

On Singular Variational Problems

Ulrich Dierkes

Abstract. We summarize recent results on singular variational problems which arise in connection with the n -dimensional analogue of the catenary problem. In addition we give some historical comments.

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Since the time of Lagrange the equation of a heavy, flexible and inextensible surface of constant mass density has been derived by several authors, see Lagrange [L, pp. 146–162], Poisson [P, §I : équation d'équilibre de la surface flexible et non élastique pp. 173–187], Cisa de Gresy [CG, pp. 274–276], and Jellett [J, pp. 349–354]. It turns out that there are several model problems available, which are due to different notions of flexibility and inextensibility, and which are all worth to be investigated. Here we are interested in the higher dimensional mathematical analogue of the catenary problem, i.e. to find a surface M of prescribed area and boundary which is of constant mass density and which has lowest center of gravity. Assuming that the surface M is given as a graph $u : \Omega \rightarrow \mathbb{R}^+$, $\Omega \subset \mathbb{R}^n$, and that the gravitation force acts in the $-x_{n+1}$ direction, this problem amounts to minimizing the integral

$$J(u) = \frac{1}{A_0} \int_{\Omega} u \sqrt{1 + |Du|^2} \, dx$$

in a class of nonnegative functions, which fulfill prescribed boundary conditions as well as the subsidiary condition

$$A(u) = \int_{\Omega} \sqrt{1 + |Du|^2} \, dx = A_0$$

for some given value of A_0 .

Note that we minimize in a class of *nonnegative* functions, because our competing surfaces may touch the "ground" $x_{n+1} = 0$, but cannot penetrate it. Hence the problem is only *singular* elliptic.

Also observe that the unconstrained problem of minimizing $J(\cdot)$ with fixed boundary and prescribed A_0 in general has no solution, even if A_0 is close to the area of the corresponding minimal surface. In fact, for given $\epsilon > 0$, $\Omega = B_1(0) \subset \mathbb{R}^2$ and arbitrary

constant boundary data, there are functions f_n which are constant along $\partial\Omega$, such that $A(f_n) = \pi(1+\epsilon)$, $J(f_n) \rightarrow -\infty$, as well as $f_n(0,0) \rightarrow -\infty$, see Nitsche [N].

Introducing a Lagrange multiplier λ the problem is reduced to the free variational problem

$$J_\lambda(u) = \int_\Omega (u + \lambda) \sqrt{1 + |Du|^2} dx \rightarrow \text{minimum},$$

in a class of nonnegative functions which is defined by boundary conditions. The Euler equation of J_λ is given by

$$(1) \quad \operatorname{div} \left[\frac{(u + \lambda) Du}{\sqrt{1 + |Du|^2}} \right] = \sqrt{1 + |Du|^2} \text{ in } \Omega,$$

or, equivalently, if $u + \lambda > 0$ and $u \in C^2(\Omega)$,

$$(1') \quad \operatorname{div} \left[\frac{Du}{\sqrt{1 + |Du|^2}} \right] = \frac{1}{(u + \lambda) \sqrt{1 + |Du|^2}} \text{ in } \Omega.$$

In this form, equation (1) has been derived by Lagrange [L, pp. 158–162], Cisa de Gresy [CG, pp. 274–276], and also Jellett [J, pp. 349–354], as the equilibrium condition for a *heavy, inextensible and flexible* surface of constant mass density, which is exposed to a vertical gravitational field. These authors base their arguments on a suitable variational principle. Later on a different approach to describe the equilibrium of a flexible surface in a force field was given by Poisson [P, pp. 173–187] who uses direct arguments from mechanics. For surfaces in \mathbb{R}^3 he introduces two independent "tensions" T and T' , which describe the forces inside the surface. Let the external force field be given by X, Y, Z , then he deduces the following equilibrium condition

$$(A) \quad Xk + \frac{\partial}{\partial x} \left[\frac{T(1+q^2)}{k} \right] - \frac{\partial}{\partial y} \left[\frac{T'pq}{k} \right] = 0,$$

$$(B) \quad Yk - \frac{\partial}{\partial x} \left[\frac{Tpq}{k} \right] + \frac{\partial}{\partial y} \left[\frac{T'(1+p^2)}{k} \right] = 0,$$

$$(C) \quad Zk + \frac{\partial}{\partial x} \left[\frac{Tp}{k} \right] + \frac{\partial}{\partial y} \left[\frac{T'q}{k} \right] = 0,$$

where we have put $p = \frac{\partial u}{\partial x}$, $q = \frac{\partial u}{\partial y}$ and $k = \sqrt{1+p^2+q^2}$.

Poisson gives special interest to the case where $T = T'$ i.e. when the tensions coincide.

In fact in this case, one easily deduces from (A), (B), and (C) the relation

$$(D) \quad Z - pX - qY + \frac{T}{k^2} \left[(1+q^2) \frac{\partial^2 u}{\partial x^2} - 2pq \frac{\partial^2 u}{\partial x \partial y} + (1+p^2) \frac{\partial^2 u}{\partial y^2} \right] = 0$$

as well as the condition

$$(E) \quad Xdx + Ydy + Zdz + dT = 0,$$

i.e. the external force must have a potential U and $T = U + c$. From (D) and (E)

Poisson deduces:

(i) The *minimal surface* equation, with

$$X = Y = Z = 0 \text{ and } T = c;$$

(ii) The equation for *capillary surfaces* by taking

$$X = Y = 0, \quad Z = \frac{a+bz}{k},$$

as the equilibrium condition of a flexible surface which is covered by a heavy fluid;

(iii) The equation of a *heavy surface in a gravitational field*, by taking $X = Y = 0$,

$Z = g\epsilon$, $g =$ gravitation constant, and ϵ denoting the density of the surface.

The tension is then given by $T = c - g\epsilon z$ and hence (D) yields

$$g\epsilon + \frac{c-g\epsilon z}{k^2} \left[(1+q^2) \frac{\partial^2 u}{\partial x^2} - 2pq \frac{\partial^2 u}{\partial x \partial y} + (1+p^2) \frac{\partial^2 u}{\partial y^2} \right] = 0 ,$$

which is equivalent to (1) or (1') if one takes $g\epsilon = 1$.

Note that, as a mechanical model, this is a simplification, however, from the geometric point of view, this problem has many interesting features. In fact (1') is an equation of mean curvature type with mean curvature given by

$$H = H(u, Du) = \{n(u + \lambda) \cdot \sqrt{1 + |Du|^2}\}^{-1} ,$$

and hence, if $\lambda = 0$, H is not a priori bounded. In addition, we think this problem may serve as a model problem for singular variational problems of, may be, more complicated nature. Let us also mention that equation (1) is of importance in architecture, since it provides a model for the so called "hanging roof" i.e. turning a positive solution u of (1) upside down, gives the optimal shape of a cupola (geodesic dome). In fact, this principle was already known to R. Hooke in 1675: "Ut pendet continuum flexile, sic stabit contiguum rigidum inversum", i.e. as hangs the flexible line, so but inverted will stand the rigid arch (see Truesdell [T, p.57]). However, Hooke seems never to have published this result (nor its proof), may be because the problem of the catenary was still unsolved at those days. (Although there was some faulty attempt by Galileo who had proposed the problem and suggested a parabola as its solution. The faulty reasoning of Galileo was criticized already in 1646 by Huygens in a letter to Mersenne.) The problem of the rigid arch was apparently brought to Hooke's attention by the architect Sir Christopher Wren, the constructor of St. Paul's cathedral, London (the foundation stone was laid in 1675 and the structure completed in 1710). A rigorous proof that the catenary and the rigid arch represent the same type of curves was given by Jakob Bernoulli in 1704 in his note book: *Thoughts, notes, and remarks*, see [T, p.83] . Even up to now this principle of turning a hanging, heavy surface upside down, is one of the basic construction tools for nowadays architects to actually obtain the shape of light-weighted domes, which are capable of spanning large area, see the interesting

construction device and many beautiful pictures of Frei Otto and collaborators in Otto [O, Bd.I p. 158-161 and Bd.II p. 26-28] .

Returning to the variational problem, we remark that by an obvious coordinate transformation we obtain the equivalent problem:

$$(P) \int_{\Omega} u \sqrt{1 + |Du|^2} \, dx \rightarrow \text{minimum, in the class } C = \{u \geq \lambda, u = \varphi \text{ on } \partial\Omega\}$$

for some suitable function φ .

Here we are primarily interested in problem (P) with $\lambda = 0$ whence the corresponding Euler equation take the form

$$(2) \quad \operatorname{div} \left[\frac{u \, Du}{\sqrt{1 + |Du|^2}} \right] = \sqrt{1 + |Du|^2}$$

or, equivalently,

$$(2') \quad \operatorname{div} \left[\frac{Du}{\sqrt{1 + |Du|^2}} \right] = \frac{1}{u \sqrt{1 + |Du|^2}}$$

Observe that this case differs considerably from the case where $\lambda > 0$ since the integrand $f(x,u,p) := u \sqrt{1 + |p|^2} \geq 0$, i.e. is only singular elliptic (or semi-coercive) .

For $n = 1$ the variational problem (P) is thoroughly investigated in the classical literature on the calculus of variations (see Bolza [B, Beispiel I]) ; the determination of the catenary with the help of extremal principles is due to Jakob Bernoulli 1697):

Let $\Omega = (a,b)$ be an open interval and assume $\varphi(a) = A, \varphi(b) = B$ are prescribed boundary values. One then has to distinguish the cases $\lambda > 0$ and $\lambda = 0$. If $\lambda > 0$ then (P) always has a solution which is either analytic, or of class $C^{1,1}$. In fact, all extremals of (P) are catenaries of the form $y(x) = \alpha \cosh\left(\frac{x-\beta}{\alpha}\right)$ with $\alpha, \beta \in \mathbb{R}$, and

hence a solution consists of a suitable arc of one such catenary, or it consists of two catenary arcs *touching* the obstacle $y = \lambda$ and the corresponding part of the obstacle itself. Physically this corresponds to a hanging chain, which either is "free" (i.e. analytic) or already touches the "ground" without becoming vertical. In case that $\lambda = 0$ problem (P) either has a regular solution, which is given by a suitable catenary $y(x) = \alpha \cosh(\frac{x-\beta}{\alpha})$, or (P) has no nonparametrically defined solution. In this case, however, the corresponding parametric variational problem (in the sense of Weierstraß) admits the so called "discontinuous" solution which has been discovered by Goldschmidt in 1831 in the celebrated paper [G]. This solution consists of suitable parametrizations of the three straight segments $\overline{(a,A) (a,0)}$, $\overline{(a,0) (b,0)}$ and $\overline{(b,0) (b,B)}$. Later on we give a BV-formulation of problem (P) where the boundary condition is replaced by a suitable penalty term in the variational integral. It is in this sense that we may interpret the Goldschmidt solution $u_0 = 0$ as an analytic minimum of the nonparametric functional E_1 , (see the Definition of the Dirichlet problem (D)) which does not assume its boundary values $\varphi(a) = A$ and $\varphi(b) = B$. Note that also the Goldschmidt solution may physically be represented by a hanging chain which touches the ground and has sufficiently large arc length.

Let us also mention that the catenary has a long history prior to its characterization by extremal principles (i.e. its center of gravity is as low as possible). In fact the history starts with Galileo, but then the matter fell asleep for at least fifty years until in 1690 the contest to determine the catenary started with Jakob Bernoulli's proposal in the *Acta Eruditorum*: "To find the curve assumed by a loose string hung freely from two fixed points". This was also the starting point of a long quarrel between the Bernoulli brothers, since the younger Johann succeeded in solving this problem, while there is no evidence that Jakob himself knew the solution in 1690 (although from 1697 to 1698 he succeeded in obtaining the general equation for a flexible line). Besides Johann Bernoulli the problem of the catenary was solved by Leibniz and Huygens (all these authors used some important previous work by Pardies!); for further information on the

interesting history we refer to the excellent explanation in Truesdell [T, in particular, p. 64–88] or Euler [E, p. 64–88].

The equation (2) or (2') and the corresponding variational problem have only recently found some interest among modern analysts, although the problem is known since the time of Lagrange. Böhme, Hildebrandt and Tausch [BHT] proved several existence results for the two-dimensional parametric problem, while Nitsche [N] gave a necessary condition on A_0 for the area-constrained problem. Using previous results from [BHT] for the parametric variational problem, Dierkes [D1] could prove existence of regular solutions for equation (2) in case $n = 2$, assuming appropriate conditions on Ω and the boundary values φ . In fact the structure of the parametric equation permits a suitable maximum principle which makes a reasoning of Rado and Kneser from minimal surface theory applicable to this situation. Typically one has the following result

Theorem 1. (Dierkes [D1]). Let $\Omega \subset \mathbb{R}^2$ be a bounded, convex, Lipschitz domain and suppose that $\varphi \in C^0(\partial\Omega, \mathbb{R}^+)$ satisfies $\varphi(x) \geq \sqrt{\delta^2 + |x|^2}$, for all $x \in \partial\Omega$ and some $\delta > 0$. Then there exists a positive solution $u \in C^\omega(\Omega, \mathbb{R}) \cap C^0(\bar{\Omega}, \mathbb{R})$ of the Dirichletproblem

$$(3) \quad \operatorname{div} \left[\frac{u \operatorname{Du}}{\sqrt{1 + |\operatorname{Du}|^2}} \right] = \sqrt{1 + |\operatorname{Du}|^2} \text{ in } \Omega .$$

$$u = \varphi \text{ on } \partial\Omega .$$

We remark that Theorem 1 implies existence of a solution for (3) provided the prescribed boundary values over the boundary of a convex set lie strictly above the cone $x_3 = \sqrt{x_1^2 + x_2^2}$. Observe also that this cone is a regular solution for (2) in $\Omega \setminus \{0\}$, which has its only singularity at zero. We claim that this cone is a weak Lipschitz solution of (2) i.e. we have for $x_3 = c(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$ the identity

$$(4) \int_{B_R(0)} \left\{ \frac{c \operatorname{Dc} \operatorname{D}\varphi}{\sqrt{1+|\operatorname{Dc}|^2}} + \varphi \sqrt{1+|\operatorname{Dc}|^2} \right\} dx_1 dx_2 = 0$$

for all $\varphi \in C_c^\infty(B_R(0))$, and arbitrary $R > 0$. Indeed this follows immediately by replacing φ in (4) by $\varphi_\delta := \varphi \eta_\delta$, where $\varphi \in C_c^\infty(B_R(0))$ and

$$\eta_\delta = \begin{cases} 1 & \text{on } B_R(0) - B_\delta(0) \\ \frac{2}{\delta} \left[r - \frac{\delta}{2} \right] & \text{on } \overline{B_\delta(0)} - B_{\delta/2}(0), \text{ with } r := \sqrt{x_1^2 + x_2^2} \\ 0 & \text{on } \overline{B_{\delta/2}(0)} \end{cases}$$

and letting $\delta \rightarrow 0$.

More generally, it follows that the cones $c_n^1(x) := \frac{1}{\sqrt{n-1}} [x_1^2 + \dots + x_n^2]^{1/2}$ are classical solutions of (2) on any domain $\Omega - \{0\} \subset \mathbb{R}^n$, which are weak Lipschitz solutions of (2) or (2') in all of Ω i.e.

$$\int_{\Omega} \left\{ \frac{c_n^1 \operatorname{D}c_n^1 \operatorname{D}\varphi}{\sqrt{1+|\operatorname{D}c_n^1|^2}} + \varphi \sqrt{1+|\operatorname{D}c_n^1|^2} \right\} dx = 0, \text{ or}$$

$$\int_{\Omega} \left\{ \frac{\operatorname{D}c_n^1 \operatorname{D}\varphi}{\sqrt{1+|\operatorname{D}c_n^1|^2}} + \frac{\varphi}{c_n^1 \sqrt{1+|\operatorname{D}c_n^1|^2}} \right\} dx = 0$$

for all $\varphi \in C_c^\infty(\Omega)$ and arbitrary $\Omega \subset \mathbb{R}^n$. These cones provide examples that the necessary condition (6) in the following existence result is optimal for large n .

Theorem 2. (Dierkes, Huisken [DH]). Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain of class $C^{2,\alpha}$, $\alpha > 0$, with nonnegative inward mean curvature. Suppose $\varphi \in C^{2,\alpha}(\overline{\Omega})$ satisfies the inequality

$$(5) \inf_{\partial\Omega} \varphi \geq n^{-1} \omega_n^{-1} (1 + \sqrt{2^{n+1}})^2 |\Omega|^{1/n}.$$

Then the Dirichletproblem (3) has a globally regular solution $u \in C^{2,\alpha}(\bar{\Omega})$. Moreover, if $u \in C^{0,1}(\bar{\Omega})$ is a weak, nonnegative, solution of (3) with Lipschitz constant L , then we have

$$(6) \sup_{\partial\Omega} \varphi \geq (1 + L^{-2})^{1/2} \frac{|A|}{H_{n-1}(\partial A)}$$

for every Caccioppoli set $A \subset \Omega$.

Remarks. 1. The necessary condition (6) is only proved for positive $u \in C^{0,1}(\bar{\Omega})$. However, an inspection of the proof in [DH, in particular p.53] shows that $u > 0$ is only needed to apply standard elliptic regularity theory, which in particular yields the sup-estimate $u \leq \sup_{\partial\Omega} \varphi$. On the other hand, this estimate can easily be derived by testing equation (3) against $\psi = \max(u - \sup_{\partial\Omega} \varphi, 0)$ and assuming $u \geq 0$ only.

2. Let us check condition (6) by taking $u(x) = c_n^1(x)$, $\Omega = B_1(0)$ and $A = B_1(0) \subset \mathbb{R}^n$. Then $L = \frac{1}{\sqrt{n-1}}$, $\sup_{\partial B_1} \varphi = \sup_{\partial B_1} u = \frac{1}{\sqrt{n-1}}$, $\frac{|A|}{H_{n-1}(\partial A)} = \frac{1}{n}$ and (6) yields $\frac{1}{\sqrt{n-1}} \geq (1 + n^{-1})^{1/2} \cdot n^{-1} = \frac{1}{\sqrt{n}}$ which is (almost) sharp for large n .

3. Because of the isoperimetric inequality it follows that

$$n^{-1} \omega_n^{-1} |\Omega|^{1/n} \geq \frac{|\Omega|}{H_{n-1}(\partial\Omega)}.$$

Thus it would be desirable to prove existence under (5) replaced by an inequality of the form

$$\inf_{\partial\Omega} \varphi \geq c \frac{|\Omega|}{H_{n-1}(\partial\Omega)} \text{ for a suitable constant } c.$$

4. The proof of Theorem 2 uses Schauders fixed point theorem together with some interesting a priori estimates including a minimum principle based on a method due to Stampacchia.

5. Another approach to the Dirichletproblem (3) via evolutionary existence techniques has been given by Stone [St].

Now let us turn to the variational problem (P). Although the integral $J(u) =$

$\int_{\Omega} u \sqrt{1 + |Du|^2}$ may be defined for functions $u \in BV(\Omega)$ (see e.g. Anzellotti [A]), it is not possible to find a minimum of J in $BV(\Omega)$. To get some idea what might be the right space to work with, let $u \in C^1(\Omega)$, $u \geq 0$, and observe that $u \sqrt{1 + |Du|^2} = \sqrt{v + \frac{1}{4}|Dv|^2}$, where $v = u^2$. This simple observation leads to the definition of the classes $BV_2^+(\Omega)$ and, more generally, $BV_{1+\alpha}^+(\Omega)$ for any $\alpha > 0$.

Definition. $BV_2^+(\Omega) := \{u \in L_2(\Omega) : u \geq 0, u^2 \in BV(\Omega)\}$, and for any $\alpha > 0$, $BV_{1+\alpha}^+(\Omega) := \{u \in L_{1+\alpha}(\Omega) : u \geq 0, u^{1+\alpha} \in BV(\Omega)\}$. Furthermore we define for $u \in BV_2^+$ or $u \in BV_{1+\alpha}^+$ resp.

$$\int_{\Omega} u \sqrt{1 + |Du|^2} := \sup \left\{ \int_{\Omega} [u g_{n+1} + \frac{1}{2} u^2 \sum_{i=1}^n D_i g_i] dx : g \in C_c^1(\Omega, \mathbb{R}^{n+1}), |g(x)| \leq 1 \right\}$$

and

$$\int_{\Omega} u^\alpha \sqrt{1 + |Du|^2} := \sup \left\{ \int_{\Omega} [u^\alpha g_{n+1} + \frac{1}{1+\alpha} u^{1+\alpha} \sum_{i=1}^n D_i g_i] dx : g \in C_c^1(\Omega, \mathbb{R}^{n+1}), |g(x)| \leq 1 \right\}.$$

This definition has several important features, some of which are reflected in the following

Proposition 1. If $0 \leq u \in H_1^1(\Omega) \cap L_{\infty}(\Omega)$ then we have $\int_{\Omega} u^\alpha \sqrt{1 + |Du|^2} = \int_{\Omega} u^\alpha \sqrt{1 + |Du|^2} dx$. Furthermore, let $u \in BV_{1+\alpha}^+(\Omega)$ and $U := \{(x, t) \in \Omega \times \mathbb{R}^+ : 0 \leq t < u(x)\}$, then we obtain

$$\int_{\Omega \times \mathbb{R}^+} x_{n+1}^\alpha |D\varphi_U| = \int_{\Omega} u^\alpha \sqrt{1 + |Du|^2}, \text{ where } \varphi_U \text{ denotes the characteristic function of } U \text{ and}$$

$$\int_{\Omega \times \mathbb{R}^+} x_{n+1}^\alpha |D\varphi_{\mathcal{H}}| := \sup \left\{ \int_{\Omega \times \mathbb{R}^+} \varphi_{\mathcal{H}} \operatorname{div}[x_{n+1}^\alpha g(x)] dx : g \in C_c^1(\Omega \times \mathbb{R}^+, \mathbb{R}^{n+1}), |g(x)| \leq 1 \right\}$$

(For details and proofs we refer to [BD] and [D2].) □

Motivated by the trace formula for BV-functions we now reformulate the variational problem (P) so as to obtain the following Dirichlet problem (D): Let $\Omega \subset \mathbb{R}^n$ be a given Lipschitz domain and suppose that $\varphi \in L_{1+\alpha}(\partial\Omega)$, $\varphi \geq 0$, are prescribed boundary values.

Minimize

$$(D) \quad E_\alpha(u) = \int_{\Omega} u^\alpha \sqrt{1 + |Du|^2} + \frac{1}{1+\alpha} \int_{\partial\Omega} |u^{1+\alpha} - \varphi^{1+\alpha}| d\mathcal{H}_{n-1}$$

in the class $BV_{1+\alpha}^+(\Omega)$.

Then we have the following result.

Theorem 3. (Bemelmans, Dierkes [BD], Dierkes [D2]). *There exists a solution $u \in BV_{1+\alpha}^+(\Omega)$ of the problem (D). If $n \leq 6$ then $u \in C^0(\Omega) \cap C^\omega(\{u > 0\})$. Furthermore, if $\alpha = 1$ and $\sup_{\partial\Omega} \varphi < \frac{|\Omega|}{\mathcal{H}_{n-1}(\partial\Omega)}$ then $|\{u = 0\}| > 0$.*

Remarks. 1. The proof of Theorem 3 uses methods from nonparametric minimal surface theory as well as the deep regularity result for one-codimensional minimizing currents in Riemannian manifolds.

2. If $\alpha = 1$, one even has the estimate $|\{u = 0\}| \geq |\Omega| - \int_{\partial\Omega} u d\mathcal{H}_{n-1}$, which shows that singular solutions really do occur under suitable boundary conditions.

Concerning boundary regularity one has the following

Theorem 4. (Dierkes [D3]). *Suppose that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain which is mean convex near $x_0 \in \partial\Omega$ and let φ be continuous at x_0 . Then any solution u of (D)*

satisfies $\lim_{x \rightarrow x_0} u(x) = \varphi(x_0)$. Furthermore if $\Omega \in C^3$ is mean convex, $0 < \varphi \in C^{1,\alpha}(\partial\Omega)$ and $u \in C^2(\Omega_\epsilon) \cap C^0(\overline{\Omega}_\epsilon)$, where $\Omega_\epsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) < \epsilon\}$, then it follows that $u \in C^{0,1}(\overline{\Omega}_\epsilon)$.

Note that the condition $u \in C^2(\Omega_\epsilon) \cap C^0(\overline{\Omega}_\epsilon)$ is satisfied, if $n \leq 6$ for example.

The proof of the continuity part of Theorem 4 uses the fact that the current $T = \partial[U]$ which is associated to the nonnegative subgraph U of a minimizer $u \in BV_{1+\alpha}^+(\Omega)$ locally minimizes mass in a submanifold $N \subset \mathbb{R}^{n+k}$. This property, a suitable maximum principle, and the fact that there are no non trivial area minimizing cones in \mathbb{R}^{n+k} having their support in a half space $H \subset \mathbb{R}^{n+k}$, eventually yields the continuity result. The global Lipschitz regularity is established with the help of suitable barrier functions. Here a new a priori estimate of the gradient for solutions of (2) or (2') is needed. We remark that this estimate does not follow from the results in [TN] or [LU].

Besides the continuity of solutions to (D) no more is known about higher interior regularity. As we have seen at the beginning, there are Lipschitz-solutions of equation (2) which are not everywhere differentiable. Hence one might conjecture that (D) also has solutions which are not differentiable. In fact the situation is even worse, since we have the following

Theorem 5. (Dierkes [D4],[D5],[D2]). Let $\alpha, R > 0, n \geq 2, n \in \mathbb{N}$ be arbitrary. Then there exist a $\delta > 0$, some $r \in (0, R)$ and a function $w \in C^{0,1/2}(T_{R,r})$, $T_{R,r} := B_R(0) - \overline{B_r(0)}$, which minimizes

$$E_\alpha(u) = \int_{T_{R,r}} u^\alpha \sqrt{1 + |Du|^2} + \frac{1}{1+\alpha} \int_{\partial T_{R,r}} |u^{1+\alpha} - w^{1+\alpha}| dH_{n-1}$$

in $BV_{1+\alpha}^+(T_{R,r})$. Furthermore, we have $w = \delta$ on $\partial B_R(0)$, $w = 0$ on $\partial B_r(0)$ and $w \notin C^{0,1/2+\epsilon}(T_{R,r})$ for any $\epsilon > 0$.

The proof of this result employs ideas from classical "field theory" developed by Weierstraß for one dimensional variational problems and extended to the higher dimensional case by H.A. Schwarz, see [S, in particular p.224] and [DHKW, chapter 2.8]. (Nowadays the word "field" has been replaced by several authors by "foliation" or "calibration", denoting more or less the same notions.). Here we work with a variant of a method due to Bombieri, De Giorgi, Giusti [BDG]. The argument is to construct a *local, singular field* in $\mathbb{R}^n \times \mathbb{R}^+$ with the help of an o.d.e. system, and finally, to prove that each element of the field merely is of class $C^{0,1/2}$ and minimizes E_α in $BV_{1+\alpha}^+$. For details we refer to [D2], [D4], [D5].

We have already mentioned that the cones c_n^1 are stationary for the integral $E_1(u) = \int u \sqrt{1 + |Du|^2} dx$. One sees immediately that the same holds true for the cones

$$c_n^\alpha(x) := \sqrt{\frac{\alpha}{n+1}} [x_1^2 + \dots + x_n^2]^{1/2} \text{ and the integral}$$

$$E_\alpha(u) = \int u^\alpha \sqrt{1 + |Du|^2} dx \text{ for any positive } \alpha.$$

Hence the question arises whether these cones are stable or even minimizing. Clearly, this requires the construction of a global, singular field about the cone c_n^α .

Theorem 6. (Dierkes [D2],[D4],[D5]). *Suppose that either*

$$\alpha + n \geq 7, \text{ and } \alpha \geq 2, n \geq 3 \text{ or } \alpha + n \geq 8, \text{ and } \alpha \geq 1, n \geq 2 \text{ hold true.}$$

Then the cones c_n^α minimize E_α in $BV_{1+\alpha}^+$. If $\alpha = 1, n \leq 6$, or $n = 2, \alpha \leq 5$ then c_n^α do not minimize E_α .

The proof of the non-minimizing property of c_n^α in the cases $\alpha = 1, n \leq 6$ or $n = 2, \alpha \leq 5$ respectively, again uses field theory. This time a field is constructed which intersects the cone c_n^α . Assuming that c_n^α were minimizing one would obtain a contradiction to the regularity result in Theorem 3 by constructing a minimizer v

which would not be differentiable in the set $\{x : v(x) > 0\}$.

Finally we are concerned with the question of stability. To this end let $\mathbb{M} \subset \mathbb{R}^n \times \mathbb{R}^+$ be an n -dimensional submanifold and let

$$E_\alpha(\mathbb{M}) := \int_{\mathbb{M}} x_{n+1}^\alpha dH_n$$

denote the parametric energy functional. The first and second variations of E_α are given by

$$(7) \quad \delta E_\alpha(\mathbb{M}, X) := \int_{\mathbb{M}} x_{n+1}^\alpha \{\operatorname{div} X + \alpha x_{n+1}^{-1} X_{n+1}\} dH_n(x)$$

and

$$\delta^2 E_\alpha(\mathbb{M}, \xi \nu) := \int_{\mathbb{M}} x_{n+1}^\alpha \{|\nabla \xi|^2 - \alpha x_{n+1}^{-2} \nu_{n+1}^2 \xi^2 - |A|^2 \xi^2\} dH_n(x)$$

respectively.

Here $X(x) = (X_1(x), \dots, X_n(x))$ is a vector field with compact support on \mathbb{M} , $\nu = (\nu_1, \dots, \nu_{n+1})$ denotes the unit normal on \mathbb{M} , $\xi \in C_c^1(\mathbb{M}, \mathbb{R})$ is arbitrary, and $|A|$ denotes the length of the second fundamental form of \mathbb{M} .

Let $C_n^\alpha := \{(y, c_n^\alpha(y)) : y \in \mathbb{R}^n\}$ denote the graph of the cone c_n^α , then an easy calculation yields

$$|A|^2 = \alpha |x|^{-2} \text{ for all } x \in C_n^\alpha - \{0\}.$$

Using the test function $X(x) := x \cdot |x|^{-2} \xi^2$, where $\xi \in C_c^1(C_n^\alpha - \{0\})$, we infer from (7) the following result:

Theorem 7. (Dierkes [D6]). C_n^α are stable, i.e. $\delta^2 E_\alpha(C_n^\alpha) \geq 0$, if $\alpha + n \geq 4 + \sqrt{8}$.

This result is in fact optimal, since we have

Theorem 8. ([Dierkes [D6]]). Suppose $C \subset \mathbb{R}^n \times \mathbb{R}^+$ is a stable n -dimensional cone with singularity at zero and let $\alpha + n < 4 + \sqrt{8}$. Then C is a hyperplane which is perpendicular to $\{x_{n+1} = 0\}$.

An important ingredient in the proof of Theorem 8 is an estimate for the Laplacian of $|A|^2$ due to Schoen, Simon, Yau [SSY]. Finally we remark that Theorems 6 and 7 yield the following interesting

Corollary. Let $n = 2$ and $\alpha \in [2 + \sqrt{8}, 5]$. Then the cones $c_2^\alpha(x_1, x_2) = \sqrt{\alpha} (x_1^2 + x_2^2)^{1/2}$ are stable but not minimizing with respect to E_α .

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