INNER PRODUCT ALGEBRAS AND THE FUNCTION THEORY OF ASSOCIATED DIRAC OPERATORS

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INTRODUCTION: The aim of this paper is to introduce a countably infinite number of algebras associated to R^n, each of which contains a generalization of the Cauchy–Riemann equations, and Cauchy integral formula. The first of these algebras is the Clifford algebra, and the associated analysis is called Clifford analysis [2]. We demonstrate that a large number of results from Clifford analysis carry over to these other algebras, including the formulae for Cauchy–Kowalewski extensions described in [9]. We utilise these formulae to describe Cauchy Kowalewski extensions of the kernel for the Fourier transform. Motivated by [4,8] this leads us to construct mutually annihilating idempotents in these algebras, and to associate new differential operators to this kernel. These idempotents enable us to construct from L^1 functions on R^{n-1} solutions of these differential operators in the upper and lower half spaces. We show that from these solutions we can construct solutions to other differential equations including the heat equation.

Inner Product Algebras: From R^n equipped with the inner product $<,>$ we can construct the Clifford algebra $A_n(1)$. By taking the orthonormal basis $e_1,\dots e_n$ of R^n we can construct the basis $1, e_1,\dots e_n,\dots,e_{j_1}\dots e_{j_r},\dots e_1\dots e_n$ of A_n, where $1 \le r \le n$ and $j_1 < \dots < j_r$. Moreover, $e_ie_j + e_je_i = -2\delta_{ij}$. One important property of $A_n(1)$ is that each non–zero vector $x \in R^n$ has a multiplicative inverse $x^{-1} = \frac{-x}{\|x\|^2}$.

The vector x^{-1} is the Kelvin inverse of the vector x. One way, [1], to construct $A_n(1)$ is to take the tensor algebra, $T(R^n)$, of R^n, ie the algebra

$$R \oplus R^n \oplus R^n \otimes R^n \oplus \ldots,$$

and factor this algebra by the two sided ideal, I_1, generated by the set

$$\{x \otimes x \oplus \langle x,x\rangle : x \varepsilon R^n\}.$$

In greater generality we can, for fixed $k \varepsilon \{1,2,\ldots,m,\ldots\}$ take the two sided ideal I_k, generated by the set $\{\overset{2k}{\otimes} x - (-1)^k \langle x,x\rangle^k : x \varepsilon R^n\}$ and construct the algebra $T(R^n)/I_k$. We shall denote this algebra by $A_n(k)$, and we shall call this algebra the k–th inner product algebra of R^n. When $k = 1$ the algebra that we get is just the Clifford algebra $A_n(1)$.

From the construction of these inner product algebras we can see that for k_1 and k_2 positive integers with $k_1 \le k_2$ there is a canonical projection $p_{k_2,k_1} : A_n(k_2) \longrightarrow A_n(k_1)$. Also there is a canonical projection $p_k : T(R^n) \longrightarrow A_n(k)$. We shall identify R^n with $p_k(R^n)$. By allowing the vectors $1,e_1,\ldots,e_1\ldots e_n$ to be an orthonormal basis for $A_n(1)$ we may use the projection $p_{k,1}$ to pull back the norm on $A_n(1)$ to obtain a pseudonorm on $A_n(k)$, for $k > 1$.

Again, each vector $x \varepsilon R^n\backslash\{0\}$ has a multiplicative inverse x^{-1} in $A_n(k)$. The inverse of x is $-x^{2k-1} \langle x,x\rangle^{-k}$. When $k \ne 1$ the element x^{-1} no longer coincides with the Kelvin inverse of x.

<u>Generalized Dirac Operators</u>: For $e_1,\ldots,e_n \varepsilon R^n \subseteq A_n(k)$ we call $D_k = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}$ the Dirac operator associated to the algebra $A_n(k)$.

When $k = 1$ this differential operator coincide with the Dirac operator used in Clifford analysis (see [2]).

Definition: Suppose that U is a domain lying in R^n and $f : U \longrightarrow A_n(k)$ is a C^1 function. Then f is a called a left $A_n(k)$ function if $D_k f(x) = 0$ for each $x \varepsilon U$.

When $k = 1$, this definition coincides with the usual definition of a left regular, or left monogenic function (see [2]).

A C^1 function $f : U \longrightarrow A_n(k)$ is called a right $A_n(k)$ function if $f(x) D_k = 0$ for all $x \varepsilon U$, where $f(x) D_k = \sum_{j=1}^{n} \frac{\partial f(x)}{\partial x_j} e_j$.

Consider the function $H_k : R^n\backslash\{0\} \longrightarrow R$, where $n > 2$ and

(i) $H_k(x) = \|x\|^{-n+2k}$ for n even and $k < n$

(ii) $H_k(x) = \log \|x\|$ for n even and $k = n$

(iii) $H_k(x) = \|x\|^{2k-n} \log \|x\| + A_k \|x\|^{2k-n}$ for n even and $k > n$, where $A_k \varepsilon R\backslash\{0\}$ and is chosen so that $\Delta_n^k H_k(x) = 0$, where Δ_n is the Laplacian is R^n.

(iv) $H_k(x) = \|x\|^{-n+2k}$ for n odd.

Then as $\Delta_n^k H_k(x) = 0$ we have from the construction of the inner product algebras that $D_k^{2k-1} H_k(x)$ is a left $A_n(k)$ function, and a right $A_n(k)$ function.

<u>Theorem</u> (Cauchy's integral formula) : Suppose that $f : U \longrightarrow A_n(k)$ is a left $A_n(k)$ function and $x_0 \in U$. Then for each compact n–dimensional manifold M, with $x_0 \in \overset{o}{M}$ and $M \subseteq U$ we have

$$f(x_0) = \int_{\partial M} B_k D_k^{2k-1} H_k(x-x_0)\, Wx\, f(x)\,,$$

where $Wx = \sum_{j=1}^{n} e_j (-1)^j d\hat{x}_j$, and $B_k, \in R\backslash\{0\}$, is a normalization constant.

<u>Outline Proof</u>: From Stokes' theorem we have that this integral is identical to

$$\int_{S^{n-1}(x_0,r)} B_k D_k^{2k-1} H_k(x-x_0)\, n(x)\, f(x)\, d\, S^{n-1}(x_0,r)\,,$$

where $S^{n-1}(x_0,r) \subseteq M$, is the (n–1) dimensional sphere centred at x_0 and of radius r, n(x) is the outward pointing vector, normal to $S^{n-1}(x_0,r)$ at x, and $dS^{n-1}(x_0,r)$ is the Lebesgue measure on $S^{n-1}(x_0,r)$. As $D_k^{2k-1} H_k(x)$ is homogeneous of degree $-n+1$ we have that

$$\lim_{r \to o} \int_{S^{n-1}(x_0,r)} B_k D_k^{2k-1} H_k(x-x_0)\, n(x)\, f(x)\, d\, S^{n-1}(x_0,r)$$

$$= \lim_{r \to o} \int_{S^{n-1}(x_0,r)} B_k D_k^{2k-1} H_k(x-x_0)\, n(x)\, f(x_0)\, d\, S^{n-1}(x_0,r)$$

$$= \lim_{r \to o} \int_{S^{n-1}(x_0,r)} B_k D_k^{2k-1} H_k(x-x_0)\, n(x)\, f(x_0)\, d\, S^{n-1}(x_0,r)\,.$$

So we only need to compute $\int_{S^{n-1}(0,1)} B_k D_k^{2k-1} H_k(x)\, x\, d\, S^{n-1}(0,1)$.

Now $B_k\, D_k^{2k-1}\, H_k\,(x)$ can be expressed at $\dfrac{P_k(x)}{\|x\|^{n-1}}$, where $P_k(x)$ is an $A_n(k)$ valued polynomial. From the symmetry of the sphere it can be seen that the integral only depends on the terms of $P_k(x)$ of odd order. Again from the symmetry of the sphere we can see that the integral

$$\int_{S^{n-1}(0,1)} P_k(x)\, x\, d\, S^{n-1}(0,1)$$

only depends on the terms of the form $(x_1^{\,2})^{j_1} \ldots (x_n^{\,2})^{j_n} \times A_{j_1 .. j_n}$, where $j_1 ... j_n \,\varepsilon\, \{0,1,...\}$ and

$A_{j_1 ... j_n} \,\varepsilon\, A_n(k)$.

So

$$\int_{S^{n-1}\,(0,1)} P_k(x)\, x\, d\, S^{n-1}(0,1) \;=\; \sum_{j_1 \ldots j_n} A_{j_1 ... j_n}\, \lambda_{j_1} \ldots \lambda_{j_n}\,,$$

where $\lambda_{j_1} \ldots \lambda_{j_n} = \int_{S^{n-1}\,(0,1)} (x_1^{\,2})^{j_1} \ldots (x_n^{\,2})^{j_n}\, d\, S^{n-1}\,(0,1)$.

A straightforward calculation now reveals that the formula $x^{2k} = \langle x,x\rangle^k$, for $x \,\varepsilon\, R^n \subseteq A_n(k)$, gives us

$$\sum_{j_1 .. j_n} A_{j_1 ... j_n}\, \lambda_{j_1} \ldots \lambda_{j_n} = 1\,. \qquad \square$$

When $k = 1$ the previous theorem gives the generalized Cauchy integral formula from Clifford analysis (see for example [2]).

For $k > 1$ the previous result contradicts [6, theorem 1].

Having obtained Cauchy integral formulae for the inner product algebras it can be seen that many results on Clifford analysis extend to these algebras.

As each vector $x \in R^n \backslash \{0\}$ has an inverse in $A_n(k)$ it is also the case that many of the main results in [9] carry through to these algebras. We shall now briefly illustrate this point.

Suppose that S is a C^1, orientable surface lying in R^n, so S is a manifold of dimension $(n-1)$. Suppose that $n : S \longrightarrow S^{n-1}$ is a Gauss map for S. Then for each $x \in S^{n-1}$ we have that

$$D_k = n(x)\, n(x)^{-1} D_k ,$$

and

$$n(x)^{-1} D_k = \frac{\partial}{\partial n(x)} + \Gamma_s(k,x) ,$$

where $\frac{\partial}{\partial n(x)}$ denotes the partial differential operator normal to S at x, while $\Gamma_s(k,x)$ is a differential operator acting over the tangent space TS_x.

When $k = 1$ the operator $\Gamma_s(k,x)$ has previously been described in [9].

The operator $\Gamma_s(k,x)$ can be seen as a generalized Dirac operator of a surface.

We may now take an open covering, $\{U_\alpha : \alpha \varepsilon I$, for some indexing set $I\}$, of S. For each $\alpha \varepsilon I$ there is an interval $(a_\alpha, b_\alpha) \subseteq R$ which contains the origin, and for each $d \varepsilon (a_\alpha, b_\alpha)$ we can construct a surface $U_{\alpha,d} = \{x + d\, n(x) : x \varepsilon U_\alpha\}$. When $d = 0$ we recover the surface U_α.

For each smooth map $\phi : R \longrightarrow U_\alpha$ we may construct the smooth map $\phi_\alpha : R \longrightarrow U_{\alpha,d} : \phi_d(t) = \phi(t) + dn(\phi(t))$. On differentiating these maps it can be seen that the tangent space of $U_{\alpha,d}$ at $x + d\, n(x)$ is a translation of the tangent space of U_α at x. Consequently $n(x)$ is a normal vector at $x + dn(x)$ to $U_{\alpha,d}$. So for each $x + dn(x) \varepsilon U_{\alpha,d}$ we get

$$D_k = n(x) \frac{\partial}{\partial n(x)} + n(x) \Gamma_{U_{\alpha,d}} (k, x + dn(x)) .$$

If $S = S^{n-1}$ then we may cover this surface by itself and the construction gives the subdivision of $R^n \backslash \{0\}$ into concentric spheres all centred at the origin.

By noticing that any homogeneous left $A_n(k)$ polynomial is an eigenvector of the operators $r \frac{\partial}{\partial r}$ and $\Gamma_{S^{n-1}(0,r)}(k,x)$, we obtain

$$D_k = n(x) \frac{\partial}{\partial r} + \frac{n(x)}{r} \Gamma_{S^{n-1}(0,1)} (k,x) .$$

In this case it is easily seen that the operator $\Gamma_{U_{\alpha,d}} (k, x + dn(x))$ depends on the variable d.

As the operator D_k has a Cauchy integral formula associated to it we can, on taking the complexification, $A_n(k)(\mathbb{C})$, of $A_n(k)$, derive analogues of the Huygens' principle integrals described in [3, 10] for n even, and greater than two. Using these integrals, and their odd dimensional analogues, it is straightforward to adapt arguments given in [7] to determine:

Theorem: (Cauchy Kowalewski theorem) Suppose that S is a real analytic, oreintable surface lying in R^n and $f : S \longrightarrow A_n(k)(\mathbb{C})$ is a real analytic function, then there is a neighbourhood U_f of S and a left $A_n(k)$ function $F : U_f \longrightarrow A_n(k)(\mathbb{C})$ such that $F|_S = f$.

□

Following [9] we could try to express F, in some neighbourhood of U_f, as a series of the form $\sum_{m=0}^{\infty} d^m \lambda_{m,k}(f)(d,x)$.

It is easily seen that $\lambda_{0,k}(f)(d,x) = f(x)$. Moreover, $\lambda_{1,k}(f)(d,x) = \Gamma_{U_{\alpha,d}}(k, x + dn(x))\, f^*(x + dn(x))$, where $f^*(x + dn(x)) = f(x)$. Continuing in this way we obtain

$$f(x + dn(x)) = \sum_{m=0}^{\infty} \frac{(-1)^m d^m}{m!} \left(\frac{\partial}{\partial d} + \Gamma_{U_{\alpha,d}}(k, x + dn(x))\right)^m f^*(n + dn(x)).$$

When $k = 1$ this formula corresponds to an expression given in [9].

One important case arises when $S = R^{n-1}$, the space spanned by $e_2, \ldots, e_n$, and the analytic function is the kernel of the Fourier transform, $e^{-i<\vec{x},\vec{t}>}$, where $\vec{x}, \vec{t} \in R^{n-1}$.

The Cauchy Kowalewski extension of this kernel to a left $A_n(k)$ function is

$$\exp(i\, x_1 e_1^{-1} \vec{t})\, e^{-i<\vec{x},\vec{t}>},$$

where multiplication is taken within the algebra $A_n(k)$. This extension is well defined on all of R^n. So is $\exp(i\, x_1 \vec{t})\, e^{-i < \vec{x},\vec{t} >}$. This function is annihilated by the operator $\tilde{D}_k = \frac{\partial}{\partial x_1} + \sum_{j=2}^{n} e_j \frac{\partial}{\partial x_j}$.

Usually one is interested in the Fourier transform $\int_{R^{n-1}} e^{-i < \vec{x},\vec{t} >} h(\vec{x})\, d\vec{t}^{\,n-1}$ where $h(\vec{t})$ belongs to some suitable function space. Suppose $h(\vec{t}) \in L^1(R^{n-1}, A_n(k))$, the $A_n(k)$ module of $A_n(k)$ valued L^1 integrable functions over R^{n-1}. Then when $k = 1$ we can, following [8], introduce the mutually annihilating idempotents $1/2\,(1 + i\frac{\vec{t}}{\|\vec{t}\|})$ and $1/2\,(1 - i\frac{\vec{t}}{\|\vec{t}\|})$.

On noting that $\frac{i\,\vec{t}}{2}(1 \pm i\frac{\vec{t}}{\|\vec{t}\|}) = \pm\frac{\|\vec{t}\|}{2}(1 \pm i\frac{\vec{t}}{\|\vec{t}\|})$ we have, [8],

$$\exp(i\, x_1 \vec{t})\, e^{-i < \vec{x},\vec{t} >} = e^{x_1\|\vec{t}\| - i < \vec{x},\vec{t} >}.\ 1/2\,(1 + i\frac{\vec{t}}{\|\vec{t}\|})$$

$$+\, e^{-x_1\|\vec{t}\| - i < \vec{x},\vec{t} >}.\ 1/2\,(1 - i\frac{t}{\|\vec{t}\|}).$$

We now have from [4,8] that (i) $\int_{R^{n-1}} e^{x_1\|\vec{t}\| - i < \vec{x},\vec{t} >}\, 1/2\,(1 + i\frac{\vec{t}}{\|\vec{t}\|})\, h(\vec{t})\, d\vec{t}^{\,n-1}$ is well defined for $x_1 < 0$ and (ii) for $x_1 > 0$ the integral

$\int_{R^{n-1}} e^{-x_1\|\vec{t}\| - i < \vec{x},\vec{t} >}\, 1/2\,(1 - i\frac{\vec{t}}{\|\vec{t}\|})\, h(\vec{t})\, d\vec{t}^{\,n-1}$ is well defined.

In both cases these functions are annihilated by the operator $\tilde{D}_1$.

We would like to obtain analogues of these observations for the cases where $k > 1$. However, for $k > 1$ the elements $1/2\,(1 \pm i\,\frac{\vec{t}}{\|\vec{t}\|})$ are no longer idempotents, nor are they mutually annihilating. But the elements $1/2\,(1 \pm i\,\frac{\vec{t}}{\|\vec{t}\|}^{k})$ are mutually annihilating idempotents of the algebra $A_n(k)$, for k odd. Moreover, when k is even the elements $1/2\,(1 + \frac{\vec{t}^{k}}{\|\vec{t}\|^{k}})$ are mutually annihilating idempotents of the algebra $A_n(k)$.

Within $A_n(k)$ we also have

$$\frac{i\vec{t}^{k}}{2}\left(1 \pm \frac{i\vec{t}^{k}}{\|\vec{t}\|^{k}}\right) = \pm\frac{\|\vec{t}\|^{k}}{2}\left(1 \pm \frac{i\vec{t}^{k}}{\|\vec{t}\|^{k}}\right) \qquad k \text{ odd}$$

and

$$\frac{\vec{t}^{k}}{2}\left(1 \pm \frac{\vec{t}^{k}}{\|\vec{t}\|^{k}}\right) = \pm\frac{\|\vec{t}\|^{k}}{2}\left(1 \pm \frac{\vec{t}^{k}}{\|\vec{t}\|^{k}}\right) \qquad k \text{ even.}$$

Consequently, within $A_n(k)$

$$\exp(i(-1)^{\ell}x_1\vec{t}^{2\ell+1})\,e^{-i<\vec{x},\vec{t}>} = e^{(-1)^{\ell}x_1\|\vec{t}\|^{2\ell+1} - i<\vec{x},\vec{t}>}\,1/2\left(1 + \frac{i\vec{t}^{k}}{\|\vec{t}\|^{k}}\right)$$
$$+\,e^{-(-1)^{\ell}x_1\|\vec{t}\|^{2\ell+1} - i<\vec{x},\vec{t}>}\,1/2\left(1 - \frac{i\vec{t}^{k}}{\|\vec{t}\|^{k}}\right),$$

where $k = 2\ell + 1$.

This function is well defined on R^n and it is annihilated by the operator $\frac{\partial}{\partial x_1} + \left(\sum_{j=2}^{n} e_j \frac{\partial}{\partial x_j}\right)^k$.

Similarly, within $A_n(k)$

$$\exp(-(-1)^\ell x_1 \vec{t}^{2\ell})\, e^{-i<\vec{x},\vec{t}>} = e^{(-1)^\ell x_1 \|\vec{t}\|^{2\ell} - i<\vec{x},\vec{t}>}\, 1/2\left(1 + \frac{\vec{t}^k}{\|\vec{t}\|^k}\right)$$

$$+\, \bar{e}^{(-1)^\ell x_1 \|\vec{t}\|^{2\ell} - i<\vec{x},\vec{t}>}\, 1/2\left(1 - \frac{\vec{t}^k}{\|\vec{t}\|^k}\right),$$

where $k = 2\ell$.

This function is well defined on R^n and it is annihilated by the operator $\frac{\partial}{\partial x_1} + \left(\sum_{j=2}^{n} e_j \frac{\partial}{\partial x_j}\right)^k$.

Simple inequalities now give us:

<u>Theorem</u>: Suppose that $h \in L^1(R^{n-1}, A_n(k))$. Then

(A) when $k = 2\ell + 1$ the integral

$$\int_{R^{n-1}} e^{(-1)^\ell x_1 \|\vec{t}\|^{2\ell+1} - i<\vec{x},\vec{t}>}\, 1/2\left(1 + i\frac{\vec{t}^k}{\|\vec{t}\|^k}\right) h(\vec{t})\, d\vec{t}^{n-1}$$

is well defined for $x_1 > 0$ and ℓ odd, and for $x_1 < 0$ and ℓ even.

(B) when $k = 2\ell + 1$ the integral

$$\int_{R^{n-1}} \bar{e}^{(-1)^{\ell} x_1 \|\vec{t}\|^{2\ell+1} - i < \vec{x}, \vec{t} >} 1/2 \left(1 - i \frac{\vec{t}^{k}}{\|\vec{t}\|^{k}}\right) h(\vec{t})\, d\vec{t}^{n-1}$$

is well defined for $x_1 > 0$ and ℓ even, and for $x_1 < 0$ and ℓ odd.

(C) when $k = 2\ell$ the integral

$$\int_{R^{n-1}} \bar{e}^{(-1)^{\ell} x_1 \|\vec{t}\|^{2\ell+1} - i < \vec{x}, \vec{t} >} 1/2 \left(1 + \frac{\vec{t}^{k}}{\|\vec{t}\|^{k}}\right) h(\vec{t})\, d\vec{t}^{n-1} \tag{1}$$

is well defined for $x_1 < 0$ and ℓ odd, and for $x_1 > 0$ and ℓ even.

(D) when $k = 2\ell$ the integral

$$\int_{R^{n-1}} e^{(-1)^{\ell} x_1 \|\vec{t}\|^{2\ell+1} - i < \vec{x}, \vec{t} >} 1/2 \left(1 - \frac{\vec{t}^{k}}{\|\vec{t}\|^{k}}\right) h(\vec{t})\, d\vec{t}^{n-1}$$

is well defined for $x_1 < 0$ and ℓ even, and for $x_1 > 0$ and ℓ odd.

All of these functions are annihilated by the operator $\frac{\partial}{\partial x_1} + \left(\sum_{j=2}^{n} e_j \frac{\partial}{\partial x_j}\right)^k$.

□

By considering the images of these functions under the projection we obtain solutions to other differential equations within $A_n(1)$. In particular when $k = 2$ we obtain solutions to the heat operator $\frac{\partial}{\partial x_1} + (\sum_{j=2}^{n} e_j \frac{\partial}{\partial x_j})^2$.

Note that when k is even only the function (1) can be projected to a function which is not identically zero.

The construction of the functions $\exp(-1(-1)^{\ell} x_1 \vec{t}^{2\ell+1}) e^{-i<\vec{x},\vec{t}>}$ and $\exp(-(-1)^{\ell} x_1 \vec{t}^{2\ell}) \bar{e}^{i<\vec{x},\vec{t}>}$ are special cases of the following constructions:

Suppose that L is a linear operator acting on a space of functions defined over a domain U in R^{n-1}. If g belongs to this space then, provided convergence is well defined on some neighourhood U_g, of U, within R^n, then the series

$$\sum_{m=0}^{\infty} \frac{(-1)^m x_1^m L^m}{m!} g(\vec{x}) \qquad (2)$$

is annihilated by the operator $\frac{\partial}{\partial x_1} + L$, for each $x_1 + \vec{x} \varepsilon U_g$.

As a special case $g : R^{n-1} \longrightarrow R^{n-1}$ is a bounded function and $T : R^{n-1} \longrightarrow R^{n-1}$ is a linear map. Then the series

$$\sum_{m=0}^{\infty} \frac{(-1)^m x_1^m}{m!} T^m g(\vec{x})$$

is well defined for each $x \varepsilon R^n$ and this function is annihilated by the operator $\frac{\partial}{\partial x_1} + T$.

Special cases of the series (2) appear in [4,5].

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