The maximum principle for degenerate parabolic PDEs with singularities

Hitoshi Ishii

1. Introduction

This is a preliminary version of [7]. Here we shall be concerned with the degenerate parabolic partial differential equation (PDE in short)

$$u_t + F(Du, D^2u) = 0 \quad \text{in } \Omega \times (0,T). \tag{1}$$

Here and in what follows Ω is a domain of $\mathbf{R}^N$, $T > 0$ is a given constant, u represents the real unknown function on $\Omega \times (0,T)$ and F is a real function on $\mathbf{R}^N \times \mathbf{S}^N$, where $\mathbf{S}^N$ denotes the set of real $N \times N$ symmetric matrices.

Recent developments have revealed that equations (1) with F having singularities or discontinuities are important in the study of generalized evolutions of hypersurfaces, especially, in the level set approach.

Chen, Giga and Goto [3] and Evans and Spruck [5] initiated the level set approach, on a firm mathematical basis, to evolutions of hypersurfaces driven by their mean curvature or by some other geometric quantitives alike. In the case of evolution by mean curvature, F turns out to be

$$F(p,X) = -\mathrm{tr}\left(I - \frac{p \otimes p}{|p|^2}\right) X.$$

Thus in their approach $F(p,X)$ is not defined for $p = 0$.

According to Angenent and Gurtin [1, 2], equations (1) with F discontinuous in a set of directions of p's arise in a mathematical model for the dynamics of a melting solid, where the boundary between the solid and liquid phases gives an evolving hypersurface.

In this model F typically has the form:

$$F(p,X) = \mathrm{tr}\left\{D^2H\left(\frac{p}{|p|}\right)\left(I - \frac{p\otimes p}{|p|^2}\right)X\right\}, \tag{2}$$

where H is a real, positively homogeneous function of degree 1. The convexity of H corresponds to the degenerate parabolicity of (1), i.e.

$$F(p,X) \le F(p,Y) \quad \text{if } X \ge Y.$$

If $H \in C^2(\mathbf{R}^N \setminus \{0\})$ and H is convex, then F has singularities only for $p = 0$ and the situation is in the case which was studied by Chen, Giga and Goto [3].

In the above model the function H in (2) describes a quantity which may be called as interface tension energy and may be nonconvex in the viewpoint of physics. Then (1) is not parabolic and the initial (-boundary) value problem for (1) is not well-posed. In this situation an appropriate replacement of H, suggested by [1, 2], is to use the convex envelope $\widehat{H}$ of H. The reader may find some arguments which give justifications for this replacement of H in [1, 2], [6]. Even if H is smooth, $\widehat{H}$ is not necessarily in $C^2(\mathbf{R}^N \setminus \{0\})$.

Motivated by the above model, Ohnuma and Sato [8] and Gurtin, Soner and Souganidis [6] recently studied PDEs (1) with $F(p,X)$ which are discontinuous in a finite number of directions of p's. This kind of singuralities are typical for $N = 2$. When $N > 2$, singularities of F typically form a continuum of directions of p's.

Here we shall establish the maximum principle for (1) with $F(p,X)$ having discontinuities in a continuum of directions of p's and indicate its application to motion of a phase interface.

2. Main results

We begin with the explanation of our assumptions on F.

(A1) There is a C^2 submanifold (without boundary) M of $S^{N-1} = \{x \in \mathbf{R}^N \mid |x| = 1\}$ of dimension $d \in \{0, \cdots, N-2\}$ such that F is continuous on $(\mathbf{R}^N \setminus \overline{R}_+ M) \times \mathbf{S}^N$.

Here and henceforth $\mathbf{R}_+$ denotes the set $(0, \infty)$ and so, $\overline{R}_+ = [0, \infty)$ and $\overline{R}_+ M = \{tx \mid t \geq 0, x \in M\}$.

We need a kind of continuity of F on the set $(\overline{R}_+ M) \times \mathbf{S}^N$. For $p \in M$ let $T_p M$ denote the tangent space of M at p and let π denote the orthogonal projection of $\mathbf{R}^N$ onto $T_p M \oplus \text{span}\{p\}$. For $p \in M$ and $t > 0$ we set

$$\mathbf{S}^N(tp) = \{X \in \mathbf{S}^N \mid \pi X \pi = X\},$$

so that $\mathbf{S}^N(p) = \mathbf{S}^N(p/|p|)$ for all $p \in \mathbf{R}_+ M$. Also, we denote by $\mathbf{S}^N(0)$ the subset $\{0\}$ of $\mathbf{S}^N$. Let $p \in M$. Note that $X \in \mathbf{S}^N(p)$ if and only if $(I - \pi)X = X(I - \pi) = 0$, with π denoting the orthogonal projection of $\mathbf{R}^N$ onto $T_p M \oplus \text{span}\{p\}$. Note also that if $X \in \mathbf{S}^N(p)$, then $tX \in \mathbf{S}^N(p)$ for all $t \in \mathbf{R}$. Moreover, observe that if $X \in \mathbf{S}^N$, $A \in \mathbf{S}^N(p)$ and $-A \leq X \leq A$, then $X \in \mathbf{S}^N(p)$. Indeed, if π is the orthogonal projection of $\mathbf{R}^N$ as above, then the inequality $-A \leq X \leq A$ yields

$$\begin{aligned}
&|\langle X(\pi\xi + (I - \pi)\eta), \pi\xi + (I - \pi)\eta\rangle| \\
&\quad = |\langle \pi X \pi \xi, \xi\rangle + 2\langle \pi X (I - \pi)\eta, \xi\rangle + \langle (I - \pi) X (I - \pi)\eta, \eta\rangle| \\
&\quad \leq \langle \pi A \pi \xi, \xi\rangle \quad \text{for all } \xi, \eta \in \mathbf{R}^N.
\end{aligned}$$

From this we deduce that $(I - \pi)X(I - \pi) = 0$ and $\pi X(I - \pi) = 0$, and furthermore, that $\pi X \pi = X$.

(A2) If $p \in \overline{R}_+ M$ and $X \in \mathbf{S}^N(p)$, then

$$F^*(p, X) = F_*(p, X).$$

Here and henceforth we use the notation:

$$F^*(\xi) = \lim_{\varepsilon \downarrow 0} \sup\{F(\eta) \mid \eta \in (\mathbf{R}^N \setminus \overline{R}_+ M) \times \mathbf{S}^N, \|\eta - \xi\| < \varepsilon\},$$

and $F_* = -(-F)^*$.

We remark that if $e_1, \cdots, e_N$ denote the standard basis of $\mathbf{R}^N$, if $e_N \in M$ and $t > 0$ and if $e_1, \cdots, e_d \in T_{e_N} M$, then

$$\begin{aligned}
X =& (x_{ij})_{1 \leq i,j \leq N} \in \mathbf{S}^N(te_N) \\
&\Longleftrightarrow \; X \in \mathbf{S}^N \;\text{ and }\; x_{ij} = 0 \;\text{ if } d < i < N \text{ and } j = 1, \cdots, N.
\end{aligned}$$

The degenerate ellipticity is statec

(A3) If $p \in \mathbf{R}^N \setminus \overline{R_+ M}$ and $X, Y \in \mathbf{S}^N$ and if $X \leq Y$, then $F(p, X) \geq F(p, Y)$.

We are now in a position to state the main theorem formulated for bounded domains Ω.

Theorem 1. *Let (A1), (A2) and (A3) hold. Assume that Ω is bounded. Let $u \in USC(\overline{\Omega} \times [0,T))$ and $v \in LSC(\overline{\Omega} \times [0,T))$ be a viscosity subsolution and a viscosity supersolution of (1), respectively. Assume that $u \leq v$ on $(\partial\Omega \times [0,T)) \cup (\overline{\Omega} \times \{0\})$. Then $u \leq v$ in $\Omega \times (0,T)$.*

The case $d = 0$ is exactly the case treated by Ohnuma and Sato [8] and Gurtin, Soner and Souganidis [6].

3. Proof of Theorem 1

Let us explain two lemmas, which are key ingredients in the proof of the above theorem.

Lemma 1. *Let $u, v \in USC(V)$, where V is an open subset of $\mathbf{R}^m$, and define $w \in USC(V \times V)$ by $w(x,y) = u(x) + v(y)$. Let $x, y \in V$, $p, q \in \mathbf{R}^m$ and $A \in \mathbf{S}^m$ satisfy*

$$\left(p, q, \begin{pmatrix} A & -A \\ -A & A \end{pmatrix}\right) \in J^{2,+}w(x,y) \quad \text{and} \quad A \geq 0.$$

Then there are $X, Y \in \mathbf{S}^m$ such that

$$(p, X) \in \overline{J}^{2,+}u(x), \ (q, Y) \in \overline{J}^{2,+}v(y),$$

$$-3\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 3\begin{pmatrix} A & -A \\ -A & A \end{pmatrix}.$$

Lemma 2. *Under assumption (A1), there is a function $\psi \in C(\mathbf{R}^N) \cap C^{1,1}(\mathbf{R}^N \setminus \{0\})$ such that*

(i) *ψ is convex and positively homogeneous of degree 1 on $\mathbf{R}^N$,*

(ii) *$\psi(x) > 0$ for $x \neq 0$,*

(iii) *ψ is twice continuously differentiable in a neighborhood of $\mathbf{R}^+ M$,*

(iv) *$x \in \mathbf{R}^+M$ if and only if $x \neq 0$, and $D\psi(x) \in \mathbf{R}^+M$ and*

(v) *for any $x \in \mathbf{R}^+M$, $D^2\psi(x) \in \mathbf{S}^N(D\psi(x))$.*

With these lemmas at hand, the proof of Theorem 1 is a rather tedious repetition of the standard argument in the theory of viscosity solutions. We refer to [7] for the proof of Theorem 1.

The idea of the proof of Lemma 2 may be explained as follows. Let $M \subset S^{N-1}$ be a C^2 submanifold as in Lemma 2. Fix any point $q \in S^{N-1}$ and a smooth strictly convex body K so that K contains the unit ball $B(0,1) \subset \mathbf{R}^N$ and so that $q \in \partial K$ and all the principal curvatures of ∂K at q vanish. Then, for each $p \in M$ we define K_p as the convex set obtained by rotating K around the origin so that the new position of q is at p. The function ψ is defined as the Minkowski functional of the ε-neighborhood of the set

$$\cup\{K_p \mid p \in M\}, \qquad \text{with } \varepsilon > 0.$$

The details of the proof may be found in [7].

4. Generalized evolution of a hypersurface

In addition to (A1) – (A3) we assume that (1) is geometric, i.e.

(A4) If $p \in \mathbf{R}^N \setminus \overline{R}_+M$, $X \in \mathbf{S}^N$ and $\lambda > 0$, $\mu \in \mathbf{R}$, then

$$F(\lambda p, \lambda X + \mu p \otimes p) = \lambda F(p, X).$$

Now we consider the initial value problem

$$\text{(IVP)} \qquad \begin{cases} u_t + F(Du, D^2u) = 0 & \text{in } \mathbf{R}^N \times (0,\infty), \\ u = g & \text{on } \mathbf{R}^N \times \{0\}, \end{cases}$$

where g is a given function on $\mathbf{R}^N$. Theorem 1 and standard arguments in viscosity solutions theory yield the following:

Theorem 2. *Assume that $g \in BUC(\mathbf{R}^N)$ and that (A1) – (A4) hold. Then there is a unique viscosity solution u of (IVP) satisfying $u \in BUC(\mathbf{R}^N \times [0,T))$ for all $T > 0$.*

Let $\mathcal{E}$ denote the set of triplets (Γ, D^+, D^-) of a closed subset Γ and two open subsets D^+, D^- of $\mathbf{R}^N$ such that

$$\Gamma \cup D^+ \cup D^- = \mathbf{R}^N \quad \text{and} \quad \Gamma,\ D^+,\ D^- \text{ are mutually disjoint.}$$

If $(\Gamma, D^+, D^-) \in \mathcal{E}$, then there is a function $g \in BUC(\mathbf{R}^N)$ such that

$$\text{(3)} \qquad \Gamma = \{g = 0\},\ D^+ = \{g > 0\} \text{ and } D^- = \{g < 0\}.$$

Conversely, if $g \in BUC(\mathbf{R}^N)$, then

$$(\{g = 0\}, \{g > 0\}, \{g < 0\}) \in \mathcal{E}.$$

The geometricity (A4) of (1) allows us to conclude the following property, the proof of which can be found in [7].

Theorem 3. *Assume that (A1) – (A4) hold. Let $g_1, g_2 \in BUC(\mathbf{R}^N)$ satisfy*

$$\{g_1 > 0\} = \{g_2 > 0\}, \quad \{g_1 < 0\} = \{g_2 < 0\} \quad \textit{and} \quad \{g_1 = g_2\}.$$

Let u_i, $i = 1, 2$, be the (unique) viscosity solutions of (IVP), with $g = g_i$, satisfying $u_i \in BUC(\mathbf{R} \times [0, T))$ for all $T > 0$. Then

$$\{u_1 > 0\} = \{u_2 > 0\}, \quad \{u_1 < 0\} = \{u_2 < 0\} \quad \textit{and} \quad \{u_1 = u_2\}.$$

Now a generalized evolution of a triplet $(\Gamma, D^+, D^-) \in \mathcal{E}$ by (1) can be defined as follows. Fix any $g \in BUC(\mathbf{R}^N)$ so that (3) holds, solve (IVP) with this initial data g and set

$$\Gamma_t = \{u(\cdot, t) = 0\}, \quad D_t^+ = \{u(\cdot, t) > 0\} \quad \text{and} \quad D_t^- = \{u(\cdot, t) < 0\}.$$

for all $t \geq 0$. Theorem 3 guarantees that the sets Γ_t, D_t^+ and D_t^- do not depend on the choice of g. The collection $\{E_t\}_{t \geq 0}$ of mappings

$$E_t : (\Gamma, D^+, D^-) \mapsto (\Gamma_t, D_t^+, D_t^-)$$

of $\mathcal{E}$ into itself is the generalized evolution of (Γ, D^+, D^-) by (1). Theorem 2 ensures the semigoup property:

$$E_0 = id_{\mathcal{E}}, \quad E_{t+s} = E_t \circ E_s \quad \text{for all } t, s \geq 0.$$

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Department of Mathematics
Chuo University
Bunkyo-ku, Tokyo 112
Japan