THE SHAPE OF INFINITY

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1. INTRODUCTION

The remarkable developments that have taken place over the last few years in the field of low-dimensional topology are well-known. In 1982 Michael Freedman proved a 5-dimensional proper h-cobordism theorem which led him to a classification of compact simply connected topological 4-manifolds, and thus to a proof of the 4-dimensional Poincaré conjecture. And more recently, Simon Donaldson proved that compact simply connected smooth 4-manifolds have intersection forms of a very restricted type. This led to the now famous result that $\mathbb{R}^4$ admits an exotic differentiable structure. Donaldson's work was based on gauge theoretical techniques deriving from elementary particle physics, which may well benefit in consequence. One may contrast this with the situation for contemporary classical general relativity which, although being primarily concerned with manifolds of dimensions 3 and 4, has yet to make any such contact with the subtleties of low-dimensional topology. My objective here therefore is to show, in the context of a fundamental class of space-times, how relativity can give rise to topological problems that are of both mathematical and physical interest.

Consider an isolated massive body with a history extending indefinitely to the past. Suppose the gravitational field is too weak to generate collapse or to give rise to orbiting null geodesics akin to those at $r = 3m$ in Schwarzschild space-time. To make the situation even simpler, assume that there is an $\mathbb{R}^3$ Cauchy surface. Then the underlying space-time manifold is diffeomorphic to (standard) $\mathbb{R}^4$ and one may reasonably assume that all endless null geodesics originate from a past null infinity $\mathcal{I}^-$ and terminate at a future null infinity $\mathcal{I}^+$. As the space-time evolves, $\mathcal{I}^+$ is exposed to data on an increasingly large region of the

Cauchy surface and may be expected to respond by exhibiting increasingly complicated geometrical and topological structure. In particular one might expect the (Weyl) curvature at $\mathcal{I}^+$ to grow away from zero and for the global topology of $\mathcal{I}^+$ to differ from that of Minkowski space. But the dogma of the subject is that neither possibility is realised. It will be seen later that this may be founded upon hypotheses which, when expressed purely in terms of the space-time, are difficult to motivate. Instead the guiding principle should be to presuppose nothing about $\mathcal{I}^+$, but rather to determine the restrictions which arise from the development of the initial data.

2. DEFINITIONS

The first task is to construct precise definitions to describe the situation to be considered. The following definition provides the means to attach a null conformal boundary to a space-time. (In general, of course, such a boundary may be empty.)

2.1. DEFINITION A C^r null asymptote of a space-time (M, g) is a C^r space-time-with-boundary $(\tilde{M}, \tilde{\mathrm{g}})$, $r \geq 3$, such that

(a) $\tilde{M} = M \cup \partial\tilde{M}$;

(b) there exists an open cover $\{\tilde{\mathcal{U}}_\alpha\}$ of $\tilde{M}$, with associated mappings $\tilde{\Omega}_\alpha : \tilde{\mathcal{U}}_\alpha \to [0, \infty)$, such that for each α, setting $\mathcal{U}_\alpha := \tilde{\mathcal{U}}_\alpha \cap M$ and $\Omega_\alpha := \tilde{\Omega}_\alpha | \mathcal{U}_\alpha$, one has $\tilde{\mathrm{g}}|\mathcal{U}_\alpha = \Omega_\alpha^2(\mathrm{g}|\mathcal{U}_\alpha)$ and $\nabla\tilde{\Omega}_\alpha | \tilde{\mathcal{U}}_\alpha \cap \partial\tilde{M} \neq 0$;

(c) $\partial\tilde{M}$ is a null hypersurface of $(\tilde{M}, \tilde{\mathrm{g}})$.

Condition (b) implies that every null geodesic of (M, g) is a null geodesic of $(\tilde{M}, \tilde{\mathrm{g}})$, and that every such null geodesic having an endpoint in $\tilde{M}$ at a point of $\partial\tilde{M}$ has infinite affine length with respect to g. One now defines future null infinity $\mathcal{I}^+$ as the set of future endpoints in $\tilde{M}$ of all future endless causal curves of (M, g). Past null infinity $\mathcal{I}^-$ is defined analogously. It is not difficult to show that $\mathcal{I}^+$ and $\mathcal{I}^-$ are disjoint, that their union is $\partial\tilde{M}$, and that each is the union of components of $\partial\tilde{M}$.

The next definition identifies the class of space-times to be considered.

2.2. DEFINITION Let (M, g) be a space-time satisfying the chronology condition (i.e. having no timelike curves). Suppose (M, g) admits a null asymptote $(\tilde{M}, \tilde{\mathrm{g}})$ such that every null geodesic of (M, g) admits future and past endpoints in $(\tilde{M}, \tilde{\mathrm{g}})$. Then (M, g) is a simple space-time and $(\tilde{M}, \tilde{\mathrm{g}})$ is an asymptotic null completion of (M, g).

This definition is equivalent to Penrose's definition of an asymptotically simple space-time [1] in the case of a null conformal boundary, exept that here, the conformal factor in the subsidiary Definition 2.1 is not required to exist globally. There are two reasons for the new terminology in Definition 2.2. First, Hawking and Ellis [2] have given a definition of an asymptotically simple and empty space-time, now commonly accepted in the literature, which includes an additional and physically mysterious condition that strong causality holds at all points of $\mathcal{I}^+$ and $\mathcal{I}^-$ in $(\tilde{M}, \tilde{\mathrm{g}})$. And second, asymptotic simplicity is an inappropriate description of structure which involves global constraints. One is therefore justified in the use of the new term 'simple space-time'.

The question arises as to whether an asymptotic null completion of a simple space-time (M, g) is in any sense unique. A proof of this would have to specify a manner in which two distinct completions $(\tilde{M}_1, \tilde{\mathrm{g}}_1)$ and $(\tilde{M}_2, \tilde{\mathrm{g}}_2)$ should be identified. Geroch [3], in this context, identifies a point $p_1 \in \tilde{M}_1$ with a point $p_2 \in \tilde{M}_2$ iff every null geodesic of (M, g) having a future (respectively past) endpoint at p_1 in $(\tilde{M}_1, \tilde{\mathrm{g}}_1)$ has a future (past) endpoint at p_2 in $(\tilde{M}_2, \tilde{\mathrm{g}}_2)$. But he overlooks the possibility that there may exist null geodesics of (M, g) having a common future endpoint in $(\tilde{M}_1, \tilde{\mathrm{g}}_1)$ but distinct future endpoints in $(\tilde{M}_2, \tilde{\mathrm{g}}_2)$. The resulting identification space may therefore fail to be Hausdorff, or even a topological manifold. Geroch's attempt at a proof of uniqueness is therefore deficient. If this is to be taken as an indication that asymptotic null completions may be non-unique, one should enquire as to the manner in which they may differ. In particular, is it possible that a simple space-time could admit distinct asymptotic null completions with non-homeomorphic future null infinities?

The following sections describe the principal properties of simple space-times and their asymptotic null completions. The proofs are given in [4].

3. CAUSAL STRUCTURE

The definition of a simple space-time (M, g) demands the absence of closed timelike curves. However the fact that all its null geodesics originate and terminate at the conformal boundary allows one to establish a stronger property.

3.1. PROPOSITION *(M, g) is strongly causal.*

There are various ways in which the definition of strong causality may be expressed. Probably the most convenient is the following: (M, g) satisfies the strong causality condition at a point $p \in M$ if every neighbourhood $\mathcal{V}_p$ of p in M contains a neighbourhood $\mathcal{V}'_p$ of p in M such that the only causal curves of (M, g) from $\mathcal{V}'_p$ to $\mathcal{V}'_p$ are those in $\mathcal{V}_p$. Unlike the definition given by Penrose [5], this has the virtue that it does not depend upon the existence of convex normal neighbourhoods to ensure that it relates purely to the global structure of (M, g). It therefore generalises naturally to space-times-with-boundary such as any asymptotic null completion $(\tilde{M}, \tilde{\mathrm{g}})$ of (M, g).

The setup considered in the Introduction assumed the presence of an $\mathbb{R}^3$ Cauchy surface. However it is of some interest that the definition of a simple space-time necessarily implies the following.

3.2. THEOREM *(M, g) has a non-compact Cauchy surface $\mathcal{C}$.*

This result implies that the simple space-time (M, g) may be considered to have a spatial infinity, defined formally as the inverse limit of sets $\tilde{M} - I(\mathcal{K}, \tilde{M})$ for all compact sets $\mathcal{K}$ of M.

Now let us turn to the causal structure of $(\tilde{M}, \tilde{\mathrm{g}})$ at $\mathcal{I}^+$. The fact that (M, g) is strongly causal implies that $(\tilde{M}, \tilde{\mathrm{g}})$ is strongly causal at all points of M. However there remains the possibility that $(\tilde{M}, \tilde{\mathrm{g}})$ violates strong causality at $\mathcal{I}^+$ or $\mathcal{I}^-$. The following result enables strong causality violation at $\mathcal{I}^+$ to be interpreted in terms of the causal structure of (M, g).

3.3. PROPOSITION *$(\tilde{M}, \tilde{g})$ violates strong causality at a point $p \in \mathcal{I}^+$ iff $M \subset I^-(p, \tilde{M})$.*

This implies that $(\tilde{M}, \tilde{g})$ is strongly causal at $\mathcal{I}^+$ iff no future endless null geodesic of (M, g) cuts the chronological future of every point of M. Analogously, $(\tilde{M}, \tilde{g})$ is strongly causal at $\mathcal{I}^-$ iff no past endless null geodesic of (M, g) cuts the chronological past of every point of M. Although it would be technically convenient to have $(\tilde{M}, \tilde{g})$ strongly causal, it is difficult to find physical motivation for either of these conditions. (Note that, since strong causality violation at $\mathcal{I}^+$ and $\mathcal{I}^-$ has been characterised entirely in terms of (M, g), if one asymptotic null completion of (M, g) violates strong causality at $\mathcal{I}^+$, then every asymptotic null completion of (M, g) must violate strong causality at $\mathcal{I}^+$.)

The strongly causal region of $\mathcal{I}^+$ is an open submanifold $\mathcal{I}_0^+$ of $\mathcal{I}^+$. It has the following basic feature.

3.4. PROPOSITION *$\mathcal{I}_0^+$ is generated by endless null geodesics of $(\tilde{M}, \tilde{g})$.*

One might expect that $\mathcal{I}_0^+$ is necessarily non-empty. Consider any set of the form $\Sigma := \dot{J}^+(\mathcal{K}, \tilde{M}) \cap \mathcal{I}^+$ for some compact set $\mathcal{K} \subset M$. The existence of a non-compact Cauchy surface $\mathcal{C}$ for (M, g) implies, fairly easily, that Σ is non-empty, and Proposition 3.3 gives that $(\tilde{M}, \tilde{g})$ is strongly causal at all points of Σ. Hence $\mathcal{I}_0^+$ contains Σ and so is indeed non-empty. A subset of $\mathcal{I}^+$ of the form Σ is called a good slice of $\mathcal{I}^+$. One may show that a good slice of $\mathcal{I}^+$ is a special case of a slice of $\mathcal{I}^+$, defined as a locally acausal, compact connected embedded topological 2-submanifold of $\mathcal{I}^+$. The fact that all good slices of $\mathcal{I}^+$ lie in $\mathcal{I}_0^+$ generalizes to the following, surprisingly difficult result.

3.5. PROPOSITION *Every slice of $\mathcal{I}^+$ is acausal, is contained in $\mathcal{I}_0^+$, and is cut by every generator of $\mathcal{I}_0^+$.*

Propositions 3.3, 3.4 and 3.5 describe aspects of the causal structure of $(\tilde{M}, \tilde{g})$ at $\mathcal{I}^+$. Analogous results apply to $\mathcal{I}^-$.

It is evident that strong causality violation at $\mathcal{I}^+$ or $\mathcal{I}^-$ is the principal source of causal pathology in $(\tilde{M}, \tilde{g})$. But there is also the possibility that $(\tilde{M}, \tilde{g})$ may fail to be causally simple. In particular, there may exist $p \in \mathcal{I}^-$ such that there is a past endless generating segment of $\mathcal{I}^+$ which is also a generator of $\dot{J}^+(p, \tilde{M})$, so that $J^+(p, \tilde{M})$ is not closed in $\tilde{M}$. Such behaviour cannot be ruled out even if $(\tilde{M}, \tilde{g})$ is strongly causal at both $\mathcal{I}^+$ and $\mathcal{I}^-$.

4. TOPOLOGICAL STRUCTURE

Given a simple space-time (M, g), with Cauchy surface $\mathcal{C}$ and an asymptotic null completion $(\tilde{M}, \tilde{g})$, the main problem is to determine all possible topologies for $\mathcal{C}$, and for the future and past null infinities $\mathcal{I}^+$ and $\mathcal{I}^-$. This problem must, of necessity, be considered in two parts. First, restrictions on the topologies of $\mathcal{C}$, $\mathcal{I}^+$ and $\mathcal{I}^-$ must be deduced from the definition of a simple space-time. And second, all remaining possibilities must be shown to be realised. Only the first of these tasks is considered here, although some effort is made to show that use is made of all available information.

4.1. AN OLD MISTAKE There is a well-known argument of Penrose [1], published in 1965, which purports to show that $\mathcal{I}^+$, and likewise $\mathcal{I}^-$, always has topology $\mathbf{S}^2 \times \mathbb{R}$. If this were correct there would be no need to proceed further. However the following discussion shows that there is a fundamental error.

Penrose begins by choosing an arbitrary point $p \in M$ and then considers the set $\Sigma := \dot{J}^+(p, \tilde{M}) \cap \mathcal{I}^+$ of $\mathcal{I}^+$. It is not difficult to see that $\dot{J}^+(p, \tilde{M})$ must be compact and hence that Σ must be compact. In general $\dot{J}^+(p, \tilde{M})$ will posess caustics which complicate matters. However it suffices for present purposes to restrict attention to the special case in which such caustics are absent. Then $\dot{J}^+(p, \tilde{M})$ is generated by null geodesics of $(\tilde{M}, \tilde{g})$ from p to $\mathcal{I}^+$, and Σ must be homeomorphic to the space of future-directed null directions at p, which is homeomorphic to $\mathbf{S}^2$. Let $\mathcal{I}^+_\Sigma$ be the open submanifold of $\mathcal{I}^+$ generated by all those generators of $\mathcal{I}^+$ which cut Σ. Since Σ is acausal in $(\tilde{M}, \tilde{g})$, $\mathcal{I}^+_\Sigma$ must be homeomorphic to $\mathbf{S}^2 \times \mathbb{R}$. According to Penrose, the compactness of Σ somehow implies that the complementary region $\mathcal{I}^* := \mathcal{I}^+ - \mathcal{I}^+_\Sigma$ lies in a separate component of $\tilde{M}$. In fact it is not difficult to envisage a generating flow on a connected $\mathcal{I}^+$ for which $\mathcal{I}^+_\Sigma$ is a proper subset of $\mathcal{I}^+$. Penrose's argument therefore fails to correctly identify the topology of $\mathcal{I}^+$.

4.2. THE KEY IDEA The following is the basis for all subsequent deductions concerning the topological structure of $\mathcal{I}^+$ and $\mathcal{C}$. It originates from Geroch [6].

Let $\mathcal{N}_+$ be the space of all future-directed null directions over $J^+(\mathcal{C}, \tilde{M})$, excluding those tangent to $\mathcal{I}^+$. Then $N_\mathcal{C} := \mathcal{N}_+ \mid \mathcal{C}$ is an S^2 bundle over $\mathcal{C}$ and $N_+ := \mathcal{N}_+ \mid \mathcal{I}^+$ is an S^2-{pt.} $\approx \mathbb{R}^2$ bundle over $\mathcal{I}^+$. Moreover $\mathcal{N}_+$ is a topological 6-manifold-with-boundary such that $\partial\mathcal{N}_+ = N_\mathcal{C} \cup N_+$, and the null geodesics of $(\tilde{M}, \tilde{\mathrm{g}})$ from $\mathcal{C}$ to $\mathcal{I}^+$ equip the triple $(\mathcal{N}_+; N_\mathcal{C}, N_+)$ with the structure of a product cobordism. One thus has a homeomorphism

$$\mathcal{C} \tilde{\times} \mathrm{S}^2 = N_\mathcal{C} \approx N_+ = \mathcal{I}^+ \tilde{\times} \mathbb{R}^2 \, .$$

Since $M \approx \mathcal{C} \times \mathbb{R}$ is assumed to be connected it follows, in particular, that $\mathcal{I}^+$ is connected.

4.3. A SPECIAL CASE It is instructive to consider first the special case in which M is orientable and $(\tilde{M}, \tilde{\mathrm{g}})$ is strongly causal at $\mathcal{I}^+$ and $\mathcal{I}^-$. The orientability assumption on $M \approx \mathcal{C} \times \mathbb{R}$ implies the orientability of $\mathcal{C}$ which, being a 3-manifold, is therefore parallelizable. Thus $TM \mid \mathcal{C}$ is a trivial bundle and so is $N_\mathcal{C}$. Let Σ be a smooth slice of $\mathcal{I}^+$. By Proposition 3.5 one has $\mathcal{I}^+ \approx \Sigma \times \mathbb{R}$ and, after a little thought, $N_+ \approx T\Sigma \times \mathbb{R}$. The homeomorphism $N_\mathcal{C} \approx N_+$ evidently becomes $\mathcal{C} \times \mathrm{S}^2 \approx T\Sigma \times \mathbb{R}$ which implies $\pi_2(\Sigma) \not\simeq 0$. Since M is orientable, so are $\tilde{M}$, $\mathcal{I}^+ \approx \Sigma \times \mathbb{R}$ and Σ, and one must have $\Sigma \approx \mathrm{S}^2$ and $\mathcal{I}^+ \approx \mathrm{S}^2 \times \mathbb{R}$. The homeomorphism $N_\mathcal{C} \approx N_+$ now reduces to $\mathcal{C} \times \mathrm{S}^2 \approx T\mathrm{S}^2 \times \mathbb{R} \approx \mathbb{R}^3 \times \mathrm{S}^2$. Geroch [6] claimed, without proof, that the homeomorphism $\mathcal{C} \times \mathrm{S}^2 \approx \mathbb{R}^3 \times \mathrm{S}^2$ implies $\mathcal{C} \approx \mathbb{R}^3$. In fact his claim is correct iff the Poincaré conjecture is true. To see this, note first that, since the homotopy groups of S^2 are finitely generated, one has $\pi_*(\mathcal{C}) \simeq 0$, which implies that $\mathcal{C}$ is contractible. Second, one shows that $\mathcal{C}$ is simply connected at infinity (i.e. for any compact set K_1 in $\mathcal{C}$ there exists a compact set $K_2 \supset K_1$ in $\mathcal{C}$ such that $\pi_1(\mathcal{C} - K_2) \to \pi_1(\mathcal{C} - K_1)$ is a trivial homomorphism). And third, one assumes the truth of the Poincaré conjecture to ensure that $\mathcal{C}$ is irreducible (i.e. every tamely embedded 2-sphere in $\mathcal{C}$ bounds a 3-ball in $\mathcal{C}$). The Loop Theorem and elementary surgery may be used to prove [4,7] that a 3-manifold is homeomorphic to $\mathbb{R}^3$ iff it is contractible, irreducible and simply connected at infinity. Thus, if the Poincaré conjecture is true, one must have $\mathcal{C} \approx \mathbb{R}^3$. If the Poincaré conjecture is false then, for any homotopy 3-sphere $\tilde{\mathbf{S}}^3$, one has $(\tilde{\mathbf{S}}^3 - \{\text{pt.}\}) \times \mathbf{S}^2 \approx \mathbb{R}^3 \times \mathbf{S}^2$ [4].

4.4. THE TOPOLOGY OF FUTURE NULL INFINITY Let us turn now to the situation in which the physically questionable condition of strong causality at $\mathcal{I}^+$ and $\mathcal{I}^-$ is not imposed. For simplicity let us temporarily restrict attention to the case $\mathcal{C} \approx \mathbb{R}^3$. The homeomorphism $N_+ \approx N_{\mathcal{C}}$ then becomes $\mathcal{I}^+ \underset{\sim}{\times} \mathbb{R}^2 \approx \mathbb{R}^3 \times \mathbf{S}^2$ which implies that $\mathcal{I}^+$ has the homotopy type of $\mathbf{S}^2$.

Choose an arbitrary point $p \in M$. Then $\Lambda := \dot{J}^+(p, \tilde{M})$ is a compact topological 3-manifold-with-boundary, with $\partial\Lambda = \dot{J}^+(p, \tilde{M}) \cap \mathcal{I}^+$ a slice of $\mathcal{I}^+$. Moreover $\Gamma := \bar{J}^+(p, \tilde{M}) \cap \mathcal{I}^+$ is a topological 3-manifold-with-boundary such that $\partial\Gamma = \partial\Lambda \subset \mathcal{I}^+_0$. By Proposition 3.3 one has $\mathcal{I}^+ - \mathcal{I}^+_0 \subset I^+(p, \tilde{M}) \cap \mathcal{I}^+ = \mathring{\Gamma}$ and hence $\mathcal{I}^+ - \mathring{\Gamma} \subset \mathcal{I}^+_0$. The generators of $\mathcal{I}^+_0$, which all cut $\partial\Gamma$, may therefore be used to define an isotopy of $\mathcal{I}^+$ onto $\mathring{\Gamma}$. This gives a homeomorphism $\mathcal{I}^+ \approx \mathring{\Gamma}$ and one obtains graded isomorphisms $\pi_*(\Gamma) \simeq \pi_*(\mathcal{I}^+) \simeq \pi_*(\mathbf{S}^2)$. Any nowhere-zero causal vector field on $\tilde{M}$ which is timelike on M, and on $\partial\tilde{M}$ is null and tangent thereto, defines a homeomorphism of $\mathring{\Lambda} = \dot{J}^+(p, \tilde{M})$ onto $\mathcal{C} \approx \mathbb{R}^3$. Hence Λ is a 3-disc. The simple connectivity of Γ therefore implies the simple connectivity of the adjunction space $\Gamma \cup_\partial \Lambda$. If $\Gamma \cup_\partial \Lambda$ was compact it would be a homotopy 3-sphere and Γ would have to be contractible. Since this is incompatible with $\pi_*(\Gamma) \simeq \pi_*(\mathbf{S}^2)$, the space $\Gamma \cup_\partial \Lambda$ must be non-compact. Hence $H^3(\Gamma \cup_\partial \Lambda) \simeq H_3(\Gamma \cup_\partial \Lambda) \simeq 0$. The Mayer–Vietoris sequence for the triple $(\Gamma \cup_\partial \Lambda, \Gamma, \Lambda)$ now gives that $H_2(\Gamma \cup_\partial \Lambda)$ is finitely generated torsion module so, by the universal coefficient theorem, one has $H_2(\Gamma \cup_\partial \Lambda) \simeq 0$. Thus $\Gamma \cup_\partial \Lambda$ has the homology of a point. By the Hurewicz isomorphism theorem one obtains $\pi_*(\Gamma \cup_\partial \Lambda) \simeq 0$ and it follows that $\Gamma \cup_\partial \Lambda$ is a contractible open 3-manifold C^3. One concludes $\mathcal{I}^+ \approx \mathring{\Gamma} \approx C^3 - \{\text{pt.}\}$.

4.5. THE MAIN RESULT Let us turn now to the completely general case. Strong causality is not assumed to hold at $\mathcal{I}^+$ or $\mathcal{I}^-$, the Cauchy surface $\mathcal{C}$ is not assumed to be homeomorphic to $\mathbb{R}^3$, the space-time manifold is not assumed to be orientable, and the Poincaré conjecture is not assumed to be true. The discussion in §4.4 gives some idea of the arguments involved.

4.5.1. THEOREM [4]:

(a) $\mathcal{C}$ is homeomorphic to the complement of a point in a homotopy 3-sphere $\tilde{\mathbb{S}}^3$;

(b) $\mathcal{I}^+$ is homeomorphic to the complement of a point in a contractible open 3-manifold C^3 which embeds in $\tilde{\mathbb{S}}^3$;

(c) $\mathcal{I}_0^+$ is homeomorphic to the complement of a point in $\mathbb{R}^3$;

(d) every slice of $\mathcal{I}^+$ is homeomorphic to $\mathbb{S}^2$ and is a strong deformation retract of $\mathcal{I}^+$;

(e) M is homeomorphic to $\mathbb{R}^4$.

Suppose the Poincaré conjecture is true. Then (a) gives that $\mathcal{C}$ is homeomorphic to $\mathbb{R}^3$, and (e) may be strengthened to the claim that M is diffeomorphic to (standard) $\mathbb{R}^4$. Moreover (b) gives that $\mathcal{I}^+$ is homeomorphic to the complement of a point in a Whitehead manifold W^3 which embeds in $\mathbb{S}^3$. (A Whitehead manifold is defined to be a contractible open 3-manifold such that every compact subspace admits a topological embedding into $\mathbb{S}^3$.) A general Whitehead manifold may be expressed as the monotone union of cubes-with-handles such that each is contained and deformable to a point in the interior of its successor. Not all such manifolds may be embedded in $\mathbb{S}^3$.

In the case of an $\mathbb{R}^3$ Cauchy surface $\mathcal{C}$, for example if the Poincaré conjecture is true, one now has a fairly clear idea of the general structure of an asymptotic null completion $(\tilde{M}, \tilde{\mathrm{g}})$ of a simple space-time (M, g). Specifically one may embed $\tilde{M}$ as an open dense submanifold-with-boundary of $\mathbb{R}^3 \times [-1, 1]$ such that

(α) $M = \mathbb{R}^3 \times (-1, 1)$;

(β) every set of the form $\mathbb{R}^3 \times \{t\}$, for $t \in (-1,1)$, is a Cauchy surface of $(M, \mathbf{g})$;

(γ) $\mathcal{I}^+$ (respectively $\mathcal{I}^-$) is the complement in $\mathbb{R}^3 \times \{1\}$ (respectively $\mathbb{R}^3 \times \{-1\}$) of a monotone intersection of cubes-with-handles such that each is contained and deformable to a point in the interior of its predecessor.

Moreover, under the identification $\mathbb{R}^3 = \mathbf{S}^3 - \{pt.\}$, spatial infinity is represented as the set $\{pt.\} \times [-1,1] \subset \mathbf{S}^3 \times [-1,1]$ identified to a point.

4.6. TOPOLOGICAL STRUCTURE OF THE BUNDLES N_+ AND $N_\mathcal{C}$ For the purposes of this section let $\mathcal{I}^+$ be identified with $C^3 - \{pt.\}$ for a contractible open 3-manifold C^3, and $\mathcal{C}$ with $\tilde{\mathbf{S}}^3 - \{pt.\}$ for a homotopy 3-sphere $\tilde{\mathbf{S}}^3$. A smooth slice Σ is a strong deformation retract of $\mathcal{I}^+$ and has the property that the bundle $N_+|\Sigma$ is equivalent to $T\mathbf{S}^2$. One may therefore characterize N_+ as the unique $\mathbb{R}^2$ bundle over $C^3 - \{pt.\}$ admiting a homotopy equivalence $i : \mathbf{S}^2 \to C^3 - \{pt.\}$ such that i^*N_+ is bundle equivalent to $T\mathbf{S}^2$. Clearly $N_\mathcal{C}$ is the unique $\mathbf{S}^2$ bundle over the contractible space $C^3 - \{pt.\}$ and is trivial.

The structure of a simple space-time demands a homeomorphism $N_+ \approx N_\mathcal{C}$ and it is not clear that this can be achieved other than in the case $C^3 = \mathbb{R}^3$, $\tilde{\mathbf{S}}^3 = \mathbf{S}^3$ for which one has $N_+ \approx T\mathbf{S}^2 \times \mathbb{R} \approx \mathbb{R}^3 \times \mathbf{S}^2 \approx N_\mathcal{C}$. In fact one can show [4] by engulfing techniques that the total space of any topological bundle of the form N_+, characterized as above, is homeomorphic to $\mathbb{R}^3 \times \mathbf{S}^2$. And Michael Freedman's 5-dimensional proper h-cobordism theorem may be used [4] to establish a homeomorphism $(\tilde{\mathbf{S}}^3 - \{pt.\}) \times \mathbf{S}^3 \approx \mathbb{R}^3 \times \mathbf{S}^2$ for any homotopy 3-sphere $\tilde{\mathbf{S}}^3$. One therefore has $N_+ \approx N_\mathcal{C}$ for all C^3 and $\tilde{\mathbf{S}}^3$. The necessity of a homeomorphism $N_+ \approx N_\mathcal{C}$ for a simple space-time therefore imposes no additional restrictions on the topologies of $\mathcal{I}^+$ and $\mathcal{C}$ beyond those already expressed by Theorem 4.5.1.

5. ASYMPTOTIC GEOMETRIC STRUCTURE

In the special case that strong causality holds at $\mathcal{I}^+$, any smooth slice Σ of $\mathcal{I}^+$ is cut by every null geodesic generator of $\mathcal{I}^+$. The compactness and simple connectivity of Σ may then be used to show [6] that the Weyl tensor of the unphysical metric $\mathbf{g}$ is zero on Σ, and therefore on the whole of $\mathcal{I}^+$. In the general case where strong causality is not assumed to hold, this argument adapts to show only that the unphysical Weyl tensor is zero on the strongly causal region $\mathcal{I}_0^+$ of $\mathcal{I}^+$. However it is possible that non-zero values may be attained on the strong causality violating region. The peeling property of gravitational radiation will consequently fail.

6. CONCLUDING REMARKS

The preceding section describes the principal causal and topological properties of simple space-times. The most important outstanding task is to construct, or at least to prove the existence of, a simple space-time admitting a future null infinity $\mathcal{I}^+$ with a topology different from $\mathbf{S}^2 \times \mathbb{R}$. The subtleties of some Whitehead manifold different from $\mathbb{R}^3$ (or a counterexample to the Poincaré conjecture!) must be reflected in the topological structure of any asymptotic null completion of such a space-time. Moreover the light cones must somehow be oriented so that the entire space-time manifold lies to the past of every point of the strong causality violating region of $\mathcal{I}^+$. A simpler initial task might be to construct a simple space-time admitting a $\mathcal{I}^+$ at which strong causality is violated, but which is homeomorphic to $\mathbf{S}^2 \times \mathbb{R}$.

ACKNOWLEDGEMENTS

Most of this work was carried out while the author was a Research Fellow in the Institute of Advanced Studies of the Australian National University. The author is grateful to the Australian National University for their hospitality and financial support.

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