|-|g|) \leq \rho(f-g)$, for every $f \in \mathcal{K}$ and $g \in \mathcal{K}$. Then $|f| \in \mathcal{L}(\rho,\mathcal{K})$, for every $f \in \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(|f|-|g|) \leq q_\rho(f-g)$, for every $f \in \mathcal{L}(\rho,\mathcal{K})$ and $g \in \mathcal{L}(\rho,\mathcal{K})$.*

Proof. It is a matter of routine application of Theorem 2.4, say.

PROPOSITION 2.20. *Let $\mathcal{K}$ be a vector space of scalar valued functions on a space Ω such that $|f| \in \mathcal{K}$ for every $f \in \mathcal{K}$. Let ρ be a monotonic integrating seminorm on $\mathcal{K}$. Then $|f| \in \mathcal{L}(\rho,\mathcal{K})$, for every $f \in \mathcal{L}(\rho,\mathcal{K})$ and the seminorm q_ρ is monotonic.*

Proof. The monotonicity of ρ implies that $\rho(|f|) = \rho(f)$, for every $f \in \mathcal{K}$. Moreover, $\rho(|f|-|g|) = \rho(||f|-|g||) \leq \rho(|f-g|) = \rho(f-g)$, for every $f \in \mathcal{K}$ and $g \in \mathcal{K}$. Hence, the assumptions of Proposition 2.19 are satisfied and so, $|f| \in \mathcal{L}(\rho,\mathcal{K})$, for every $f \in \mathcal{L}(\rho,\mathcal{K})$. Then it is again a matter of routine to deduce that $q_\rho(|f|) = q_\rho(f)$, for every $f \in \mathcal{L}(\rho,\mathcal{K})$.

Now, let $f \in \mathcal{L}(\rho,\mathcal{K})$ and $g \in \mathcal{L}(\rho,\mathcal{K})$ be functions such that $|f| \leq |g|$. Let $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ be ρ-Cauchy sequences of functions from $\mathcal{K}$, converging ρ-almost everywhere to the functions f and g , respectively. Let $h_n = \frac{1}{2}(f_n+g_n-||f_n|-|g_n||)$, for every $n = 1,2,\ldots$. Then $|h_n-h_m| \leq |f_n-f_m| + |g_n-g_m|$, for any integers $n \geq 1$ and $m \geq 1$, so that the sequence $\{h_n\}_{n=1}^\infty$ is ρ-Cauchy. Moreover, the sequence $\{h_n\}_{n=1}^\infty$ converges ρ-almost everywhere to the function $|f|$.

Because $\rho(h_n) \le \rho(g_n)$ for every $n = 1,2,\ldots,$ by Theorem 2.4,

$$q_\rho(f) = q_\rho(|f|) = \lim_{n\to\infty} \rho(h_n) \le \lim_{n\to\infty} \rho(g_n) = q_\rho(g) .$$

PROPOSITION 2.21. *Let $\mathcal{K}$ be a vector space of scalar valued functions on a space Ω such that $|f| \in \mathcal{K}$ for every $f \in \mathcal{K}$. Let ρ be a monotonic integrating seminorm on $\mathcal{K}$ such that $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$. Assume that each ρ-bounded monotonic sequence of real valued functions from $\mathcal{K}$ is ρ-Cauchy.*

(i) *Let $\{f_n\}_{n=1}^\infty$ be a ρ-bounded monotonic sequence of real valued functions from $\mathcal{K}$. Then the sequence $\{f_n(\omega)\}_{n=1}^\infty$ is convergent for ρ-almost every $\omega \in \Omega$. If, moreover, f is a function on Ω such that*

$$f(\omega) = \lim_{n\to\infty} f_n(\omega) \tag{G.1}$$

for ρ-almost every $\omega \in \Omega$, then $f \in \mathcal{K}$ and

$$\lim_{n\to\infty} \rho(f - f_n) = 0 . \tag{G.2}$$

(ii) *Let $g \in \mathcal{K}$ be a real valued function and $f_n \in \mathcal{K}$, $n = 1,2,\ldots,$ arbitrary functions such that $|f_n| \le g$ for every $n = 1,2,\ldots$ and the sequence $\{f_n(\omega)\}_{n=1}^\infty$ is convergent for ρ-almost every $\omega \in \Omega$. Let f be a function on Ω such that* (G.1) *holds for ρ-almost every $\omega \in \Omega$. Then $f \in \mathcal{K}$ and* (G.2) *holds.*

Proof. (i) By Theorem 2.4, the sequence $\{f_n\}_{n=1}^\infty$ has a ρ-almost everywhere convergent subsequence. Hence, because of its monotonicity, the sequence $\{f_n(\omega)\}_{n=1}^\infty$ converges for ρ-almost every $\omega \in \Omega$. By Theorem 2.4, if f is a function on Ω such that (G.1) holds for ρ-almost every $\omega \in \Omega$, then $f \in \mathcal{K}$ and (G.2) holds.

(ii) Let the function f_n, $n = 1,2,\ldots,$ be real valued. Let

$$g_n(\omega) = \lim_{m\to\infty}(\sup\{f_j(\omega) : n \le j \le m\}) , \quad h_n(\omega) = \lim_{m\to\infty}(\inf\{f_j(\omega) : n \le j \le m\}) ,$$

for every $n = 1,2,\ldots$ and $\omega \in \Omega$. Because the seminorm ρ is monotonic, by (i) we have $g_n \in \mathcal{K}$ and $h_n \in \mathcal{K}$, for every $n = 1,2,\ldots$. Also, the sequences $\{g_n\}_{n=1}^\infty$,

$\{h_n\}_{n=1}^{\infty}$ and $\{g_n-h_n\}_{n=1}^{\infty}$ are monotonic and ρ-bounded, $h_n(\omega) \le f(\omega) \le g_n(\omega)$ for every $n = 1,2,\ldots$ and ρ-almost every $\omega \in \Omega$ and

$$\lim_{n\to\infty} h_n(\omega) = f(\omega) = \lim_{n\to\infty} g_n(\omega)$$

for ρ-almost every $\omega \in \Omega$. Therefore, by (i), $f \in \mathcal{K}$ and $\rho(g_n-h_n) \to 0$, as $n \to \infty$. Because $\rho(f-h_n) \le \rho(g_n-h_n)$ and $\rho(f_n-h_n) \le \rho(g_n-h_n)$, for every $n = 1,2,\ldots,$ we have (G.2).

EXAMPLE 2.22. Let $\Omega = [0,1]$, $\mathcal{K} = C([0,1])$ and $\rho(f) = \sup\{|f(\omega)| : \omega \in \Omega\}$, for every $f \in \mathcal{K}$. Then ρ is a monotonic integrating norm on $\mathcal{K}$ such that $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$. However, not every ρ-bounded monotonic sequence of real valued functions from $\mathcal{K}$ is ρ-Cauchy.

If the space $\mathcal{K}$ and the seminorm ρ satisfy the assumptions of Proposition 2.21, we say that they have the Lebesgue property.

H. Let B be a set of integrating gauges; each gauge $\beta \in B$ is defined on a nontrivial family, $\mathcal{K}_\beta$, of functions on a space Ω_β. The spaces Ω_β, $\beta \in B$, are assumed to be pair-wise disjoint.

Let $\mathcal{J}$ be a vector lattice or real valued functions on B and α a monotonic integrating seminorm on $\mathcal{J}$.

Let

$$\Omega = \bigcup_{\beta\in B} \Omega_\beta .$$

Let $\mathcal{K}$ be the family of all functions f on Ω such that, for every $\beta \in B$, the restriction, $f_\beta = f|\Omega_\beta$, of f to Ω_β belongs to $\mathcal{K}_\beta$ and the function φ_f on B, such that $\varphi_f(\beta) = \beta(f_\beta)$ for every $\beta \in B$, belongs to $\mathcal{J}$.

Let

$$\rho(f) = \alpha(\varphi_f)$$

for every $f \in \mathcal{K}$.

PROPOSITION 2.23. *The functional ρ is an integrating gauge on $\mathcal{K}$.*

Proof. Let $f \in \mathcal{K}$ and let c_j be numbers and $f_j \in \mathcal{K}$ functions, $j = 1,2,\ldots,$ such that

$$\sum_{j=1}^{\infty} |c_j| \rho(f_j) = \sum_{j=1}^{\infty} |c_j| \alpha(\varphi_{f_j}) < \infty$$

and the equality (A.2) holds for each $\omega \in \Omega$ for which the inequality (A.3) does. By Proposition 2.7,

$$\varphi_f(\beta) = \beta(f_\beta) \le \sum_{j=1}^{\infty} |c_j| \beta((f_j)_\beta) = \sum_{j=1}^{\infty} |c_j| \varphi_{f_j}(\beta)$$

for each $\beta \in \mathrm{B}$, because these gauges are integrating. Let

$$\psi(\beta) = \sum_{j=1}^{\infty} |c_j| \varphi_{f_j}(\beta) ,$$

for every $\beta \in \mathrm{B}$ such that

$$\sum_{j=1}^{\infty} |c_j| \varphi_{f_j}(\beta) < \infty ,$$

and let $\psi(\beta) = \varphi_f(\beta)$, for every $\beta \in \mathrm{B}$ such that

$$\sum_{j=1}^{\infty} |c_j| \varphi_{f_j}(\beta) = \infty .$$

Then $\psi \in \mathcal{L}(\alpha,\mathcal{J})$ and $0 \le \varphi_f \le \psi$. Therefore, by Proposition 2.19,

$$\rho(f) = \alpha(\varphi_f) = q_\alpha(\alpha_f) \le q_\alpha(\psi) \le \sum_{j=1}^{\infty} |c_j| \alpha(\varphi_{f_j}) = \sum_{j=1}^{\infty} |c_j| \rho(f_j) .$$

So, by Proposition 2.7, the gauge ρ is integrating.

In practice, the most useful choice of $\mathcal{J}$ is perhaps the space $l^1(\mathrm{B})$, or the space $l^\infty(\mathrm{B})$, with its natural norm (Example 2.15 with weight $W(\beta) = 1$ for each $\beta \in \mathrm{B}$).

J. The basic way of showing that a positive additive set function is in fact σ-additive is to exploit compactness and regularity of some sort or another, that is, to use the Alexandrov theorem or some of its generalizations. (See Section 1F.) In this section a similar means for showing that a gauge is integrating is presented.

Let $\mathcal{Q}$ be a quasiring of sets in a space Ω. (See Section 1D.) Let ρ be a gauge on $\mathcal{Q}$.

Let us call the gauge ρ very sub-additive if the inequality

$$\rho(X) \leq \sum_{j=1}^{n} c_j \rho(X_j)$$

holds for any set $X \in \mathcal{Q}$, any $n = 1,2,\ldots$, and any sets $X_j \in \mathcal{Q}$ and numbers $c_j \geq 0$, $j = 1,2,\ldots,n$, such that

$$X(\omega) \leq \sum_{j=1}^{n} c_j X_j(\omega)$$

for every $\omega \in \Omega$.

The use of the adverb "very" in this definition is dictated by a certain caution: it is a warning that a gauge may rather unexpectedly fail to be very sub-additive.

EXAMPLE 2.24. Let $\Omega = \mathbb{R}^2$ and let $\mathcal{Q}$ be the family of all intervals $X = (u_1,v_1]\times(u_2,v_2]$ with $u_1 \leq v_1$ and $u_2 \leq v_2$. Let $f = 2(0,3]\times(0,3] - 3(1,2]\times(1,2]$ and

$$\mu(X) = \int_X f \, d\iota ,$$

for every $X \in \mathcal{Q}$, where ι is the two-dimensional Lebesgue measure. The gauge ρ, defined by

$$\rho(X) = \sup\{ |\mu(X \cap Z)| : Z \in \mathcal{Q} \}$$

for every $X \in \mathcal{Q}$, is not very sub-additive. In fact, the interval $X = (0,3]\times(0,3]$ is equal to the union of the intervals $X_1 = (1,2]\times(0,3]$, $X_2 = (0,3]\times(1,2]$, $X_3 = (0,1]\times(0,1]$, $X_4 = (2,3]\times(0,1]$, $X_5 = (2,3]\times(2,3]$ and $X_6 = (0,1]\times(2,3]$, but $\rho(X) = 15$, $\rho(X_1) = \rho(X_2) = 3$ and $\rho(X_3) = \rho(X_4) = \rho(X_5) = \rho(X_6) = 2$.

The property of being very sub-additive is rather advantageous though, because it allows us to use the following property of regularity to prove that a gauge is integrating.

Assuming that Ω is a topological space, the gauge ρ is said to be regular if, for every set $X \in \mathcal{Q}$ and $\epsilon > 0$, there is

(i) an open set $U \supset X$ and a set $Y \in \mathcal{Q}$ such that $U \subset Y$ and $\rho(Y) - \rho(X) < \epsilon$; and

(ii) a compact set $K \subset X$ and a set $Z \in \mathcal{Q}$ such that $Z \subset K$ and $\rho(X) - \rho(Z) < \epsilon$.

PROPOSITION 2.25. *A very sub-additive and regular gauge on a quasiring of sets in a topological space is integrating.*

Proof. Let $\mathcal{Q}$ be a quasiring of sets in a topological space Ω. Let ρ be a very sub-additive and regular gauge on $\mathcal{Q}$.

Let $X \in \mathcal{Q}$ and, for every $j = 1,2,\dots$, let $X_j \in \mathcal{Q}$ be a set and c_j a number such that

(J.1) $$X(\omega) = \sum_{j=1}^{\infty} c_j X_j(\omega)$$

for every $\omega \in \Omega$ for which

(J.2) $$\sum_{j=1}^{\infty} |c_j| X_j(\omega) < \infty .$$

Our aim is to show that

(J.3) $$\rho(X) \le \sum_{j=1}^{\infty} |c_j| \rho(X_j) .$$

Let $0 < \epsilon < 1$. Let K be a compact set and Z a set in $\mathcal{Q}$ such that $Z \subset K \subset X$ and $\rho(X) - \epsilon < \rho(Z)$. For every $j = 1,2,\dots$, let U_j be an open set and Y_j a set in $\mathcal{Q}$ such that $X_j \subset U_j \subset Y_j$ and $|c_j|\rho(Y_j) < |c_j|\rho(X_j) + \epsilon 2^{-j}$. Let $n \ge 1$ be an integer such that

$$\sum_{j=1}^{n} |c_j| Y_j(\omega) \ge (1-\epsilon) Z(\omega)$$

for every $\omega \in \Omega$. Then

$$(1-\epsilon)(\rho(X)-\epsilon) < (1-\epsilon)\rho(Z) \leq \sum_{j=1}^{n} |c_j| \rho(Y_j) <$$

$$< \sum_{j=1}^{n} (|c_j| \rho(X_j) + \epsilon 2^{-j}) < \sum_{j=1}^{\infty} |c_j| \rho(X_j) + \epsilon .$$

Hence, (J.3) holds and, by Proposition 2.7, the gauge ρ is integrating.

K. The first proposition of this section represents a method of producing new integrating gauges from ones already guaranteed to be integrating. Recall that a quasiring, $\mathcal{Q}$, is said to be multiplicative if $X \cap Y \in \mathcal{Q}$ for any sets $X \in \mathcal{Q}$ and $Y \in \mathcal{Q}$. (See Section 1D.)

Clearly, a gauge, σ, on a quasiring of sets, $\mathcal{Q}$, is monotonic if and only if $\sigma(X) \leq \sigma(Y)$ for any sets $X \in \mathcal{Q}$ and $Y \in \mathcal{Q}$ such that $X \subset Y$. (See Section G.)

PROPOSITION 2.26. *Let σ be an integrating monotonic gauge on a multiplicative quasiring, $\mathcal{Q}$, of sets in a space Ω. Let φ be a real valued, continuous, strictly increasing and concave function on the interval $[0,\infty)$ such that $\varphi(0) = 0$. Let $\rho(X) = \varphi(\sigma(X))$ for every $X \in \mathcal{Q}$.*

Then ρ is an integrating gauge on $\mathcal{Q}$.

Proof. Let $X \in \mathcal{Q}$ be a set, c_j numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,\ldots,$ such that the equality (J.1) holds for every $\omega \in \Omega$ for which the inequality (J.2) does. Our aim is to show that (J.3) holds.

Without loss of generality, we will assume that $X_j \subset X$, for every $j = 1,2,\ldots,$ because, if the sets X_j are replaced by $X_j \cap X$, then the equality (J.1) remains valid for every $\omega \in \Omega$ satisfying (J.2) and $\rho(X_j \cap X) = \varphi(\sigma(X_j \cap X)) \leq \varphi(\sigma(X_j)) = \rho(X_j)$, by the monotonicity of φ and σ. We will also assume that

(K.1) $$\sum_{j=1}^{\infty} |c_j| \rho(X_j) < \infty$$

and that $\sigma(X_j) \neq 0$, for some $j = 1,2,\ldots$.

Let $s = \sup\{\sigma(X_j) : j = 1,2,\dots\}$. By the assumption just made and the monotonicity of σ, we have $0 < s \le \sigma(X)$. Let $k = \varphi(s)/s$. Then $k\sigma(X_j) \le \varphi(\sigma(X_j)) = \rho(X_j)$, for every $j = 1,2,\dots$, because the function φ is concave. Therefore

$$t = \sum_{j=1}^{\infty} |c_j|\sigma(X_j) \le k^{-1} \sum_{j=1}^{\infty} |c_j|\rho(X_j) < \infty .$$

By Proposition 2.7, $\sigma(X) \le t$, because the gauge σ is integrating, and so, $s \le t$. Consequently, by the monotonicity and concavity of φ, we have

$$\rho(X) = \varphi(\sigma(X)) \le \varphi(t) \le kt = k \sum_{j=1}^{\infty} |c_j|\sigma(X_j) \le \sum_{j=1}^{\infty} |c_j|\rho(X_j) .$$

So, (J.3) holds. But, if (K.1) does not hold, then (J.3) is trivially true. Moreover, if $\sigma(X_j) = 0$ for every $j = 1,2,\dots$, then, by Proposition 2.7, $\sigma(X) = 0$, because the gauge σ is integrating. Hence, (J.3) holds also in this case, and, by Proposition 2.7, the gauge ρ is integrating.

Typically, a non-negative σ-additive set function is used in the role of the gauge σ in Proposition 2.26.

The second proposition of this section says that if ρ is an integrating gauge on a quasiring of sets, then the assumptions of Proposition 2.10 are satisfied, that is, the seminorm, σ, defined in that proposition is integrating.

PROPOSITION 2.27. *Let $\mathcal{Q}$ be a quasiring of sets in a space Ω and let ρ be an integrating gauge on $\mathcal{Q}$. Then, for every real valued function $f \in \mathrm{sim}(\mathcal{Q})$,*

$$q_\rho(f) = \inf \sum_{j=1}^{n} |c_j|\rho(X_j) ,$$

where the infimum is taken over all expressions of f in the form

$$f = \sum_{j=1}^{n} c_j X_j ,$$

with arbitrary $n = 1,2,\dots$, real numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1,2,\dots,n$.

Proof. Let $\mathcal{R}$ be the ring of sets generated by $\mathcal{Q}$. (See Section 1D.) For every set $Y \in \mathcal{R}$, let

$$\sigma(Y) = \min \sum_{j=1}^{n} \rho(X_j) ,$$

where the minimum is taken over all expressions of the set Y in the form

$$Y = \sum_{j=1}^{n} e_j X_j ,$$

with arbitrary $n = 1,2,\ldots,$ arbitrary choices of $e_j = \pm 1$ and sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots,n$. Then, for every real valued function $f \in \mathrm{sim}(\mathcal{Q})$, there exist unique integers $k \geq 0$ and $\ell \geq 0$, sets $Y_i \in \mathcal{R}$, $Z_j \in \mathcal{R}$ and real numbers $c_i > 0$, $d_j > 0$, $i = 1,2,\ldots,k$, $j = 1,2,\ldots,\ell$, such that $Y_1 \cap Z_1 = \emptyset$, $Y_{i-1} \subset Y_i$, $Z_{j-1} \subset Z_j$, for $i = 2,\ldots,k$ and $j = 2,\ldots,\ell$, and

$$f = \sum_{i=1}^{k} c_i Y_i - \sum_{j=1}^{\ell} d_j Z_j .$$

For any function so expressed, let

$$\sigma(f) = \sum_{i=1}^{k} c_i \sigma(Y_i) + \sum_{i=1}^{\ell} d_j \sigma(Z_j) .$$

Now, $\mathcal{L}(\sigma,\mathrm{sim}(\mathcal{Q})) = \mathcal{L}(\rho,\mathcal{Q})$ and $q_\sigma(f) = q_\rho(f)$, for every $f \in \mathcal{L}(\sigma,\mathrm{sim}(\mathcal{Q}))$. In fact, $\mathcal{Q} \subset \mathrm{sim}(\mathcal{Q})$ and $\rho(X) = \sigma(X)$, for every $X \in Q$. On the other hand, if $f_j \in \mathrm{sim}(\mathcal{Q})$, $j = 1,2,\ldots,$ are functions such that

$$\sum_{j=1}^{\infty} \sigma(f_j) < \infty ,$$

then there exist numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots,$ such that

$$\sum_{j=1}^{\infty} |c_j| \rho(X_j) < \infty , \quad \sum_{j=1}^{\infty} c_j X_j(\omega) = \sum_{j=1}^{\infty} f_j(\omega)$$

and also

$$\sum_{j=1}^{\infty} |c_j| X_j(\omega) = \sum_{j=1}^{\infty} |f_j(\omega)| ,$$

for every $\omega \in \Omega$. It follows that σ is a seminorm on (real) $\mathrm{sim}(\mathcal{Q})$ and that $L(\rho,\mathcal{Q})$ can be identified with the completion of $\mathrm{sim}(\mathcal{Q})$ in (the norm induced by) σ . Consequently, $q_\rho(f) = \sigma(f)$, for every $f \in \mathrm{sim}(\mathcal{Q})$.