BOUNDARY REGULARITY FOR SOLUTIONS OF THE EQUATION OF PRESCRIBED GAUSS CURVATURE

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We describe a recent boundary regularity result for the equation of prescribed Gauss curvature

(1) $$\det D^2u = K(x)(1+|Du|^2)^{(n+2)/2}$$

in the case that the gradient of the solution is infinite on some relatively open portion of the boundary of the domain. To see how this situation arises, we recall that to solve the Dirichlet problem for (1) on a smooth, uniformly convex domain $\Omega \subset \mathbb{R}^n$ we need two conditions on the function K. First, we need

(2) $$\int_\Omega K < \omega_n ,$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$, to obtain a bound for the maximum modulus of the solution in terms of its boundary values, and second, we need

(3) $$K(x) \le \mu \operatorname{dist}(x,\partial\Omega)$$

for some positive constant μ to obtain a boundary gradient estimate. We then have the following theorem (see [4]).

THEOREM 1 *Let Ω be a $C^{1,1}$ uniformly convex domain in $\mathbb{R}^n$ and let $K \in C^{1,1}(\Omega)$ be a positive function satisfying (2) and (3). Then the Dirichlet problem*

(4) $$\det D^2u = K(x)(1+|Du|^2)^{(n+2)/2} \quad in\ \Omega ,$$
$$u = \varphi \quad on\ \partial\Omega$$

has a unique convex solution $u \in C^2(\Omega) \cap C^{0,1}(\bar\Omega)$ for every $\varphi \in C^{1,1}(\partial\Omega)$.

If the condition (3) is weakened (but (2) is maintained) it is not generally possible to solve the Dirichlet problem (4) in the classical sense, but it is possible to find a convex solution of (1) which satisfies the boundary condition $u\big|_{\partial\Omega} = \varphi$ in a certain optimal sense (see [6]). In general this is satisfied in the classical sense at some points of $\partial\Omega$, while at other points it is not, and we have

$$(5)\qquad \liminf_{\substack{x\to x_0\\ x\in\Omega}} u(x) < \varphi(x_0)$$

at such a point $x_0 \in \partial\Omega$. If K does not grow too fast near $\partial\Omega$ (for example $K \in L^n(\Omega)$) we find that

$$(6)\qquad \lim_{\substack{x\to x_0\\ x\in\Omega}} |Du(x_0)| = \infty .$$

In fact, if K is strictly positive in a neighbourhood of x_0, then u is continuous up to the boundary near x_0 (see [7], Corollary 3.11), so (5) and (6) hold not just at x_0 but for all boundary points sufficiently close to x_0.

If instead of (2) we have the extremal condition

$$(7)\qquad \int_\Omega K = \omega_n ,$$

it is not possible to solve the Dirichlet problem (4) in any reasonable sense. In this case there is a convex generalized solution u of (1) which is unique up to additive constants, and under a restriction on the growth of K near $\partial\Omega$ (for example $K \in L^n(\Omega)$) it follows that (6) holds for all $x_0 \in \partial\Omega$, and consequently $u \in C^2(\Omega)$ if $K \in C^{1,1}(\Omega)$ is positive (see [5], [7]).

Now let us return to the boundary regularity question. We have recently proved the following result (see [8]).

THEOREM 2 *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and suppose that for some* $x_0 \in \partial\Omega$, *which we may take to be the origin,* $\Gamma = \partial\Omega \cap B_{R_0}$ *is a connected,* $C^{2,1}$ *uniformly convex portion of* $\partial\Omega$. *Suppose that* $K \in C^{1,1}(\bar{\Omega})$ *is a positive function and* $u \in C^2(\Omega)$ *is a convex solution of* (1) *satisfying* (6) *for each* $x_0 \in \Gamma$. *Then there is a number* $\rho \in (0,R_0)$, *depending only on* n, R_0, Γ, $\sup_\Omega K$, $\inf_\Omega K > 0$ *and* $\|K\|_{C^{1,1}(\bar{\Omega})}$, *such that the following hold :*

(i) $u \in C^{0,1/2}(\overline{\Omega \cap B_\rho})$ *and for any* $x,y \in \overline{\Omega \cap B_\rho}$ *we have*

$$|u(x) - u(y)| \le C_1 |x-y|^{1/2} . \tag{8}$$

(ii) $\operatorname{graph}\left[u\big|_{\overline{\Omega \cap B_\rho}}\right]$ *is a* $C^{2,\alpha}$ *hypersurface for any* $\alpha \in (0,1)$, *and we have*

$$\|\nu\|_{C^{1,\alpha}(((\overline{\Omega \cap B_\rho}) \times \mathbb{R}) \cap \operatorname{graph} u)} \le C_2 \tag{9}$$

where $\nu = \dfrac{(Du, -1)}{(1+|Du|^2)^{1/2}}$ *is the normal vectorfield to* graph u .

(iii) $u\big|_{\overline{\Gamma \cap B_\rho}}$ *is of class* $C^{1,\alpha}$ *for any* $\alpha \in (0,1)$ *and we have*

$$\|u\|_{C^{1,\alpha}(\overline{\Gamma \cap B_\rho})} \le C_3 . \tag{10}$$

C_1 *depends only on* n, R_0, Γ, $\sup_\Omega K$, $\inf_\Omega K > 0$ *and* $\|K\|_{C^{1,1}(\bar{\Omega})}$, *and* C_2, C_3 *depend in addition on* α .

If Γ and K are more regular, we obtain correspondingly better regularity assertions in (ii) and (iii); in particular, if Γ and K are C^∞, then $\nu\big|_{\overline{\Omega \cap B_\rho}}$ and $u\big|_{\overline{\Gamma \cap B_\rho}}$ are C^∞ .

We briefly sketch the main ideas involved in the proof of Theorem 2. First we need to get some control on the behaviour of u near Γ. A key ingredient here is a regularity result ([7], Corollary 3.11) which tells us that $u \in C^{0,\alpha}(\overline{\Omega \cap B_{R_0/2}})$ for some small $\alpha = \alpha(n) > 0$, and

$$|u(x) - u(y)| \leq C|x-y|^{\alpha} \tag{11}$$

for all $x,y \in \overline{\Omega \cap B_{R_0/2}}$, where C depends only on n, R_0, Γ and $\inf_{\Omega} K > 0$. It is interesting that this boundary regularity result makes no use of the boundary condition (6), or indeed, of any boundary condition.

It is also crucial to our argument to estimate carefully how fast Du blows up near Γ. We have ([8], Lemma 2.2)

$$|Du(x)| \geq C\,\mathrm{dist}(x,\partial\Omega)^{-1/2} \tag{12}$$

for all $x \in B_{R_0/2} \cap \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) < \epsilon_1\}$ for some positive constants C and ϵ_1 depending only on n, R_0, Γ and $\sup_{\Omega} K$.

The estimates (11) and (12) are the main preliminary estimates we need. Once we have these it is quite straightforward to show that on $B_{R_0/2} \cap \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) < \epsilon_1\}$ we have

$$|\delta u|^{1+\beta} \leq C|Du| \tag{13}$$

for some small positive $\beta = \beta(n)$, where δu is the tangential gradient of u relative to Γ. From this we easily get a Hölder estimate for the normal vector field ν of the form

$$|\nu(x) - \nu(x_0)| \leq C|x-x_0|^{\gamma} \tag{14}$$

for all $x_0 \in \Gamma \cap B_{R_0/2}$, $x \in \overline{\Omega \cap B_{R_0/2}}$, where $\gamma = \gamma(n)$ is a small positive constant.

Once we have (14) we can express $\operatorname{graph} u$ near 0 as the graph of a convex function w over a subdomain of the vertical tangent plane to $\Gamma \times \mathbb{R}$ at 0. We find that w satisfies an equation

$$\det D^2 w = \tilde{K}(y,w)(1+|Dw|^2)^{(n+2)/2} \tag{15}$$

in $D = \{y \in \mathbb{R}^n : |y| < R,\ y_n > \omega(y')\}$ for suitable controlled $R > 0$ and $\omega \in C^{0,\alpha}(\overline{B_R^{n-1}})$ with $\omega(0) = 0$. Here the positive y_n direction corresponds to the negative x_{n+1} direction in the original coordinates, $y' = (y_1,\dots,y_{n-1})$, and we have assumed without loss of generality that $u(0) = 0$. On the "free boundary" $\Sigma = B_R \cap \{y_n = \omega(y')\}$ we have the two conditions

$$\tag{16} \begin{gathered} w = \psi \text{ on } \Sigma\ , \\ D_n w = D_n \psi = 0 \text{ on } \Sigma\ , \end{gathered}$$

where $\psi \in C^{2,1}(B_R)$ is the function representing $\Gamma \times \mathbb{R}$ near 0 as a graph over the vertical tangent plane at 0. Finally, from (11) and the uniform convexity of Γ, and from (14), we obtain the important dilation estimates

$$C_0|y-y_0|^{\delta} \le |Du(y) - Du(y_0)| \le C_1|y-y_0|^{\gamma} \tag{17}$$

for all $y \in \bar{D}$, $y_0 \in \Sigma$, where $\gamma = \gamma(n) > 0$, $\delta = \delta(n) \ge 2$, and C_0, C_1 are controlled positive constants.

The free boundary problem (15), (16) is similar to the free boundary problems studied by Kinderlehrer and Nirenberg [1]. However, their regularity results cannot be applied since we do not have sufficient regularity of Σ and of w near Σ. We would need $\Sigma \in C^1$ and $w \in C^2(D \cup \Sigma)$ to apply their results; all we have so far is $\Sigma \in C^{0,\alpha}$ and $w \in C^2(D) \cap C^1(D \cup \Sigma)$. Nevertheless, their technique can be used. We introduce $z = Dw$ as the new independent variables and define the Legendre transform w^* of w by

$$w^*(z) = \sum_{k=1}^{n} y_k D_k w(y) - w(y) .$$

Then $w^* \in C^2(B^+_{R_1}) \cap C^{0,1}(\overline{B^+_{R_1}})$ is a convex solution of a Monge-Ampère equation

$$\det D^2 w^* = f(z,w^*,Dw^*)$$

in a half-ball $B^+_{R_1} = \{z \in \mathbb{R}^n : |z| < R_1 , z_n > 0\}$ of controlled radius (by virtue of (17)), and f is positive and of class $C^{1,1}$. Furthermore, by virtue of (16) we have $w^* = \psi^*$ on the flat boundary position of $B^+_{R_1}$, where ψ^* is a $C^{2,1}$ uniformly convex function. Using minor modifications of standard techniques in the theory of Monge-Ampère equations we can then show that $w^* \in C^{2,\alpha}(\overline{B^+_\sigma})$ for all $\alpha \in (0,1)$ for some controlled positive σ. This gives us the assertions of Theorem 2 for the original function u by going back through the various coordinate transformations.

It is interesting to compare Theorem 2 with an analogous result of Lin [2], [3] for the equation of prescribed mean curvature. On this basis it seems reasonable to conjecture that similar results will eventually prove to be true for more general curvature equations.

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