On Mean Curvature Flow of Surfaces in Riemannian 3-Manifolds

Klaus Ecker

University of Melbourne, Australia

In this talk I would like to consider hypersurfaces $(M_t)_{t\in[0,T)}$ which move by mean curvature in a Riemannian manifold N^{n+1}. The hypersurfaces are described by a one-parameter family

$$x_t = x(\cdot,t) : M^n \to N^{n+1}$$

of smooth immersions of an n - dimensional manifold M^n without boundary, with images $M_t = x_t(M^n)$, satisfying

$$\frac{d}{dt}x(p,t) = -H(p,t)\,\nu(p,t) \qquad p \in M^n\,,\ t\in(0,T). \tag{1}$$

Here $H(p,t)$ and $\nu(p,t)$ denote mean curvature and a choice of unit normal of the hypersurface M_t at $x(p,t)$.

Mean curvature flow arises naturally as the steepest descent flow for the area functional which explains its great significance due to its possible applications to minimal surface theory in Riemannian manifolds. Indeed, from (1) one easily derives the evolution equation for the surface element of M_t,

$$\frac{d}{dt}\mu_t = -H^2\mu_t \tag{2}$$

see [H1], which immediately yields

$$\mathcal{H}^n(M_t) \le \mathcal{H}^n(M_0) \tag{3}$$

for all $t \ge 0$ where $\mathcal{H}^n$ denotes n - dimensional Hausdorff-measure.

Without any special assumptions on M_0, such as convexity (see [H1], [H2]), a solution of (1) will in general develop singularities in finite time before it 'disappears', as for example Grayson's axially symmetric dumbbell surface ([Gr]).

In [B], K. Brakke has studied the regularity behaviour of n - dimensional varifolds moving by mean curvature in $\mathbf{R}^{n+k}, k \ge 1$. For smooth solutions of (1) with $T < \infty$, his result implies that if there exists a unit density rectifiable varifold M_T such that $M_t \to M_T$ as $t \to T$ in the measure-theoretic sense in $B_\rho(x_0)$ for some $x_0 \in \mathbf{R}^{n+1}$

then M_T is a smooth hypersurface in a neighbourhood of every point $x \in B_{\frac{\rho}{2}}(x_0) \sim S_T$ where the singular set S_T satisfies $\mathcal{H}^n(S_T) = 0$.

For applications of mean curvature flow it is very important to obtain a regularity theorem which does not assume unit density for the first singular surface as one cannot a priori rule out singularity formation based on the local merging of two nearly flat pieces of surface which are very close to each other and are connected by many very small catenoidal necks the number of which increases up to the first singular time. This is analogous to problems arising in the regularity theory for stationary varifolds where double sheeted examples of bounded mean curvature surfaces with single density catenoidal necks give reason for concern. The catenoidal necks connect a dense set of holes in each sheet of surface but the double density set where the two sheets touch has positive $\mathcal{H}^2$-measure, see for example [B, 6.1].

Moreover, one should try to improve the estimate on the dimension of the singular set. The best result one can expect here is that the singular set has locally finite $\mathcal{H}^{n-1}$-dimensional measure in view of the examples of a cylinder which contracts to a line and a symmetric thin torus which contracts to a circle in finite time.

In [AAG], Altschuler, Angenent and Giga proved that the first singular set (at time T) of hypersurfaces of rotation about an axis in $\mathbf{R}^{n+1}$ consists of isolated points unless, of course, M_T has zero area as in the case of the shrinking cylinder. They showed, in fact, that these isolated singularities only form at a finite number of times before the hypersurface 'disappears'. This result suggests the following question:

Suppose $M_0 \subset \mathbf{R}^3$ is compact, connected and at the first singular time T there holds $\liminf_{t \to T} \mathcal{H}^2(M_t) > 0$. Does this imply that only finitely many singularities form?

In [EH], it was proved that local control on the norm of the second fundamental form $|A|^2$ in the form

$$\sup_{[T-\rho^2, T)} \sup_{M_t \cap B_\rho(x_0)} |A|^2 \le C_0$$

for some $\rho > 0$ at some point $x_0 \in \mathbf{R}^{n+1}$ implies

$$\sup_{[T-(\frac{\rho}{2})^2, T)} \sup_{M_t \cap B_{\frac{\rho}{2}}(x_0)} |\nabla^m A|^2 \le C_m$$

for all $m \ge 0$ and hence M_t can be smoothly extended beyond the time T in a neighbourhood of x_0. This estimate carries over to Riemannian manifolds almost immediately, see [E2].

In [CS] and [An1]-[An4], Choi and Schoen and Anderson proved an ϵ-regularity theorem involving a curvature integral for minimal surfaces in 3-manifolds which they then used to prove compactness theorems for minimal surfaces of fixed genus.

The main result presented in this talk is a parabolic analogue of their theorem. It gives a local estimate for $|A|^2$, assuming that $\int_{M_t} |A|^2$ is sufficiently small locally in space and time. In particular, it implies the existence of a constant $\epsilon_0 > 0$ such that for every point x in the singular set S_T and sufficiently small $\rho > 0$, there exists a sequence of times $t_k \to T$ such that

$$\lim_{t_k \to T} \int_{M_{t_k} \cap B_\rho(x)} |A|^2 > \epsilon_0 .$$

Let me outline how this result could possibly lead to an alternative proof of partial regularity in the case $n = 2$.

We first use the evolution equation (3) for the area and the Gauss-Bonnet formula to obtain an estimate of the form

$$\int_0^T \int_{M_t} |A|^2 \leq c < \infty$$

for compact surfaces M_t and $T < \infty$ where c depends on T, $\mathcal{H}^2(M_0)$, the genus of M^2 and a bound on the Ricci curvature of N^3. By a standard covering argument this implies that the sets

$$S_T^\alpha \equiv \{x \in N^3, \limsup_{\rho \to 0} \rho^{-2} \int_{T-\rho^2}^T \int_{M_t \cap B_\rho(x)} |A|^2 > \alpha\}$$

satisfy $\mathcal{H}^2(S_T^\alpha) = 0$ for any $\alpha > 0$.

Now note that

$$\operatorname{dist}(x\,,\, M_t) \leq 2\sqrt{T-t}$$

for all $x \in \overline{M}_T$ in view of the sphere comparison argument in [B, Th.3.7]. This implies by the strong maximum principle that

$$\int_{M_t \cap B_\rho(x_0)} |A|^2 > 0$$

for $t \in [T - \frac{1}{16}\rho^2, T)$.

The main, as yet incomplete, step in the argument is to show $S_T \subset \bigcup_{\alpha > 0} S_T^\alpha$ which would then imply $\mathcal{H}^2(S_T) = 0$ in view of the covering argument mentioned above. To establish this, one has to prove that $\int_{M_t \cap B_\rho(x)} |A|^2$ is bounded below by a constant $\gamma = \gamma(\epsilon_0, x) > 0$ on a set of times of measure at least $\beta\rho^2$ where $\beta = \beta(\epsilon_0, x) > 0$. This in turn requires good control of the change of $\int_{M_t \cap B_\rho(x)} |A|^2$ during times when this quantity is small.

A step in this direction is made in [E1, Corollary 2.6], where it is shown that if $\int_{M_t \cap B_\rho(x)} |A|^2$ is less than or equal to ϵ_0 on a given time interval then this quantity taken over any smaller ball does not change much during this time. The requirement of having to go to a smaller ball in this estimate seems to be the remaining technical difficulty.

Let me now state the main result.. The proof given here is based on a modification of a scaling technique used in [CS] and [St]. For an alternative proof we also refer to [E2].

Theorem. *Let $(M_t)_{t\in[0,T)}$ be a solution of (1). Then there exist constants $\epsilon_0 > 0$ and $C_0 > 0$ depending only on the geometry of N^3 such that for any $x_0 \in N^3$ and $0 < \rho < \min\{\epsilon_0, \sqrt{T}\}$ with $\sup_{[T-\rho^2,T)} \mathcal{H}^2(M_t \cap B_\rho(x_0)) \le \epsilon_0$ the inequality*

$$\sup_{[T-\rho^2,T)} \int_{M_t \cap B_\rho(x_0)} |A|^2 \le \epsilon_0$$

implies the estimate

$$\sigma^2 \sup_{[T-(\rho-\sigma)^2,T)} \sup_{M_t \cap B_{\rho-\sigma}(x_0)} |A|^2 \le C_0$$

for all $\sigma \in [0,\rho]$.

The smoothness of M_t up to time T in a neighbourhood of x_0 is then an immediate consequence of the evolution equation for the derivatives of A (Lemma 7.2, [H2]) and a straightforward generalization of the interior estimates in [EH, Theorem 3.7], see [E2] for more details.

Corollary. *Under the assumptions of the theorem the estimate*

$$\sup_{[T-(\frac{\rho}{2})^2,T)} \sup_{M_t \cap B_{\frac{\rho}{2}}(x_0)} |\nabla^m A|^2 \le C_m$$

holds for any integer $m \ge 0$ where C_m depends only on m, ρ and the geometry of N^3.

Remark. (i) The area condition of the theorem enters due to the presence of the covariant derivatives of the curvature tensor of N^3 in the evolution equation for $|A|^2$ as well as due to an application of a version of Moser's mean value inequality for subsolutions of parabolic operators (see the proposition below) which relies heavily on the Sobolev inequality ([HS], [MS]) on surfaces in Riemannian manifolds. In particular, no area condition is required in the case where N^3 is complete, simply connected, locally symmetric and has nonpositive sectional curvatures.

If M^2 is compact, the area condition is automatically satisfied near every $x_0 \in N^3$ for sufficiently small ρ, possibly depending on x_0, in view of (3). To obtain independence of x_0 in the choice of ρ one needs some local control on the area of M_t in the form obtained in [E1] in the case of $\mathbb{R}^3$.

(ii) In the case of minimal surfaces (which are stationary solutions of (1)), the theorem reduces to an earlier result of Choi and Schoen ([CS]) and M. Anderson ([An4]). Note furthermore that for a compact minimal surface M in N^3, the monotonicity formula implies $\mathcal{H}^2(M \cap B_\rho(x_0)) \le c\,\rho^2$ where c depends on the curvatures of N^3. The area condition therefore reduces to a smallness assumption for ρ in this case.

For the proof of the theorem we need the following straightforward generalization of Moser's mean value inequality to mean curvature flow which we state for general dimensions. To establish this, one essentially follows Moser's proof in [M] with cut-off functions supported in $B_\rho(x_0)$ but uses the Sobolev inequality of [HS] (which requires the area condition stated below), Young's inequality and (2) to control the term involving $|H|$, see [E1].

Proposition. *Let N^{n+1} be a Riemannian manifold with sectional curvatures bounded from above by κ_+ and injectivity radius denoted by inj_N. Let $(M_t)_{t\in[0,T)}$ be a solution of (1) in N^{n+1} satisfying*

$$\kappa_+ \left(\omega_n^{-1} \sup_{[T-\rho^2,T)} \mathcal{H}^n(M_t \cap B_\rho(x_0))\right)^{\frac{2}{n}} \le (1-a)^{2/n},$$

$$\begin{aligned}\left(\omega_n^{-1} \sup_{[T-\rho^2,T)} \mathcal{H}^n(M_t \cap B_\rho(x_0))\right)^{\frac{1}{n}} &\le (1-a)^{1/n} \kappa_+^{-1/2} \sin\left(\frac{1}{2}\kappa_+^{1/2}\,\mathrm{inj}_N\right), && \text{if } \kappa_+ \ge 0\\ &\le \frac{1}{2}(1-a)^{1/n}\,\mathrm{inj}_N, && \text{if } \kappa_+ \le 0\end{aligned}$$

in some ball $B_\rho(x_0)$ where $a \in (0,1)$ is arbitrary. Let $f \ge 0$ satisfy

$$\left(\frac{d}{dt} - \Delta\right) f \le 0$$

in $[T-\rho^2, T) \times B_\rho(x_0)$. Then the mean value inequality

$$\sup_{[T-\frac{\rho^2}{4},T)} \ \sup_{M_t \cap B_{\frac{\rho}{2}}(x_0)} f \le c(n,a)\,\rho^{-n-2} \int_{T-\rho^2}^{T} \int_{M_t \cap B_\rho(x_0)} f$$

holds.

For the proof of the theorem we also recall the following consequence of the evolution equation for the curvature of M_t given in [H2]:

Lemma. *The second fundamental form $A = (h_{ij})$ of M_t satisfies the inequality*

$$\left(\frac{d}{dt} - \Delta\right) |A|^2 \le 2|A|^4 + c(n)(\kappa_0 + \kappa_1^{2/3})|A|^2 + c(n)\kappa_1^{4/3}$$

where κ_0 and κ_1 denote bounds on $|\mathrm{Riem}_N|$ and $|\nabla^N \mathrm{Riem}_N|$ respectively.

Proof of Theorem. Since M_t is smooth in $[0,T)$ and ϵ_0 and C_0 are independent of T, we may assume without loss of generality that

$$\lambda^2 \equiv \max_{\sigma\in[0,\rho]}\left(\sigma^2 \sup_{[T-(\rho-\sigma)^2,T]}\ \sup_{M_t\cap B_{\rho-\sigma}(x_0)} |A|^2\right) < \infty.$$

Otherwise we replace T by $T-\delta$ for $\delta\in(0,T)$ and then let $\delta\to 0$.

Let $\sigma_0\in(0,\rho]$ such that

$$\sigma_0^2 \sup_{[T-(\rho-\sigma_0)^2,T]}\ \sup_{M_t\cap B_{\rho-\sigma_0}(x_0)} |A|^2 = \lambda^2$$

and choose $\tau_0\in[T-(\rho-\sigma_0)^2,T]$ and $y_0\in M_{\tau_0}\cap B_{\rho-\sigma_0}(x_0)$ for which

$$\sigma_0^2\,|A|^2(y_0,\tau_0) = \lambda^2.$$

Note, in particular, that

$$\sup_{[T-(\rho-\frac{\sigma_0}{2})^2,T]}\ \sup_{M_t\cap B_{\rho-\frac{\sigma_0}{2}}(x_0)} |A|^2 \le 4\,|A|^2(y_0,\tau_0)$$

and hence

(4)
$$\sup_{[\tau_0-(\frac{\sigma_0}{2})^2,\tau_0]}\ \sup_{M_t\cap B_{\frac{\sigma_0}{2}}(y_0)} |A|^2 \le 4\,|A|^2(y_0,\tau_0).$$

If the estimate $\sigma_0^2\,|A|^2(y_0,\tau_0)\le 4$ holds, the statement of the theorem is true. We therefore have to show that the reverse inequality

(5)
$$\sigma_0^2\,|A|^2(y_0,\tau_0)\ge 4$$

leads to a contradiction for sufficiently small ϵ_0.

To this end we rescale the metric g on N^3 by setting $\tilde g=\lambda_0^2\,g$ for $\lambda_0=|A|(y_0,\tau_0)$. The family of surfaces

$$(\widetilde{M}_t)\equiv(M_{\lambda_0^{-2}t+\tau_0})$$

is a solution of (1) with respect to $\tilde g$ for $t\in[-\lambda_0^2\tau_0,\ \lambda_0^2(T-\tau_0)]$.

From inequalities (4) and (5), we infer

(6)
$$\sup_{[-1,0]}\ \sup_{\widetilde{M}_t\cap\widetilde{B}_1(y_0)} |\widetilde{A}|^2\le 4$$

where quantities with respect to $\tilde g$ are indicated by a tilde. We also note that

(7)
$$|\widetilde{A}|^2(y_0,0)=1.$$

The evolution inequality for $|A|^2$ implies

$$\left(\frac{d}{dt} - \widetilde{\Delta}\right) |\widetilde{A}|^2 \le c(|\widetilde{A}|^2 + \tilde{\kappa}_0 + \tilde{\kappa}_1^{2/3})|\widetilde{A}|^2 + \tilde{\kappa}_1^{4/3}.$$

In view of (6), the function $u = |\widetilde{A}|^2 + \tilde{\kappa}_1^{2/3}$ therefore satisfies the inequality

$$\left(\frac{d}{dt} - \widetilde{\Delta}\right) u \le c\, u \tag{8}$$

in the ball $\widetilde{B}_1(y_0)$ for $t \in [-1, 0]$ where c depends on the curvatures of N^3 with respect to $\tilde{g}$ as above. The area assumption of the theorem implies in view of (5) that for small enough ϵ_0 depending on κ_0 and inj_N the area conditions of the proposition are satisfied for $\widetilde{M}_t$ in $[-1, 0] \times \widetilde{B}_1(y_0)$. We can then apply the mean value inequality to $f = e^{-ct}u$ to infer

$$u(y_0, 0) \le c \int_{-1}^{0} \int_{\widetilde{M}_t \cap \widetilde{B}_1(y_0)} u$$

where c depends exponentially on $\tilde{\kappa}_0$ and $\tilde{\kappa}_1^{2/3}$.

In particular, we conclude from (7) that

$$1 = |\widetilde{A}|^2(y_0, 0) \le c \left(\sup_{[-1,0]} \int_{\widetilde{M}_t \cap \widetilde{B}_1(y_0)} |\widetilde{A}|^2 + \tilde{\kappa}_1^{2/3} \sup_{[-1,0]} \mathcal{H}^2(\widetilde{M}_t \cap \widetilde{B}_1(y_0)) \right).$$

Rescaling this inequality and using (5) again implies the estimate

$$1 \le c \left(\sup_{[T-\rho^2, T)} \int_{M_t \cap B_\rho(x_0)} |A|^2 + \kappa_1^{2/3} \sup_{[T-\rho^2, T)} \mathcal{H}^2(M_t \cap B_\rho(x_0)) \right)$$

where c depends exponentially on $(\kappa_0 + \kappa_1^{2/3})\rho^2$. In view of the assumptions of the theorem this yields a contradiction for sufficiently small ϵ_0. Note that ϵ_0 depends only on κ_0, κ_1 and inj_N.

[AAG] S.Altschuler, S.Angenent and Y.Giga, Mean curvature flow through singularities for surfaces of rotation, preprint

[An1] M.T.Anderson, Curvature estimates for minimal surfaces in 3-manifolds, Ann.Sci. Ecole Norm.Sup., 18:1 (1985)

[An2] M.T Anderson, Local estimates for minimal submanifolds in dimension greater than two, in Geometric measure theory and the calculus of variations (Arcata, Calif. 1984) 131-337, Proc.Symp.Pure Math. 44 AMS, Providence, R.I.1986

[An3] M.T.Anderson, The compactification of a minimal submanifold in Euclidean space by the Gauss map, preprint

[An4] M.T.Anderson, Remarks on curvature integrals and minimal varieties, in Complex differential geometry and nonlinear differential equations (Brunswick, Maine 1984) 11-18, Contemp.Math 49, AMS, Providence, R.I.1986

[B] K.A.Brakke: The Motion of a Surface by its Mean Curvature, Math. Notes Princeton, NJ, Princeton University Press 1978

[CS] H.I.Choi and R.Schoen, The space of minimal embeddings of a surface into a three-dimensional manifold of positive Ricci-curvature, Invent.math. **81**, 387-394 (1985)

[E1] K.Ecker, On regularity for mean curvature flow of hypersurfaces, to appear in Calculus of Variations and PDE

[E2] K.Ecker, An ϵ-regularity theorem for mean curvature flow of surfaces in 3-manifolds, preprint

[EH] K. Ecker and G. Huisken, Interior estimates for hypersurfaces moving by mean curvature, Invent.math. **105**, 547-569 (1991)

[Gr] M.Grayson, A short note on the evolution of a surface by its mean curvature, Duke Math.J. **58**, 555 - 558 (1989)

[HS] D.Hoffman and J.Spruck, Sobolev and isoperimetric inequalities for Riemmannian submanifolds, Comm.Pure Appl.Math. **27**, 715-727 (1974)

[H1] G.Huisken, Flow by mean curvature of convex surfaces into spheres, J.Differ.Geom. **20**, 237-266 (1984)

[H2] G.Huisken, Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature, Invent.math. **84**, 463-480 (1986)

[MS] J.H.Michael and L.M.Simon, Sobolev and mean value inequalities on generalized submanifolds of $\mathbb{R}^n$, Comm.Pure Appl.Math. **26**, 361-379 (1973)

[M] J.Moser, A Harnack inequality for parabolic differential equations, Comm.Pure Appl. Math. **17**, 101-134 (1964)

[St] M.Struwe, On the evolution of harmonic mappings of Riemannian surfaces, Comm. Math. Helv. **60**, 558-581 (1985)

Department of Mathematics
The University of Melbourne
Parkville Vic. 3052 Australia

e-mail: ecker@mundoe.maths.mu.oz.au