SOLVABILITY OF DIFFERENTIAL OPERATORS ON SEMIRADIAL SEMIDIRECT PRODUCTS

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Let $G = N \rtimes K$ be the semidirect product of a simply connected nilpotent Lie group by a compact Lie group. Let $\underline{\underline{n}}_i$ be the derived series of $\underline{\underline{n}}$, defined by $n_0 = n$, $\underline{\underline{n}}_i = [\underline{\underline{n}}_{i-1}]$ for $i \geq 1$. We shall say that the semidirect product is **semiradial** if we can write $\underline{\underline{n}}_i = \underline{\underline{n}}^i + \underline{\underline{n}}_{i-1}$ as a K-space, in such a way that the K-invariants in $\underline{\underline{n}}^i$ form a commutative algebra.

Let P be a bi-K-invariant, left N invariant differential operator on G, and consider the **partial Fourier coefficients** of P, $(P_\wedge)_{\wedge \in \widehat{K}}$. These are K-invariant, $\mathcal{B}(\mathcal{H}_\wedge)$-valued differential operators on N defined for $\phi \in C^\infty(N)$ by

$$(P_\wedge \phi)(n) = P(\phi \otimes \wedge).$$

In [2], the following result was proved.

Theorem 1. *Let G and P be as above and suppose that Ω is a relatively compact set in G.*

Suppose further that for each integer a there is a constant C so that for all $\wedge \in \widehat{K}$

$$\| P_\wedge \|' \geq CN(\wedge)^{-a}. \qquad (**)$$

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Then P has a fundamental solution on Ω.

In the above inequality, $N(\wedge)$ denotes the constant $(\wedge + \delta,\ \wedge + \delta) - (\wedge, \wedge)$, where $\wedge$ is the highest weight associated to $\wedge$, $(\quad)$ is the Killing form, and $\|\cdot\|'$ denotes a certain norm on the K-invariant $\mathcal{B}(\mathcal{H}_\wedge)$-valued operators which will be defined below.

(Actually, the above theorem was proved for a semiradial semidirect product of a solvable group by a compact group.)

It is of some interest to ask whether one can remove all mention of the set Ω from the statement of the above theorem: that is, can one prove that P is globally solvable rather than semiglobally solvable. The purpose of this note is to answer this question in the affirmative. In fact, one has

Theorem 2. *Let G and P be as in Theorem 1. If (**) holds then P has a global fundamental solution on G.*

This theorem will be proved below. The basic techniques will be to show that G is P-convex. Actually, the case where N is abelian was proved in [2]; P-convexity in this case was established in [3]. In [2], we also showed that if N has one-dimensional centre, theorem 2 holds. The case of biinvariant operators on a direct product was done in [1].

Before proving theorem 2, we need to define the norm $\|\cdot\|'$; this involves an analysis of the structure of the K-invariant $\mathcal{B}(\mathcal{H}_\wedge)$-valued operators on N. In fact, let ${}^iQ_1, \cdots, {}^iQ_{d_i}$ be a basis for the K-invariant polynomials in $S(\underline{\underline{n}}_i)$. (By semiradiality, the ${}^iQ_j$'s commute for fixed i.) One can show that there are pairwise orthogonal vectors $P_1, \cdots, P_s$ in $\mathcal{B}(\mathcal{H}_\wedge)$ and corresponding harmonic polynomials $H_1, \cdots, H_s$ in $S(\underline{\underline{n}}_i)$ so that the invariants in

$\mathcal{B}(\mathcal{H}_\wedge) \otimes S(\underline{n}_i)$ all have the form $\sum_{j=1}^{s} p_j P_j({}^iQ_1, \cdots, {}^iQ_{d_i})H_j$, where the P_j's are polynomials in d_i variables. The K-invariants in $\mathcal{B}(\mathcal{H}_\wedge) \otimes S(\underline{n})$ are generated (as an algebra) by these, and hence each one is expressible as a sum $\sum A_\alpha Q^\alpha H_\alpha$, where for a multi index α in $\mathbb{N}^{d_1+d_2+\cdots+d_r}$, Q^α denotes $\Pi_{i=1}^r \Pi_{j=1}^{d_i} ({}^iQ_j)^{\alpha_{i+j}}$, $A_\alpha \in \mathcal{B}(\mathcal{H}_\wedge)$, and H_α is one of a finite number of operators formed from the products of the H_j's.

The coefficient α is called a **winning coefficient** if $A_\alpha \neq 0$ and α is maximal in the order on $\mathbb{N}^{d_1+\cdots+d_r}$ which is obtained by taking the lexiographic order on each $\mathbb{N}^{d_i}$ and forming the product order on $\mathbb{N}^{d_1+\cdots d_r}$. The norm $\| \cdot \|'$ is defined by $\| \sum A_\alpha Q^d H_\alpha \|' = \sum \| A_\alpha \|_{H.S.}$, the sum being taken over all winning coefficients.

Proof of Theorem 2.

Consider the partial Fourier coefficients $(P_\wedge)_{\wedge \in \widehat{K}}$ of our operator P. As noted in proposition 5.2 of [1], it will suffice to show that every compact set $L \subseteq N$ is contained in a compact set $\tilde{L} \subseteq N$ which is $P_\wedge$-full for each $\wedge \in \widehat{K}$. This will be proved by induction.

In [2], we proved the result for abelian groups and for nilpotent groups whose centre has dimension one. To complete an inductive argument, it is sufficient to consider the case where the centre has dimension strictly greater then one and reduce it to groups of lower dimension.

Thus let Z_1 and Z_2 be two linearly independent elements of the centre of $\underline{n}$. Let N_i = exp $\underline{n}/Z_i$, for $i = 1, 2$ so that $N_i = N/Z_i$ is a nilpotent group of dimension dim$N - 1$. Let $p_i : N \to N_i$ be the canonical projection. For $u \in \mathbb{D}(N)$, define $u_i \in \mathbb{D}(N_i)$ by

$$u_i(hD_i) = \int_{D_i} u(hz_i)dz_i.$$

If $P \in \mathbb{U}(\underline{n})$ the restriction of P to the D_i-invariant functions defines an element of $\mathbb{U}(\underline{n}_i) \otimes$

$\mathcal{B}(\mathcal{H}_\wedge)$ denoted P_i. It is easy to see that $(Pu)_i = P_i u_i$. Further, extending Z_i to a basis $\{X_1, \cdots, X_{n-1}, \ Z_i\}$ for $\underline{\underline{n}}$, one has $P = Q_0 + Q_1 Z_i + \cdots + Q_i Z_i^m$ where the Q_j belong to $\mathsf{U}(\underline{\underline{n}}) \otimes \mathcal{B}(\mathcal{H}_\wedge)$ are polynomials in the X's only. Then $P_i = Q_0$.

Notice that each $P_\wedge$ can be written in the form $P_\wedge = Z_i^{\alpha_{\wedge,i}} Q_{\wedge,i}$ for $i \in \{1,2\}$, where $\alpha_{\wedge,i} \in \mathbb{N}$, and $Q_{\wedge,i}$ is a $\mathcal{B}(\mathcal{H}_\wedge)$-valued operator on N not divisible by Z_i (i.e. $(Q_{\wedge,i})_0 \neq 0$).

Let L be a compact subset of H; $p_i(L)$ is a compact subset of N_i and by the inductive hypothesis applied to N_i, there exists a compact set $L_i \subseteq N_i$ which is $(Q_{\wedge,i})_i$ full for each $\wedge \in \hat{K}$, such that $p_i(L) \subseteq L_i$. (If L is empty, L_i may be taken empty also.) By a lemma due to D. Wigner, [4], $p_i^{-1}(L_i)$ is a subset of N which is $Q_{\wedge,i}$-full for each $\wedge$. Now the argument of [1] 8.6 shows that $p_i^{-1}(L_i)$ is Z_i-full and $P_\wedge$-full for all $\wedge \in \hat{K}$.

I claim that the set $\tilde{L} = p_1^{-1}(L_1) \cap p_2^{-1}(L_2)$ is a compact subset of N, containing L and $P_\wedge$-full for each $\wedge \in \hat{K}$. The compactness follows from lemma 6.5 of [1].

This completes the inductive step and hence proves theorem 2.

Bibliography

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