TOEPLITZ OPERATORS ON FOCK SPACES

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1. INTRODUCTION

This work gives a brief exposition of recent progress made in the theory of Toeplitz operators in Bargmann-Segal (Fock) space. Such operators have been studied in several papers of Berezin in the early seventies [3],[4] and also in [8]. Substantial advances in understanding their properties have been made in recent years due to the works of Berger and Coburn [5], [6]. Other related results are contained in [9], [10]. Despite the fact that there is a natural equivalence between Toeplitz operators in Fock space and pseudo-different operators in $L^2(\mathbb{R}^n)$, their study requires some specific methods. One of such methods, introduced in [3], is based on the idea of Berezin symbol of operators acting in the Fock space. This method has been successfully employed in [6] and [9]. Topics such as the theory of Toeplitz forms over Fock spaces developed in [12], and attempts to generalize the theory for Fock space over general Hilbert space [2], [11] are related, but we shall not discuss them here. This brief report almost certainly misses other works on Toeplitz operators of which we are not aware (done mainly by physicists).

The paper is divided into three parts. The first part introduces the Segal-Bargmann-Fock space $H^2(\mu)$ and its relation to $L^2(\mathbb{R}^n)$, Toeplitz operators in $H^2(\mu)$, and the Berezin symbol of operators acting in $H^2(\mu)$. The second part deals with bounded Toeplitz operators. The third part is devoted to unbounded Toeplitz operators in $H^2(\mu)$.

The material of this work is based mainly on the following papers [5], [6], [7], [8], [9], [10].

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Throughout the paper we use the following notation. For $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$ - the space of n complex variables we write: $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)$, where $\bar{z}_j$ is the usual conjugate in $\mathbb{C}$, $|z|^2 = |z_1|^2 + \ldots + |z_n|^2$, and $zw = z_1 w_1 + \ldots + z_n w_n$. For $k = (k_1, \ldots, k_n)$, an n-tuple of non-negative integers, we write $k! = k_1! \ldots k_n!$, $|k| = k_1 + \ldots + k_n$, $z^k = z_1^{k_1} \ldots z_n^{k_n}$. If $\alpha \in \mathbb{N}^n$ and if f is a function on $\mathbb{C}^n$ we denote $D^\alpha f = \dfrac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}}$.

2. THE PRELIMINARIES

Let μ be the Gaussian measure on $\mathbb{C}^n$ given by

$$d\mu = (2\pi)^{-n} e^{-|z|^2/2} \, dV$$

where dV is the Euclidean volume on $\mathbb{C}^n = \mathbb{R}^{2n}$.
Denote by $H^2(\mu)$ the closed subspace of $L^2(\mu)$ (the μ-square integrable functions in $\mathbb{C}^n$) of all entire functions. This space has been introduced by I.E. Segal and employed by V. Bargmann in [1]. $H^2(\mu)$ has the reproducing kernel

$$e_\alpha(z) = e^{z\bar{\alpha}/2},$$

so that for $f \in H^2(\mu)$, $f(a) = (f, e_a)$, where $(.,.)$ is the usual scalar product in $L^2(\mu)$.

An orthonormal basis in $H^2(\mu)$ consists of the functions

(1) $\quad f_k(z) = \{2^{|k|} k!\}^{-1/2} z^k, \; k \in \mathbb{N}^n.$

It follows that the set $\mathbb{P}$ of polynomials in the z_j is a dense subspace in $H^2(\mu)$. Similarly as for $L^2(\mu)$, the multiplicativity of $d\mu(z)$ implies a natural isometry

$$H^2(\mu) \cong H^2(\mathbb{C}, \mu_1) \otimes \ldots \otimes H^2(\mathbb{C}, \mu_1),$$

where μ_1 stands for the Gaussian measure on $\mathbb{C}$.

Following Bargmann note that the map

$$A_1 : h_k \longrightarrow f_k$$

where $h_k(x) = [\Pi 4^k (k!)^2]^{-1/4} (-1)^k e^{x^2/2} \left[\frac{d}{dx}\right] e^{-x^2}$ is the orthonormal basis of Hermite functions in $L^2(\mathbb{R})$, establishes a natural isometry of $L^2(\mathbb{R})$ and $H^2(\mu_1)$. Moreover A_1 can be written in integral form as:

$$(2) \qquad A_1 f(z) = \Pi^{-1/4} \int f(\alpha) \exp\left[-\frac{1}{4}\left(z^2+\alpha^2\right)+\sqrt{2}\, z\, \alpha\right] d\alpha$$

Hence the mapping $A = A_1 \otimes \dots \otimes A_1$ induces isometry from $L^2(\mathbb{R}^n)$ onto $H^2(\mu)$ and it is given as an explicit integral kernel operator. Moreover, the mapping A carries the creation operator $\left(x_j - \frac{\partial}{\partial x_j}\right)$ in $L^2(\mathbb{R}^n)$ into a simple operator in $H^2(\mu)$, namely

$$(3) \qquad A^{-1} \left(x_j - \frac{\partial}{\partial x_j}\right) A = T_{z_j} .$$

where T_{z_j} is the operator of multiplication by the co-ordinate function z_j in $H^2(\mu)$.

The third part of this paper contains an extension (due to Guillemin) of (3) for general pseudo-differential operators in $L^2(\mathbb{R}^n)$.

The appearance of Toeplitz operators T_{z_j} is therefore quite natural in this context.

Now we define the Toeplitz operator T_ψ with a general symbol ψ. Let $P : L^2(\mu) \longrightarrow H^2(\mu)$ be the orthogonal projection onto $H^2(\mu)$. For a measurable function ψ on $\mathbb{C}^n$ such that the set of all $f \in H^2(\mu)$ for which $\psi f \in L^2(\mu)$ is dense in $H^2(\mu)$, we define the Toeplitz operator by

$$(4) \qquad T_\psi f = P(\psi \cdot f).$$

This operator is in general unbounded in $H^2(\mu)$. For essentially bounded ψ, it is obviously bounded and $\|T_\psi\| \le \|\psi\|_\infty$. In the third part of this work we shall consider other possible definitions of unbounded Toeplitz operators.

Now recall the definition of the Berezin Symbol of an operator A in $H^2(\mu)$ [3]. Let $k_\alpha(z) = \exp\left[\frac{z\alpha}{2} - \frac{|\alpha|^2}{4}\right]$. Suppose that $k_\alpha \in D(A)$ - the domain of A, for every $\alpha \in \mathbb{C}^n$.

The function

(5) $\tilde{A}(\alpha) = (Ak_\alpha, k_\alpha)$

is called the Berezin symbol of A.

Note that for the Toeplitz operator T_ψ one can compute $\tilde{T}_\psi$ explicitly

$$\tilde{T}_\psi(\alpha) = (2\Pi)^{-n} \int \psi(z) e^{-\frac{|z-\alpha|^2}{2}} dV(z).$$

Hence the Berezin symbol of T_ψ is also the solution of the heat equation on $\mathbb{R}^{2n}$ at the time $t = \frac{1}{2}$ with initial values ψ. General properties of the Berezin symbol of arbitrary operators can be found in [3].

3. BOUNDED TOEPLITZ OPERATORS

In this part of the paper some results are presented concerning bounded Toeplitz operators in $H^2(\mu)$. They are chosen here to illustrate the usefulness of the Berezin symbol in the study of Toeplitz operators in $H^2(\mu)$.

For brevity we denote $\tilde{T}_\psi$ by $\tilde{\psi}$ and this should not cause confusion later. For a function ψ on $\mathbb{C}^n$ and $q \in \mathbb{R}$, we denote by $\psi(q\cdot)$ the function $z \to \psi(qz)$. We start with a result which gives a sufficient condition for a function ψ to define bounded T_ψ in $H^2(\mu)$ [9].

PROPOSITION 3.1

If $\psi(\sqrt{2}\cdot)$ is bounded then T_ψ is bounded and

$$\|T_\psi\| \le 8\Pi^n \|\tilde{\psi}(\sqrt{2}\cdot)\|_\infty .$$

Conversely, if T_ψ is bounded then $\tilde{\psi}(\cdot)$ is bounded on $\mathbb{C}^n$.

The proof of this Proposition is based on the Schur test for the integral operator $PM_\psi P$, where M_ψ denotes the operator of multiplication by ψ.

REMARK

For positive ψ the above result can be stated shortly: T_ψ is bounded iff $\tilde{\psi}(\cdot)$ is bounded. We do not formulate results concerning compactness of T_ψ or when $T_\psi \in C_p$ (the Schatten -v.Neumann class). They are also expressed in terms of the behaviour if $\tilde{\psi}(q.)$ at infinity, see [9].

Instead we turn to the question for which symbols f, g, $T_f T_g - T_{fg}$ is compact . For the classical Toeplitz operators in the unit disc the above difference is compact, provided that f or g is continuous. In our context the answer is much more difficult and has been found by Berger and Coburn in [6]. It is also given in terms of the behaviour of $\tilde{\psi}$ at infinity. First, however, recall their definition of ESV space

$$ESV = \{f \in L^{\infty}(\mathbb{C}^n),\ \lim_{R\to\infty} \sup_{\substack{|z-w|<1\\|z|>R}} |f(z)-f(w)| = 0\}$$

For example $\exp(i\sqrt{|z|}) \in ESV$.

Now for f and g such that $|\tilde{f}|^2$ and $|\tilde{g}|^2$ are bounded and continuous we have [6].

THEOREM 3.2

For f and g as above $T_f T_g - T_{fg}$ is compact if $\tilde{f} \in ESV$ and $\widetilde{|f - \tilde{f}|^2}(z) \xrightarrow[|z|\to\infty]{} 0$ or $\tilde{g} \in ESV$ and $\widetilde{|g - \tilde{g}|^2}(z) \xrightarrow[|z|\to\infty]{} 0$.

This result shows that one can obtain a symbol calculus of Toeplitz operators modulo the ideal of compact operators, provided their symbols behave properly at infinity. The main ideas of the proof of the Theorem is to note that

(a) $\lim_{|z|\to\infty} (f - \tilde{f}^{(m)})(z) = 0$, for $m \in N$, where $\tilde{f}^{(m)} := \widetilde{\tilde{f}^{(m-1)}}$, $\tilde{f}^{(o)} = f$,

(b) $\tilde{f}^{(m)}$ is Lipchitz continuous with modulus of continuity converging to 0 as $m \longrightarrow \infty$,

(c) If $K(\cdot)$ is a uniformly bounded weakly measureable, compact operator valued function and ρ is a finite positive measure then $\int K(z)\,d\rho(z)$ is also compact.

The proof also relies on the relationship between the Berezin symbol of operators and an averaging operation over a representation of the

Heisenberg group. Bargmann defined in [2] the following mapping of $\mathbb{C}$ into unitary operators in $H^2(\mu)$

$$\mathbb{C}^n \ni a \longrightarrow W_a = e^{iT(\operatorname{Re} z\bar{a})} = e^{-|a|^2/4}\, T\{e^{i \operatorname{Re} za}\}$$

Applying the identities

$$W_a W_b = \exp[i \operatorname{Re} a\bar{b}/2] W_{a+b}$$

one can check that the mappings $a \longrightarrow W_a$ and $a \longrightarrow W_a^*$ are strongly continuous.

For a bounded operator A on $H^2(\mu)$ one defines an averaging operation by

$$\hat{A} = \int W_a^* A W_a \, d\mu(a) \ .$$

The relation between $\hat{A}$ and $\tilde{A}$ is given by [6].

PROPOSITION 3.3

For any bounded operator A on $H^2(\mu)$ we have

$$\hat{A} = T_{\tilde{A}}$$

PROOF

By the definition of W_a we have

$$W_a k_z = \exp[i \operatorname{Re} z\bar{a}/2] k_{z+a}$$

Hence

$$\tilde{\hat{A}} = \tilde{T}_{\tilde{A}} \quad \text{and so} \quad \hat{A} = T_{\tilde{A}} \ .$$

The last implication depends on the fact that the Berezin symbol determines the operator uniquely i.e. $\tilde{B}(\cdot) \equiv 0$ implies that $B = 0$.

We conclude this section with the following recent result of Gautrin [7].

THEOREM 3.4

The set of bounded Toeplitz operators is dense in the operator norm topology in the space of all bounded operators on $H^2(\mu)$.

The proof of this result depends on the technique of tensor product of nuclear spaces (the space of entire functions of exponential type).

Other results on bounded Toeplitz operators in $H^2(\mu)$ (for example concerning their spectral properties) are also contained in [6], [9].

4. UNBOUNDED TOEPLITZ OPERATORS

In this section we slightly change the measure μ (for technical reasons),

$$d\mu = \pi^{-n} \exp(-|z|^2)\ dV .$$

It follows that the reproducing kernel

$e_\alpha(z) = e^{z\alpha}$ and $f_k(z) = z^k/\sqrt{k!}$.

Not much is known about unbounded Toeplitz operators. We present here only a few results taken from [10].

It turns out that even the definition of unbounded Toeplitz operators in $H^2(\mu)$ is not unique. Namely we may associate, for a given measurable the following three operators in $H^2(\mu)$

(a) $T_\psi f = P(\psi \cdot f)$, $D(T_\psi) = \{f \in H^2(\mu) : \psi f \in L(\overset{2}{\mu})\}$

(b) $\Pi_\psi f(z) = \int \psi(a) f(a)\ e^{z\bar{\alpha}} d\mu(\alpha)$,

provided the integral exists and belongs to $H^2(\mu)$, as a function of z.

(c) Let $D(S_\psi) = \{f \in H^2(\mu) : \psi f = h+r\ ,\ h \in H^2(\mu)\ \int r\bar{p} d\mu = 0,\ \forall p \in \mathcal{P}\}$

Put $S_\psi f = h$.

Note that S_ψ is well defined. Indeed, suppose that $h_1 + r_1 = \psi h = h_2 + r_2$ where $h_k \in H^2(\mu)$ and $\int r_k \bar{p} d\mu = 0$. Hence $r_1 - r_2 = h_2 - h_1 \in H^2(\mu)$ and is orthogonal to $\mathcal{P}$. Since $\mathcal{P}$ is dense in $H^2(\mu)$ it follows $r_1 - r_2 = 0 = h_2 - h_1$.

The next result given below more or less explains the natural appearance of the definitions of S_ψ and Π_ψ.

Let E denote the linear span of $\{e_z ; z \in \mathbb{C}^n\}$.

PROPOSITION 4.1

If $P \subset D(T_\psi)$ (respect. $E \subset D(T_\psi)$) and $T = T_\psi | \mathbb{P}$ (respect. $T_1 = T_\psi | E$)

then $T^* = S_{\bar{\psi}}$ (respect. $T_1{}^* = \Pi_{\bar{\psi}}$), where $\bar{\psi}$ denotes the bar of ψ.

The proof follows by direct but careful checking of the definitions [10].

The relationship between the above operators explains the next

PROPOSITION 4.2

For any measurable ψ we have

(i) $T_\psi \subseteq \Pi_\psi \subseteq S_\psi$.

(ii) If ψ is entire and $P \subset D(T_\psi)$ then

$$T_\psi = S_\psi .$$

REMARK

However for $\psi(z) = \operatorname{Re} z^3$ $(m = 1)$ we have the strict inclusion $T_\psi \subseteq S_\psi$. In the case of the unit disc Toeplitz operators with a bounded holomorphic symbol Ψ are bounded and $T_\psi^* = T_{\bar{\psi}}$. Since there are no nontrivial bounded entire functions the problem of computing T_ψ^* is much more complicated. Nevertheless we have [10].

THEOREM 4.3

Let ψ be an entire function for which Π_ψ is densely defined. Suppose that for any $h \in D(\Pi_\psi)$ there exists $\xi > 0$ such that

$$(+) \qquad \sum_s \frac{\|D^s \psi \cdot h\|^2}{s!} (1 + \xi)^{|s|} < +\infty$$

then $\Pi_{\psi}^{*} = \Pi_{\bar{\psi}}$.

EXAMPLE.

$$\text{Let } \psi(z) = \sum_{k=1}^{N} p_k(z) e^{z\lambda_k},$$

where $p_k \in \mathbb{P}$ and $\lambda_k \in \mathbb{C}^n$. One can check that ψ satisfies (+) with any $\xi > 0$.

Naturally the next question is when is T_ψ with a real symbol ψ selfadjoint (or essentially selfadjoint)? This is not true even for simple functions like $\psi_0 = \text{Re}z^3$, in which case T_ψ is not selfadjoint. Note however, that T_ψ with real valued ψ commutes with the natural conjugation $C : L^2(\mu) \to L^2(\mu)$ given by $Cf(z) = \overline{f(\bar{z})}$. Hence such T_ψ must have equal deficiency indices. We don't consider here other examples of positive results about selfadjointness. The interested reader is referred to [10].

There are also many questions concerning the spectral properties of unbounded Toeplitz operators. For example for which symbol ψ is the resolvent $R(\lambda, T_\psi)$ compact.

Here is a class of symbols with this property.

PROPOSITION 4.4

If T_ψ is closed, densely defined and there exists $c \geq 0$ such that

(E) $\text{Re}(T_\psi f, f) \geq c \|\text{grad } f\|^2$, $f \in D(T_\psi)$,

$$\text{where } \|\text{grad } f\|^2 = \sum_{i=1}^{n} \int \left|\frac{\partial f}{\partial z_i}\right|^2 d\mu$$

then $R(\lambda, T_\psi)$ is compact.

REMARK

The last result holds for arbitrary closed operator A satisfying the assumption (E) i.e. put A in the place of T_ψ.

We conclude this brief report by recalling the equivalence of PSDO (pseudodifferential operators) and Toeplitz operators, see [8].

Let

$$[A_w(X,D)f](x) = (2\Pi)^{-n} \int\int A\left(p, \frac{x+y}{2}\right) e^{ip(x-y)} f(y)\,dy\,dp$$

be the PSDO in $L^2(\mathbb{R}^n)$ with the symbol $(A(p,q) : \mathbb{R}^{2n} \to \mathbb{C}$. This is the Weyl method of quantization.

Suppose that the operator $\mathbb{A}^{-1} A_w(X,D) \mathbb{A}$ is a Toeplitz operator with a symbol ψ. Then, by a result of Guillemin [8, p.187] we have the following relation between $\psi(\cdot)$ and $A(.,.)$

$$A(x,y) = e^{-\Delta/2} \psi(x+iy) ,$$

$$\text{where } \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial z_i \partial \bar{z}_i} , \quad z = x + iy$$

Therefore in order to represent a given PSDO with its Weyl symbol $A(.,.)$ as a Toeplitz operator we have to solve the equation

$$\psi = e^{\Delta/2} A ,$$

which is, in general, not possible.

This explains why it is not efficient, in general, to transfer the results found in the theory of PSDO to the theory of Toeplitz operators in $H^2(\mu)$.

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