A Geometric Measure Theory Approach To Exterior Problems For Minimal Surfaces

Michael Grüter

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1 Introduction

The classical Plateau Problem in minimal surface theory is the following question:

Suppose $\Gamma \subset \mathbb{R}^3$ is a simple closed Jordan curve. Does there exist a surface $M \subset \mathbb{R}^3$ which has Γ as its boundary, $\partial M = \Gamma$, such that its area $|M|$ is as small as possible?

Using parametric methods the Plateau Problem was solved around 1930 independently by Douglas and Radó. Their solution was parametrized over a disk.

The same problem as well as its higher dimensional analogue was also investigated using Geometric Measure Theory (GMT). In the classical three dimensional case the GMT-approach gives the following beautiful result:

There exists an embedded smooth oriented submanifold $M \subset \mathbb{R}^3$ with boundary Γ, such that M has least area among all oriented surfaces S with $\partial S = \Gamma$.

Recently, starting with the work by Tomi and Ye [TY], corresponding exterior problems for minimal surfaces have attracted attention. In their paper Tomi and Ye concentrate on the parametric method. The reader is referred to Tomi's paper in this volume.

The GMT-approach to exterior problems I want to present here has its origins in the proof of the Positive Mass Conjecture in general relativity by

Schoen and Yau. In fact, Kuwert in [KE1] gave a detailed proof of the so called Positive Energy Theorem as outlined in [SR] by Schoen. The method used there was later generalized by Kuwert in his thesis [KE2].

Here, I want to give an application of this method to an exterior free boundary problem using previous work of mine on regularity questions for minimal surfaces with free boundaries. More details can be found in [GM3], see also [KE2].

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2 Some Remarks On GMT

In GMT a smooth oriented submanifold $M^n \subset \mathbb{R}^{n+k}$ of locally finite measure is considered as a linear functional $[\![M]\!]$ on differential n-forms:

$$[\![M]\!] (\omega) = \int_M \omega,$$

ω being a smooth n-form having compact support. The object $[\![M]\!]$ is called an *n-current*, as is any linear functional on the space of these n-forms. The space of n-currents in an open set $U \subset \mathbb{R}^{n+k}$ is denoted by $\mathcal{D}^n(U)$. More interesting in geometrical problems are the so called *(integer multiplicity) rectifiable* currents which can be thought of as subsets of a countable union of oriented n-dimensional submanifolds. In particular, they possess an n-dimensional tangent plane almost everywhere which has to be oriented and to be counted with multiplicities. Forgetting the orientation one obtains the so called *rectifiable n-varifolds.*

A fundamental result in the theory of rectifiable currents is the following.

Compactness Theorem (Federer and Fleming 1960)

Suppose $U \subset \mathbb{R}^{n+k}$ is open and that a sequence $\{T_k\}$ of rectifiable n-currents in U is given. If for any $W \subset\subset U$ we have

$$\sup_k \{\mathbf{M}_W(T_k) + \mathbf{M}_W(\partial T_k)\} < \infty$$

then $T = \lim_{k\to\infty} T_k$ is a rectifiable n-current.

Remark

Here, ∂T denotes the *boundary* of the current T, defined by $\partial T(\omega) := T(d\omega)$, and $\mathbf{M}_W(T)$ is the *mass* (= area with multiplicities) of the current T in W.

For more on GMT the reader is referred to [MF] and [FH].

3 An Exterior Free Boundary Problem

Suppose, we are given a closed smooth hypersurface $M^n \subset \mathbb{R}^{n+1}$ and a prescribed direction $\nu \in S^n$.

Find a minimal surface $\Sigma^n \subset \mathbb{R}^{n+1}$ such that the following is true:

(a) $\partial\Sigma \subset M$;

(b) Σ intersects M orthogonally along $\partial\Sigma$;

(c) Σ = graph u over $\Pi \sim D$, where Π is an n-plane orthogonal to ν and $D \subset \Pi$ an n-dimensional ball;

(d) Σ has the prescribed normal ν at infinity.

Outline of the proof of existence.

After a possible rotation and translation we may assume without loss of generality that $\nu = e_{n+1}$ and that M and $\mathbb{R}^n$ intersect transversely. Denote by $p = p_{\mathbb{R}^n}$ the orthogonal projection onto $\mathbb{R}^n$ and write $\xi = p(x)$ for $x \in \mathbb{R}^n$.

First, we choose a large ball B_{R_0} centered at the origin such that M is contained in its interior. For $\sigma > R_0$ we let $\Gamma_\sigma = [\![\, \partial D_\sigma]\!]$, D_σ being the n-disk in $\mathbb{R}^n$ centered at the origin.

Next, consider the class

$\mathcal{C}(M, \sigma) = \{T : T$ a rectifiable n-current, spt T compact, $\text{spt}(\partial T - \Gamma_\sigma) \subset M\}$. From standard arguments it follows that we get the existence of a $\Sigma_\sigma \in \mathcal{C}(M, \sigma)$ minimizing area within $\mathcal{C}(M, \sigma)$. Of course, the underlying idea is to let $\sigma \to \infty$ and obtain the existence of a limit surface which is a good candidate for a solution to our problem.

Since M and $\mathbb{R}^n$ intersect transversely we obviously have

$$\mathbf{M}(\Sigma_\sigma) \leq \omega_n \sigma^n - a_0, \tag{1}$$

for some $a_0 > 0$ not depending on σ. Here, ω_n is the volume of the unit n-ball.

Now, by considering the current $\Sigma_\sigma - [\![D_\sigma]\!]$ and invoking the constancy theorem one can show that for any measurable set

$$U' \subset A(R_0, \sigma) := \{\xi \in \mathbb{R}^n : R_0 < |\xi| < \sigma\}$$

$$\mathcal{H}^n(spt\ \Sigma_\sigma \cap p^{-1}(U')) \geq \mathcal{L}^n(U'). \tag{2}$$

But then, (1) and (2) imply for $U \subset A(R_0, \sigma)$:

$$\mu_\sigma(p^{-1}(U)) + \mu_\sigma(p^{-1}(clos\ D_{R_0})) \leq \mathcal{L}^n(U) + \omega_n R_0^n - a_0, \tag{3}$$

μ_σ denoting the measure associated with Σ_σ.

From this inequality one easily deduces the *uniform* (in σ) *mass-estimate* for any $\tau < \sigma$:

$$\mathbf{M}(\Sigma_\sigma \llcorner p^{-1}(clos\ D_\tau)) = \mu_\sigma(\{x : |\xi| \leq \tau\}) \leq \omega_n \tau^n - a_0. \tag{4}$$

To apply Federer's and Fleming's Compactness Theorem we still have to estimate uniformly the mass of the free boundary. But, from results in [GM1] it follows that we have

$$\mathbf{M}(\partial\Sigma_\sigma - \Gamma_\sigma) \leq c_M(1 + \mathbf{M}(\Sigma_\sigma \llcorner p^{-1}(clos\ D_{R_0}))) \leq c_{M,R_0}.$$

In this estimate we have also used (4).

We now get the existence of a subsequence $\sigma_j \to \infty$ such that the corresponding surfaces $\Sigma_j = \Sigma_{\sigma_j}$ have a limit

$$\Sigma_j \to \Sigma.$$

For this limit Σ we have

- $\partial\Sigma \subset M$,
- Σ is minimizing ,
- $\mu_{\Sigma_j} \to \mu_\Sigma$;

and from (4) we get for any $\tau > 0$

$$\mathbf{M}(\Sigma \llcorner p^{-1}(clos\ D_\tau)) \leq \omega_n \tau^n - a_0.$$

But this inequality obviously shows that Σ is nontrivial ($\neq$ hyperplane).

By [GM2] the minimality of Σ also implies that Σ and M intersect orthogonally along $\partial\Sigma$. We still have to check that Σ satisfies properties (c) and (d) of our problem.

First, we note that Σ_σ is stationary, so that the convex hull property and the fact that

$$spt\ \partial\Sigma_\sigma \subset (M \cup \partial D_\sigma) \subset B_{R_0} \cup \partial D_\sigma$$

give the following *uniform height estimate* :

$$|z| \leq R_0 \ for \ x = (\xi, z) \in spt\ \Sigma_\sigma,$$

and of course $|\xi| \leq \sigma$.

Next, consider the *excess* $E(x,\rho)$ given by

$$E(x,\rho) = \frac{1}{2\rho^n}\int_{B_\rho(x)} |p_{T_y\Sigma_\sigma} - p|^2 \, d\mu_\sigma(y).$$

By a kind of Caccioppoli inequality and the height estimate we get

$$E(x,\rho) \leq c_n R_0^2 \rho^{-n-2} \mu_\sigma(B_\rho(x)).$$

Using inequality (3) for $U = D_\rho(\xi) = p(B_\rho(x))$ we infer

$$E(x, \rho/2) \leq c_{n,R_0}\rho^{-2}, \tag{5}$$

which is valid for $x = (\xi, z) \in \ spt\ \Sigma_\sigma$ and $R_0 + \rho \leq |\xi| \leq \sigma - \rho$.

Now, the fundamental idea is to apply Allard's famous regularity theorem on *large balls.* For, note that by choosing $\rho \geq \rho_0(R_0)$ in (5) we can make the excess as small as we want.

Since the excess was measured with respect to $\mathbb{R}^n$ Allard's theorem gives

$$spt\ \Sigma_\sigma \cap \{x : R_1 < |\xi| < \frac{2}{3}\sigma\} = \ graph\ u_\sigma$$

for $R_1 = R_0 + \rho_0$ and a suitable $C^{1,\alpha}$-function

$$u_\sigma : A(R_1, \frac{2}{3}\sigma) \to \mathbb{R}.$$

From the excess estimate we get the additional information

$$|\nabla u_\sigma| \leq \frac{c}{\rho} \tag{6}$$

for $\rho_0 \leq \rho \leq \frac{2}{3}\sigma$.

But u_σ is a solution of the nonparametric minimal surface equation, so that by (6) we may uniformly estimate all derivatives of u_σ.

By letting $\sigma \to \infty$ we get $u_\sigma \to u$ such that

$$u : A(R_1, \infty) \to \mathbb{R}$$

is a solution of the minimal surface equation. Furthermore

- $spt\ \Sigma \cap \{x : |\xi| > R_1\} = \ graph\ u,$
- $|\nabla u(\xi)| \to 0 \ \ as\ |\xi| \to \infty.$

More precisely, for any $0 < \tau < 1$ we get

$$u - h_\infty = O(|\xi|^{-\tau}) \quad if\ n = 2,$$

$$u - h_\infty - \frac{g_\infty}{|\xi|^{n-2}} = O(|\xi|^{2-n-\tau}) \quad if\ n \geq 3.$$

Thus, we have finally obtained a solution to our exterior free boundary problem.

Remark

From well known regularity results (interior, free boundary) it follows that for $n \leq 6$ the solution Σ is smooth everywhere and embedded.

4 Further Results

In [KE2] Kuwert also treats the exterior Plateau Problem using the same approach. For $n = 2$ a modification of the method produces solutions having a catenoidal end. In this case, the approximating solutions are obtained by solving a suitable obstacle problem (a given catenoid plus a vertical translation of it are used as barriers). This is necessary because a catenoid is not minimizing in $\mathbb{R}^3$.

Of course the same procedure works in dimensions $n \geq 3$. But the solutions obtained in this way will have a *planar end* because the higher dimensional catenoids have the form

$$C = \{x \in \mathbb{R}^{n+1} : x^{n+1} = \pm h(|p(x)|), |p(x)| \geq a > 0\}$$

where h is given by

$$h(r) = a \int_1^{r/a} \frac{ds}{\sqrt{s^{2(n-1)} - 1}} \ .$$

But since $\lim_{r \to \infty} h(r) < \infty$ for $n \geq 3$ these catenoids lie between two planes!

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Fachbereich Mathematik, Universität des Saarlandes, D-6600 Saarbrücken
Germany