RECENT EXISTENCE AND UNIQUENESS RESULTS IN SHADING ANALYSIS

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Abstract. A smooth object depicted in a monochrome image will often exhibit brightness variation, or shading. Of interest in computer vision is the problem of how object shape may be recovered from such an image. When the imaging conditions are such that an overhead point-source illuminates a smooth Lambertian surface, the problem may be formulated as that of finding a solution to an eikonal equation. This article will focus on the existence and uniqueness of such solutions, reporting recent results obtained. With regard to existence, shading patterns are exhibited for which there is no corresponding object shape. Specifically, a necessary and sufficient condition is presented for a circularly-symmetric eikonal equation to admit exclusively unbounded solutions; additionally, a sufficient condition is given for an eikonal equation to have no solution. In connection with uniqueness, we consider eikonal equations, defined over a disc, such that the Euclidean norm of the gradient of any solution is circularly-symmetric, vanishes exactly at the disc centre, and diverges to infinity as the circumference of the disc is approached. Contrary to earlier expectations in the area, a class of such eikonal equations is shown to possess simultaneously circularly-symmetric and non-circularly-symmetric bounded smooth solutions.

1. INTRODUCTION

The eikonal equation

$$u_x^2 + u_y^2 = \mathcal{E}(x, y), \tag{1}$$

which arises naturally in wavefront analysis and in the development of special methods for integrating Hamilton's equations (the Jacobi-Hamilton method), has long attracted the attention of physicists and mathematicians. More recently, there has been a resurgence of interest in the eikonal equation as a result of its applicability in an area of computer vision. Issues considered in the latter context are those of existence and uniqueness of solutions to an eikonal equation over a given domain. In this paper, we offer insight into these issues by presenting a number of (non-)existence and (non-)uniqueness results of significance for the foundations of computer vision.

A monochrome photograph of a smooth object will typically exhibit brightness variation, or *shading*. Of interest to researchers in computer vision is the problem of how object shape may be extracted from image shading. This *shape-from-shading problem* has been shown by Horn ([6]; see also Horn and Brooks [9, pp. 123-172], where the same article appears in a collection of seminal papers in the field) to correspond to that of solving a first-order partial differential equation. Specifically, one seeks a function $u(x, y)$, representing surface depth in the direction of the z-axis, satisfying the *image irradiance equation*

$$R(u_x, u_y) = E(x, y)$$

over Ω. Here, R is a known function (the so-called *reflectance map)* capturing the illumination and surface reflecting conditions, E is an image formed by (orthographic) projection of light along the z-axis onto a plane parallel to the xy-plane, and Ω is the image domain.

An interesting case obtains when the reflectance map is specified so as to correspond to the situation in which an overhead, distant point-source illuminates a *Lambertian surface.* A small portion of such a surface acts as a perfect diffuser appearing equally bright from all directions. At first, this might seem to imply that Lambertian surfaces cannot exhibit other than constant shading. However, a curved object will, in general, receive illumination that differs in strength across the surface due to surface foreshortening, and it is this that will be responsible for variation in image brightness. If a small surface portion with normal direction $(-u_x, -u_y, 1)$ is illuminated by a distant, overhead point-source of unit power in direction $(0, 0, 1)$, then, according to Lambert's law, the emitted radiance and, in view of the aforementioned assumptions, the reflectance map are given by the cosine of the angle between the two directions, namely $(u_x^2 + u_y^2 + 1)^{-1/2}$. Thus, if $E(x, y)$ denotes the corresponding image, the image irradiance equation for the above situation takes the form

$$(u_x^2 + u_y^2 + 1)^{-1/2} = E(x, y).$$

Noting that $0 < E(x, y) \leq 1$, we may safely let $\mathcal{E}(x, y) = \big(E(x, y)\big)^{-2} - 1$ and rewrite the above equation as (1).

Given an image, the natural question arises as to whether it actually corresponds to a physically-realisable shape. For Lambertian shading with illumination conditions as above, this reduces to the problem of solving (1) over a given domain. It was Horn [7] who first posed this problem and who coined the term *impossible shading* for a brightness pattern that could not be the image of a smooth surface.

In this article, we present two different classes of images for which there are no genuine shapes. Initially, we reveal a class of images for which only unbounded (and therefore physically-unrealisable) shapes exist. Next, we present a class of images exhibiting shading for which neither bounded nor unbounded shapes exist. This portion of the article will refine an approach due to Horn [8].

Given an image of some particular shape, another question arises as to whether it could also be the image of other shapes. For Lambertian surfaces illuminated by an overhead point-source, this reduces to the problem of finding all solutions of (1) over some domain. Note that if u is a solution of (1), then so too is any member of the family $\pm u + k$, where k is an arbitrary constant. Thus, the image of the surface S formed by the graph of u will be preserved under either a depth-shift of S along the z-axis, the inversion of S with respect to the xy-plane, or a combination of these transformations. These surfaces may clearly be said to possess a common shape. Of interest in computer vision is the situation of essential uniqueness in which a family of the type specified above constitutes, within some class of functions, the complete set of solutions to an equation of the form given in (1).

Uniqueness of this kind has been demonstrated for equation (1) in which

$$\mathcal{E}(x,y) = \frac{x^2+y^2}{1-x^2-y^2} \,.$$

Deift and Sylvester [5], and independently Brooks [1], proved that $\pm(1-x^2-y^2)^{1/2} + k$ are the only C^2 solutions to this equation over the unit disc $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$. All of these solutions are hemispherical in shape. Interestingly, this result fails in the class of C^1 solutions.

In an effort to obtain a more general result, Bruss ([4]; see also [9, pp. 69-88]), in perhaps the major work in the uniqueness area, asserted the following: if R is a positive number, $D(R)$ is the disc in the xy-plane with radius R centred at the origin, and f is a continuous function on $[0,R)$ of class C^2 over $(0,R)$ such that

(i) $f(0) = 0$ and $f(r) > 0$ for $0 < r < R$,

(ii) $\lim_{r\to 0} f'(r) = 0$, $\lim_{r\to 0} f''(r)$ exists and is positive,

(iii) $\lim_{r\to R} f(r) = +\infty$,

then all solutions of class C^2 to (1) in $D(R)$ with

$$\mathcal{E}(x,y) = f(\sqrt{x^2+y^2}) \tag{2}$$

take the form

$$\pm \int_0^{\sqrt{x^2+y^2}} \sqrt{f(\sigma)}\, d\sigma + k,$$

and so are circularly-symmetric with common shape. Here, conditions (i) and (ii) ensure that the origin is the only (singular) point at which $\mathcal{E}$ vanishes to second order, while condition (iii) implies that the Euclidean norm of the gradient of any solution to (1) diverges to infinity as the circumference of $D(R)$ is approached. In this paper, we shall show that this assertion is invalid. Specifically, we shall reveal a class of functions f, having the above properties, for which the corresponding eikonal equations have a bounded, non-circularly-symmetric solution of class C^2.

As we are concerned here only with the reporting of results, theorems in this paper will not be accompanied by proofs. For extended versions of this work, including proofs, the interested reader is referred to [2] and [3].

2. EXISTENCE RESULTS

2.1. IMAGES WITHOUT BOUNDED SOLUTION

Let R be either a positive number or $+\infty$. Let f be a non-negative continuous function on the interval $[0, R)$ vanishing exactly at zero. Consider equation (1) with $\mathcal{E}$ given by (2). With this special form of $\mathcal{E}$, the class of circularly-symmetric solutions is readily determined. Each solution in this class takes the form $\pm U + \text{const}$, where

$$U(x, y) = \int_0^{\sqrt{x^2+y^2}} \sqrt{f(\sigma)}\, d\sigma.$$

Note that it is critical that f vanish at zero so as to ensure the differentiability of U at the origin of the xy-plane. Our eikonal equation may also admit non-circularly-symmetric solutions. The function $u(x, y) = xy$ provides an example of such a solution when $f(r) = r^2$ and $R = +\infty$. Unlike the class of circularly-symmetric solutions, the class of all non-circularly-symmetric solutions is not easily specified.

A condition on f guaranteeing that all solutions to the corresponding eikonal equation are unbounded may readily be formulated. Clearly, in the class of circularly-symmetric solutions this sufficient condition is

$$\int_0^R \sqrt{f(\sigma)}\, d\sigma = +\infty. \tag{3}$$

It is less evident, though true, that the same condition is sufficient in the general case. In fact, we have the following:

THEOREM 1. *Let f be a non-negative continuous function on $[0, R)$ vanishing exactly at zero and satisfying (3). Then there is no bounded C^1 solution in $D(R)$ to (1) with $\mathcal{E}$ given by (2).*

Interestingly, condition (3) is not only sufficient but also necessary for the unboundedness of all solutions to the equation in question. We have the following theorem:

THEOREM 2. *Let f be a non-negative continuous function in $[0, R)$ vanishing exactly at zero and satisfying*

$$\int_0^R \sqrt{f(\sigma)}\, d\sigma < +\infty.$$

Then every solution in $D(R)$ to (1) with $\mathcal{E}$ given by (2) is bounded. Moreover, if u is any such solution, then

$$\sup_{(x,y)\in D(R)} u(x,y) - \inf_{(x,y)\in D(R)} u(x,y) \leq 2\int_0^R \sqrt{f(\sigma)}\, d\sigma.$$

Observe that whether the integral $\int_0^R \sqrt{f(\sigma)}\, d\sigma$ is finite or infinite depends exclusively on the behaviour of f near R. The integral will be infinite if, for example, $f(r)$ diverges to infinity sufficiently rapidly as r tends to R. This means that, in the context of real images of Lambertian surfaces illuminated by an overhead point-source, a circularly-symmetric image cannot be derived from a genuine shape if it gets dark too quickly as the image boundary is approached. Note also that the above integral may be finite or infinite under the condition that R is finite and $\lim_{r\to R} f(r) = +\infty$, which implies that the Euclidean norm of the gradient of any solution to (1) diverges to infinity as the circumference of $D(R)$ is approached. This is of interest in computer vision in that it relates to the notion of an *occluding boundary*. The following examples show that the integral may be finite or infinite with the above condition being met: if $R = \pi/2$ and $f(r) = \tan^2 r$, then the integral is infinite, and so no bounded solutions to (1) can exist; on the other hand, if $R = 1$ and $f(r) = r^2(1-r^2)^{-1}$ (the image of the unit sphere centered at the origin), then the integral is finite, and so all solutions to (1) must be bounded.

Comparison of Theorems 1 and 2 reveals the following remarkable dichotomy: either all solutions to equation (1) with $\mathcal{E}$ given by (2) are bounded, or all solutions are unbounded, according to whether the integral $\int_0^R \sqrt{f(\sigma)}\, d\sigma$ is finite or infinite, respectively. The question then arises as to whether there is an eikonal equation having both an unbounded and a bounded solution. This is answered in the affirmative when we note that, in the semidisc $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1, x > 0\}$, the bounded function $\arctan(yx^{-1})$ and the unbounded function $\ln\sqrt{x^2+y^2} + 1$ both satisfy the eikonal equation $u_x^2 + u_y^2 = (x^2+y^2)^{-1}$. The graphs of these functions are displayed in Figures 1a and 1b, respectively.

2.2. IMAGES WITHOUT SOLUTION

We now establish the existence of images $\mathcal{E}$ for which there is no solution to equation (1). In addition, we offer some insight into the result. The theorem presented below is a refinement of that due to Horn [8]; its proof, to be found in [2], elaborates an outline also due to Horn.

THEOREM 3. *Let Ω be a bounded domain in the xy-plane with boundary $\partial\Omega$ being a piecewise C^1 curve of length $\ell_{\partial\Omega}$. Let (x_0, y_0) be a point in Ω and r be a positive number such that the closed disc $\bar{D}(x_0, y_0, r)$ of radius r centered at (x_0, y_0) is contained in Ω.*

Suppose $\mathcal{E}$ is a non-negative continuous function on the closure of Ω, positive in Ω, such that

$$4r\sqrt{\mathcal{E}_1} > \ell_{\partial\Omega}\sqrt{\mathcal{E}_2}, \tag{4}$$

where $\mathcal{E}_1 = \min\{\mathcal{E}(x,y) : (x,y) \in \bar{D}(x_0,y_0,r)\}$ and $\mathcal{E}_2 = \max\{\mathcal{E}(x,y) : (x,y) \in \partial\Omega\}$. Then there is no C^1 solution to (1) in Ω.

Note that the theorem is of local character: if Ω is a subset of a domain Δ and $\mathcal{E}$ is a non-negative function on Δ whose restriction to Ω satisfies (4) for some choice of $\bar{D}(x_0,y_0,r)$ in Ω, then, obviously, there is no C^1 solution to (1) in Δ. Reformulated in terms of Lambertian shading, this locality property can be expressed as saying that no genuine image can admit too dark a spot on too bright a background, assuming that the background does not contain a point having unit brightness. The precise balance between the qualifications "too dark" and "too bright" is, of course, given by condition (4). An example of shading without shape is given in Figure 2.

Further insight may be gained by considering the following. Suppose that a planar rubber sheet is inclined slightly away from the horizontal, and that a coin is glued to the underside of the sheet. Imagine twisting the coin so as to make a portion of the sheet more steeply inclined (see Figure 3a). An image of the sheet will now exhibit a dark area surrounded by a bright background. This area may be made arbitrarily dark by a further twisting of the coin, while the background may be brightened by having the sheet inclined more closely to the horizontal. We therefore appear to be in a position to formulate a contradiction to Theorem 3. However, in attempting to generate a specific counter-example in this way, it soon becomes apparent that the image of the steep area cannot be made sufficiently dark and large without the surface exhibiting at least one stationary point where $u_x = u_y = 0$ (see Figure 3b). Such a point would result in a violation of the condition in Theorem 3 that $\mathcal{E} > 0$ in Ω. The theorem therefore survives intact.

3. UNIQUENESS RESULTS

3.1. SOLUTIONS OVER QUADRANTS AND DISCS

The construction of non-circularly-symmetric solutions to eikonal equations with $\mathcal{E}$ given by (2) will be divided into several steps. The graph of any such solution will take the form of a saddle having four regions of monotonicity spread out over four quadrants in the xy-plane determined by the lines $x = \pm y$. First, we shall construct a portion of a typical solution over the quadrant containing the positive x-halfaxis; the three remaining portions will easily be generated from this one. Next, we shall specify a class of functions f for which the portions over all four quadrants can be smoothly pasted together and shall describe the corresponding process of glueing. Finally, we shall discuss the differentiability properties of the solutions obtained.

We now undertake the first stage of the construction.

THEOREM 4. *Let R be either a positive number or $+\infty$. Let f be a positive function of class C^2 on $(0, R)$ such that*

$$\lim_{r\to 0} f(r) = 0, \tag{5}$$

$$\lim_{r\to 0} \frac{f'(r)}{r} = 2, \tag{6}$$

and

$$r\left[f''(r)f(r) - \left(f'(r)\right)^2\right] + f(r)f'(r) \geq 0 \tag{7}$$

for $0 < r < R$. Then there is a unique solution u of class C^2 to (1), with $\mathcal{E}$ given by (2), defined over the quadrant

$$Q_1(R) = \{(x,y) \in \mathbb{R}^2 : |y| < x,\ 0 < x < R\},$$

such that u is positive in the upper xy-halfplane and vanishes at the positive x-halfaxis. Moreover, $u(x,-y) = -u(x,y)$ for each (x,y) in $Q_1(R)$.

Proceeding to the next stage of the construction, let R be either a positive number or $+\infty$, and let

$$\begin{aligned} Q_2(R) &= \{(x,y) \in \mathbb{R}^2 : |x| < y, 0 < y < R\},\\ Q_3(R) &= \{(x,y) \in \mathbb{R}^2 : |y| < -x, -R < x < 0\},\\ Q_4(R) &= \{(x,y) \in \mathbb{R}^2 : |x| < -y, -R < y < 0\}. \end{aligned}$$

Given a positive function f of class C^2 on $(0,R)$ satisfying (5), (6), and (7), let u be the solution to (1), with $\mathcal{E}$ given by (2), defined over $Q_1(R)$ that has the properties stated in Theorem 4. Let

$$U(x,y) = \begin{cases} u(x,y), & \text{if } (x,y) \in Q_1(R);\\ u(y,x), & \text{if } (x,y) \in Q_2(R);\\ u(-x,-y), & \text{if } (x,y) \in Q_3(R);\\ u(-y,-x), & \text{if } (x,y) \in Q_4(R);\\ \int_0^{\sqrt{2}|x|} \sqrt{f(\sigma)}\,d\sigma, & \text{if } -R < x = y < R;\\ -\int_0^{\sqrt{2}|x|} \sqrt{f(\sigma)}\,d\sigma, & \text{if } -R < x = -y < R. \end{cases}$$

We have the following.

THEOREM 5. *Let R be either a positive number or $+\infty$. Let f be a positive function of class C^2 over $(0,R)$ satisfying (5), (6), and (7). Suppose, moreover, that for some $0 < r_0 < R$, f is of class C^4 over $[0, r_0)$ and of class C^5 over $(0, r_0)$, and that $f^{(5)}$ is bounded in $(0, r_0)$. Then U is a solution to (1), with $\mathcal{E}$ given by (2), of class C^1 over $D(R)$ and of class C^2 over $D(R)\backslash\{(0,0)\}$.*

It is interesting to consider whether or not the solution U is of class C^2 over the entire disc $D(R)$. The following theorem specifies certain conditions on the function f that must be met for the answer to be in the affirmative.

THEOREM 6. *Let R be either a positive number or $+\infty$. Let f be a positive function of class C^2 over $(0, R)$ and, for some $0 < r_0 < R$, of class C^4 over $[0, r_0)$ satisfying (5), (6), and (7). Suppose that U is of class C^2 over $D(R)$. Then $f'''(0) = f^{(4)}(0) = 0$.*

We conclude this section with a simple sufficient condition for U to be of class C^2 over $D(R)$.

THEOREM 7. *Let R be either a positive number or $+\infty$. Let f be a positive function that is of class C^2 over $(0, R)$, satisfies (7), and, for some $0 < r_0 < R$, $f(r) = r^2$ whenever $0 \leq r < r_0$. Then U is of class C^2 over $D(R)$.*

3.2. REFINEMENTS

We now specify certain classes of functions f to which the results of the previous section are applicable. One of these classes will be used to generate a counter-example to Bruss' assertion mentioned in the introduction.

THEOREM 8. *Let R be a positive number. Let $g : (0, R) \to [0, 1)$ be a function of class C^2 such that g' and g'' are non-negative, g' is bounded in $(0, r_0)$ for some $0 < r_0 < R$, and $\lim_{r \to 0} g(r) = 0$. Then the function f defined by*

$$f(r) = \frac{r^2}{1 - g(r)} \quad (0 < r < R) \tag{8}$$

is of class C^2 and satisfies (5), (6), and (7).

Notice that if we let $R = 1$ and $g(r) = r^2$ for $0 < r < 1$, then the function f given by (8), namely $r^2(1 - r^2)^{-1}$, corresponds to the image of the unit hemisphere. Let U be the corresponding (non-circularly-symmetric) solution to (1) with $\mathcal{E}$ as in (2) (see Figure 4). Since $f^{(4)}(0) = 1$, it follows from Theorem 6 that U is not of class C^2. Of course, this result can independently be inferred from uniqueness results, mentioned in the introduction, due to Deift and Sylvester, and Brooks.

Let R be a positive number. Let r_0 and r_1 be such that $0 < r_0 < r_1 < R$. Let $\varphi : (0, R) \to [0, 1]$ be a continuous function vanishing on $(0, r_0]$ and equal to 1 on $[r_1, R)$. For each $0 < r < R$, set

$$g(r) = c \int_0^r \varphi(x)(r - x)\, dx,$$

where

$$c = \left[\int_0^R \varphi(x)(R - x)\, dx \right]^{-1}.$$

Clearly, g is of class C^2 and, for each $0 < r < R$,

$$g'(r) = c \int_0^r \varphi(x)\, dx$$

and $g''(r) = c\varphi(r)$. Accordingly, g meets the conditions specified in Theorem 8. Let f be the function given by (8) and U be the corresponding solution to (1) in which $\mathcal{E}$ is given by (2). Then, $\lim_{r\to R} g(r) = 1$ and so $\lim_{r\to R} f(r) = +\infty$. Since g vanishes on $(0, r_0)$, it follows that $f(r) = r^2$ for $0 < r \leq r_0$. Thus, by Theorem 7, U is of class C^2 over $D(R)$. A straightforward computation shows that $\int_0^R \sqrt{f(r)}\, dr < +\infty$. This jointly with Theorem 2 implies that U is bounded.

It is now clear that our goal expressed in the introduction is achieved: the pair (f, U) provides a desired counter-example to Bruss' assertion.

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REFERENCES,

[1] M. J. Brooks, Two results concerning ambiguity in shape from shading, *in* "Proceedings of the National Conference on Artificial Intelligence, Washington, D.C., August 22-26, 1983" (The American Association for Artificial Intelligence, Sponsor), pp. 36-39.

[2] M. J. Brooks, W. Chojnacki and R. Kozera, Shading without shape, *Quart. Appl. Math.*, in press.

[3] M. J. B rooks, W. Chojnacki and R. Kozera, Circularly-symmetric eikonal equations and non-uniqueness in computer vision, *J. Math. Anal. Appl.*, in press.

[4] A. R. Bruss, The eikonal equation: some results applicable to computer vision, *J. Math. Phys. (5)* **23** (1982), 890-896.

[5] P. Deift and J. Sylvester, Some remarks on the shape-from-shading problem in computer vision, *J. Math. Anal. Appl. (1)* **34** (1981), 235-248.

[6] B. K. P. Horn, Obtaining shape from shading information, *in* "The Psychology of Computer Vision" (P. H. Winston, Ed.), pp. 115-155, McGraw Hill, New York, 1975.

[7] B. K. P. Horn, *private communication,* 1987.

[8] B. K. P. Horn, *private communication,* 1989.

[9] B. K. P. Horn and M. J. Brooks, Eds., "Shape from Shading", MIT Press, Cambridge, Mass., 1989.

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