Second Order Elliptic Equations with Venttsel Boundary Conditions

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1 Introduction

In this work we study the following boundary value problems. Let $\Omega \subset R^n$ be a bounded domain with smooth boundary. We define the tangential gradient operator on the boundary manifold $\partial\Omega$ by $\delta_i = D_i - \nu_i \nu_k D_k$, where $\nu = (\nu_1, \ldots, \nu_n)$ is the inner normal vector to $\partial\Omega$. We consider the problem

$$Lu \equiv a^{ij} D_{ij} u + b^i D_i u + cu = f, \qquad \text{in } \Omega, \tag{1}$$

$$Bu \equiv \alpha^{ij} \delta_i \delta_j + \beta^i D_i u + \gamma u = g, \qquad \text{on } \partial\Omega. \tag{2}$$

The equation (1) is elliptic in the usual sense [2] and the boundary condition (2) is called Venttsel if the following conditions are satisfied:

$$\alpha^{ij} \eta_i \eta_j \geq 0, \qquad \text{for } \eta \in R^n \text{ s.t. } \eta \perp \nu, \tag{3}$$

and

$$\beta^i \nu_i \geq 0. \tag{4}$$

If the inequality in (3) is strict the boundary condition B is called elliptic, otherwise it is called degenerate elliptic; if the inequality in (4) is strict the boundary condition B is called oblique, otherwise it is called degenerate oblique.

Without loss of generality we may assume that $\{\alpha^{ij}\}$ has the property

$$\alpha^{ij} \nu_j \equiv 0, \qquad \text{on } \partial\Omega, \text{ for } i = 1, 2, \ldots, n.$$

Therefore when we locally flatten the boundary our boundary value problem becomes

$$a^{ij} D_{ij} u + b^i D_i u + cu = f, \qquad \text{in } B^+, \tag{5}$$

$$\alpha^{st} D_{st} u + \beta^i D_i u + \gamma u = g, \qquad \text{on } B^0 \tag{6}$$

where $B^+ = \{x \in R^n |\ |x| < 1, x_n > 0\}$,$B^0 = \{x \in R^n |\ |x| < 1, x_n = 0\}$, and the repeated indices s, t indicate the summation from 1 to $n-1$.

This type of boundary condition originally came from probability theory (see for example Venttsel [10]), and also occurs in three-dimensional water wave theory, (see Shinbrot [9]), and in the engineering problem of "hydraulic fracturing" of oil wells, (see Cannon and Meyer [1]).

Our goal in this work is to study the existence and uniqueness of the solution of the boundary value problem. The results about the linear problems are simple and stated in the next section. When the equation and the boundary condition are nonlinear, like the model in the three-dimensional water wave theory, the problem becomes difficult. In such a case, some general theorems on the a priori estimates of the solutions and the existence in both classical and general sense are stated in the remaining sections.

2 Linear Problems

In the case of linear problem (1) and (2), the existence and uniqueness results are obtained as follows.

Assume that the boundary condition is elliptic. By considering the boundary condition 6 as an elliptic equation, we can obtain the Schauder estimate on the boundary

$$|u|_{2,\alpha;\partial\Omega} \leq C(|g|_{\alpha;\Omega} + |u|_{1,\alpha;\Omega}),$$

where, and from now on, C is a constant independent of the solutions. Then by applying the Schauder global estimate to the Dirichlet problem of (1) with $C^{2,\alpha}$ boundary data, and using the interpolation inequality, we have

$$|u|_{2,\alpha;\Omega} \leq C(|u|_0 + |g|_{\alpha;\Omega} + |f|_{\alpha;\Omega}).$$

This estimate enables us to apply the method of continuity to get the existence theorem, provided the norm $|u|_0$ is bounded. The $|u|_0$ bound is a consequence of the maximum principle which is the only good property that Venttsel boundary value problem shares with the usual boundary value problems. The uniqueness also follows from the maximum principle. For the Schauder estimates and the interpolation inequality we refer [2].

Theorem 1 *Suppose that L and B are uniformly elliptic and that $c, \gamma < 0$. Then the problem (1) ,(2) has a unique solution in $C^{2,\alpha}$.*

The work in this section is a collaboration with N.S.Trudinger. (see [6]).

3 Local estimates

It is interesting and important to study the local behavior of the solutions near the boundary, such as supremum and oscillation. This observation leads us to establish the Aleksandrov-Bakelman type maximum principles and weak Harnack inequalities for the solutions of Venttsel boundary problems. These estimates are found to be crucial in the later work on the nonlinear problems, because, as is well known, the existence theory in nonlinear case depends strongly upon the a priori estimates of the solutions, such as Hölder estimate which can be obtained quickly from a weak Harnack inequality.

Let us consider the problem (5),(6) with flat boundary, and assume that L is uniformly elliptic and B is Venttsel, i.e. $\{\alpha^{st}\}$ is an $(n-1)\times(n-1)$ symmetric non-negative definite matrix, and $\beta^n \geq 0$. In the case when the boundary condition is uniformly elliptic we have

Theorem 2 *Suppose L and B are uniformly elliptic and u is a solution of*

$$Lu \geq f, \qquad \text{in } R^n_+, \quad Bu \geq g, \qquad \text{on } \partial R^n_+. \tag{7}$$

Suppose also that $u \leq 0$ in $R^n_+ \setminus B^+_R$. Then we have

$$\sup_{B^+_R} u^+ \leq C(\|g/\Delta^*\|_{n-1} + \|f/D^*\|_n)R,$$

where C is a constant depending only on n, $\|b^i/D^*\|_n$. $D^* = (det\{a^{ij}\})^{1/n}$ and $\Delta^* = (det\ \{\alpha^{st}\})^{1/n-1}$.

If the boundary operator B is degenerate elliptic and the degeneracy is of type I by which we mean

$$\{\alpha^{st}\}_{(n-1)\times(n-1)} = \{\alpha^{st}\}_{h\times h} \oplus \{O\}_{(n-h-1)\times(n-h-1)},\ 0 < h < n-1,$$

we have

Theorem 3 *Suppose L is uniformly elliptic , B is degenerate elliptic of type I and oblique. Suppose also that $b^i \equiv 0$, and $\beta^s \equiv 0$. If u is a solution of*

$$Lu \geq f, \qquad \text{in } R^n_+, \quad Bu \geq g, \qquad \text{on } \partial R^n_+. \tag{8}$$

and $u \leq 0$ in $R^n_+ \setminus B^+_R$. Then we have

$$\sup_{B^+_R} u^+ \leq C(\|G\|_h + \|f/D^*\|_n)R,$$

where $C = C(n)$ and

$$G(x_1, x_2, \ldots, x_h) = \sup_{(x_{h+1},\ldots,x_{n-1})} g(x_1, x_2, \ldots, x_{n-1})/\Delta^*_h.$$

The idea to prove theses theorems is similar to the proof of the original version of Aleksandrov maximum principle, i.e. we consider the upper contact set of the solutions and the normal mapping defined on it, then we estimate the measure of the normal image of the contact set. The difference in our case is that we have introduced here a new notion of *boundary contact set* which enables us to handle the Venttsel boundary condition.

With the help of the Aleksandrov maximum principles above we can obtain the following weak Harnack inequalities.

Theorem 4 *Suppose that L and B are uniformly elliptic. Then there exists a constant $p > 0$ such that for any non-negative solution u of*

$$Lu \leq f, \text{ in } B^+_R, \qquad Bu \leq g, \text{ on} B^0_R,$$

we have the estimate

$$\left(\frac{1}{|B^0_{R/2}|}\int_{B^0_{R/2}} |u|^p dx'\right)^{1/p} \leq C\left(\inf_{B^0_{R/2}} u + R\|g/\kappa\|_{n-1.B^0_R} + R\|f/\lambda\|_{n,B^+_R}\right),$$

where $x' = (x_1, x_2, \ldots, x_{n-1})$.

Theorem 5 *In the above theorem, if B is degenerate elliptic of type II, then the estimate is*

$$\left(\frac{1}{|B^0_{R/2}|}\int_{B^0_{R/2}} |u|^p dx'\right)^{1/p} \leq C\left(\inf_{B^0_{R/2}} u + R\|G\|_{h,B^h_R} + R\|f/\lambda\|_{n,B^+_R}\right),$$

where G is defined as before and B^h_R is the h-dimensional ball.

The work stated in this section is on the publication [7].

4 Quasilinear Problems — Elliptic Boundary Conditions

The quasilinear problem that we are concerned with is

$$a^{ij}(x,u,Du)D_{ij}u + b(x,u,Du) = 0, \qquad \text{in } \Omega, \tag{9}$$

$$\alpha^{ij}(x.u.\delta u)\delta_i\delta_j u + \beta(x,u,Du) = 0, \qquad \text{on } \partial\Omega. \tag{10}$$

The boundary condition is of Venttsel type if

i) $\{\alpha^{ij}\}$ is as before,

ii) $D_p\beta \cdot \nu \geq 0$.

Under the natural structure conditions of Ladyzhenskaya and Uraltseva:

(A) $\lambda I \leq \{a^{ij}(x,z,p)\} \leq \Lambda I$ and $\Lambda \leq \lambda\mu(|z|)$,

(A1) $|b(x,z,p)| \leq \lambda\mu_1(|z|)(1+|p|^2)$,

(A2) $|D_x a^{ij}|, |D_z a^{ij}|, |D_p a^{ij}|(1+|p|) \leq \lambda\mu_1(|z|)$,

$|D_x b|, |D_z b|, |D_p b|(1+|p|) \leq \lambda\mu_1(|z|)(1+|p|^2)$, and the corresponding conditions on the boundary operator:

(B) $\kappa|\eta|^2 \leq \alpha^{ij}\eta_i\eta_j \leq K|\eta|^2$, for $\eta \perp \nu$, and $K \leq \kappa\mu(|z|)$; $D_p\beta\cdot\nu \geq \chi$, $\chi > 0$ is a constant,

(B1) $|\beta(x,z,p)| \leq \kappa\mu_2(|z|)(1+|p|)$,

(B2) $|D_x\alpha^{ij}|, |D_z\alpha^{ij}|, |D_p\alpha^{ij}|(1+|p|) \leq \kappa\mu_2(|z|)$,

$|D_x\beta|, |D_z\beta| \leq \kappa\mu_2(|z|)(1+|p|)$, $|D_p \leq \kappa\mu_2(|z|)$, where μ, μ_1, μ_2 are nondecreasing functions, the following theorem is obtained.

Theorem 6 *The problem (9) and (10) has a solution $u \in C^{2,\alpha}(\overline{\Omega})$ provided $\sup_\Omega |u|$ is bounded.*

Outline if the proof. First we establish the $C^{1,\alpha}$ a priori estimate for the solutions u in the following steps:

(i)By the weak Harnack inequalities obtained in the last section, we get the C^α estimate near the boundary.

(ii) A maximum principle argument then yields the tangential gradient estimate.

(iii) The C^α estimate near the boundary for the tangential gradient is proved in a similar way as in step (i).

(iv) Finally the global $C^{1,\alpha}$ estimates follows from a result of Lieberman [4].

Since the maximum principles are valid in the case of Venttsel boundary value problems , the estimate for $\sup_\Omega |u|$ can be obtained by assuming that $D_z b, D_z\beta \leq -c_0$ for some positive constant c_0. The uniqueness of the solution also follows from a comparison principle under some extra assumptions.

For more detail about the theorem and the proof see the publication [8].

5 Quasilinear Problems–Degenerate Elliptic Boundary Conditions

Here we consider the equation (9) with the boundary condition (10). When we flatten the boundary locally the problem becomes

$$a^{ij}(x,u,Du)D_{ij}u + b(x,u,Du) = 0, \qquad \text{in } B^+ \tag{11}$$

$$\alpha^{st}(x.u.D'u)D_{st}u + \beta(x,u,Du) = 0, \qquad \text{on } B^0 \tag{12}$$

and we assume that the boundary condition has only the following two types of degenerate points:

Type A: x is a degenerate point of Type A if $\{\alpha^{st}\}$ has a rank h, $0 < h < n-1$, and

$$\{\alpha^{st}\} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is a positive definite minor with the smallest eigen-value $\kappa > 0$ and

$$A_{22} - \frac{1}{\delta\kappa}A_{21}A_{12} \geq 0, \qquad \text{for some } \delta \in (0,1).$$

Type B: x is a degenerate point of Type B if

$$0 \leq \{\alpha^{st}(y)\} \leq \eta(y)I,$$

where $\eta(y) \leq C|x-y|$ in a neighborhood of x.

To obtain the existence of classical solutions, our strategy is to make use of elliptic regularization, i.e. to add an ϵ times of Laplacian to the boundary condition and solve the approximate problem first, then by letting $\epsilon \to 0$ to get the solution. For this purpose we need to set up an ϵ independent $C^{2,\alpha}$ estimate to ensure that the approximate sequence coverges in a proper function space. To get the estimate, besides the natural structure conditions we have to make some additional assumptions:

(AA) In a neighborhood of degenerate points the following equations hold

$$\frac{\partial^2 a^{ij}}{\partial p_k \partial p_l}\nu^j\nu^k\nu^l = 0, \ \forall i, \text{ and } \frac{\partial a^{ij}}{\partial p_k}\nu^i\nu^j\nu^k = 0;$$

(BB) In the same neighborhood as in (AA)

$$\frac{\partial^2 \beta}{\partial p_k \partial p_l}\nu^k\nu^l = 0.$$

Theorem 7 *Under the assumptions stated above and that the functions α^{ij} are independent of the arguments z and p,the problem (9), (10) has a $C^{2,\alpha}$ solution.*

The main technique used in obtaining the $C^{2,\alpha}$ estimates is the maximum principle argument, which is an analogue of that of Lieberman and Trudinger [6].

An application of this existence theorem is to the water wave problem. A similar result is obtained under some weaker assumptions than that of Korman [3].

The assumptions (AA) and (BB) and seem to be artificial, (although (AA) is satisfied by the mean curvature operator, and (BB) is satisfied by any β which is linear in the normal derivative $D_\nu u$). Also the restriction that the functions α^{ij} are independent of z and p is too strong. To avoid these assumptions, let us consider the general solutions in viscosity sense.

Definition 1 *A function $u \in C(\overline{\Omega})$ is said to be a viscosity subsolution (supersolution) of (9), (10) if $\phi \in C^2(\overline{\Omega})$ is an arbitrary function and $u - \phi$ attains its maximum (minimum) at $x_0 \in \overline{\Omega}$, then*

$$a^{ij}(x_0, u(x_0), D\phi(x_0))D_{ij}\phi(x_0) + b(x_0, u(x_0), D\phi(x_0)) \geq (\leq) 0,$$

when $x_0 \in \Omega$, *and*

$$\alpha^{ij}(x_0)\delta_i\delta j\phi(x_0) + \beta(x_0, u(x_0), D\phi(x_0)) \geq (\leq)0,$$

when $x_0 \in \partial\Omega$.

A function $u \in C(\overline{\Omega})$ *is said to be a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.*

Theorem 8 *Suppose the structure conditions (A), (A1), (A2), (B),(B1) and (B2) are satisfied. Suppose all the degenerate points are either type A or type B. Suppose also that there is a positive constant* c_0 *such that* $D_z b$, $D_z\beta \leq -c_0$. *Then there exists a viscosity solution.*

The proof of this theorem is also based on the procedure of elliptic regularization. However this time we need only the convergence in $C(\overline{\Omega})$ so that the estimate is simple.

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