RESOLVING THE TRANSMISSIVITY ZONATION IN A CONFINED AQUIFER

R. S. Anderssen and S.-S. Chow

ABSTRACT

For aquifers having a zonation structure with the transmissivity varying smoothly and slowly over each zone, a common approach in determining transmissivity is to presume a known zonation structure and seek a constant approximation to the transmissivity over each zone. However, this procedure is not always acceptable as it may lead to instability in the estimation process.

In this paper, we discuss how one may simultaneously determine the zonation structure and a piecewise constant representation of the transmissivity by adapting the linear functional strategy proposed by Anderssen and Dietrich (1987). The implementation of this idea results in several adaptive and highly parallelizable procedures for the parameter identification problem. Some stability results and a generalization of the method using a Petrov-Galerkin interpretation are also described.

1. Introduction.

The problem of parameter identification plays a crucial rôle in the study of groundwater flows in aquifers. For the modelling equations to be completely specified, it is necessary to have sufficiently accurate information regarding the various parameters used to describe the flow system. For example, before one performs any simulation studies on the problem of steady groundwater flow in a confined aquifer, it is necessary to have available an accurate representation of the transmissivity within the aquifer.

It has long been recognized that the parameter identification problem is inherently ill-posed and often complicated by the fact that in most practical situations only a sparse amount of measured data is available for one to employ to ascertain the parametric values of the flow system. Moreover, in practice, boundary information for the aquifer is difficult if not impossible to determine, since the location of the boundary of a confined aquifer is not usually known with great certainty. Consequently, in developing numerical procedures for this type of problem, one should only consider methods which first of all possess a stabilizing property in order to deal with the ill-posedness, secondly, generate an accurate but not "overly sophisticated" representation of the parameters, and finally does not require an excessive amount of (or better still, any) boundary information.

In the transmissivity identification problem arising from the study of steady flow of groundwater in a confined aquifer Ω, one typically assumes that the transmissivity T is a piecewise constant function over Ω (i.e., over each of the disjoint subregions of Ω, T assumes a constant value), and then proceeds to determine these constant values. This approach is the zonation method (Coats *et al.* [6], Emsellem and de Marsily [10], Yeh and Yoon [14], and Cooley [7,8]).

In most practical problems, particularly those involving aquifers in uncomplicated geological formations, the piecewise constant assumption is a valid one. Moreover, the amount of data available is unable to support more than a simple approximation, such as a piecewise constant. However, as observed by Sun and Yeh [12] and Yeh [13], the associated problem of determining the zonation structure is often neglected. The zones over which the transmissivity is constant are often assumed known and only the constant values of T are estimated. This practice is seldom acceptable, since the available geographical field data are in most instances insufficient to determine the zonation structure. Any *a priori* assumptions about the zonation structure that are not sufficiently accurate may lead to instability in the estimation of the piecewise constant structure of T (Sun and Yeh [12]). Such difficulties are avoided, if one aims to determine simultaneously the subregions over which the transmissivity is constant and the value of these constants.

In this paper, we discuss how the linear functional strategy proposed by Anderssen and Dietrich [2] (see also Anderssen [1]) may be adapted to simultaneously determine a piecewise constant representation of the transmissivity and the zonation structure of the aquifer. The inherent stabilizing property of the linear functional strategy is highly desirable in tackling the parameter identification problem. Moreover, since the linear functional strategy is a localized procedure, applicable to any subregion of Ω, the lack of data on the boundary does not cause any problem in the implementation of this methodology.

2. Linear Functional Strategy.

The governing equation of steady flow in a confined aquifer Ω is given by

$$- \operatorname{div} (T \operatorname{grad} \phi) = q, \tag{1}$$

where T is the transmissivity of the aquifer, ϕ the piezometric head, and q the source term (Bear [4], Yeh [15]). In the parameter identification problem, ϕ and q are known and T is to be determined.

By multiplying both sides of (1) by a test function W and integrating by parts over a subdomain Λ of Ω, we have the weak formulation

$$\int_\Lambda T \text{ grad } \phi \cdot \text{ grad } W dx = \int_\Lambda qW dx + \int_{\partial\Lambda} T \text{ grad } \phi \cdot \mathbf{n} W ds, \qquad (2)$$

where $\partial\Lambda$ denotes the boundary of Λ and $\mathbf{n}$ denotes the unit outward normal on Λ.

Following Anderssen and Dietrich [2], we now develop a linear functional strategy to determine the transmissivity in the subregion Λ where T assumes a constant value T_0. By utilizing a test function W that satisfies

$$W = 0 \text{ on } \partial\Lambda,$$

we may compute T_0 over Λ from (2) as

$$T_0 = ((q, W)) \,/\, ((\text{ grad } \phi, \text{ grad } W)), \qquad (3)$$

where

$$((q, W)) = \int_\Lambda qW dx,$$

and

$$((\text{ grad } \phi, \text{ grad } W)) = \int_\Lambda \text{ grad } \phi \cdot \text{ grad } W dx.$$

For confined aquifers with constant transmissivity zones, this strategy may be employed to determine the zonation structure of the aquifer as well as the constant values associated with the various zones. The **basic idea** is that if the transmissivity is constant over a region Λ, then it must also be constant over any subregion Λ' contained in Λ. Furthermore, if the transmissivity is only approximately constant over Λ, (3) may still be employed to compute a representative constant approximate value for the transmissivity over the subregion Λ. So, by evaluating the transmissivity values over Λ and Λ', (possibly making use of several different test functions,) and comparing the values obtained from applying the linear functional strategy, we would be able to decide whether the transmissivity is indeed constant over Λ and if so, what the value of the transmissivity is. In this manner,we would be able to construct a reasonable picture of the zonation structure as well as the constant values that the transmissivity takes over the various zones.

Note that the implementation of this idea along with the linear functional strategy would allow us to develop adaptive and highly parallelizable procedures for the parameter identification problem. Moreover, the estimation process has inherent stability associated with the use of bounded linear functionals (Anderssen [1]), thus enabling us to treat the ill-posedness of the problem with some degree of confidence.

3. Simultaneous Zonation and Transmissivity Identification.

3.1 Global Source Term.

To begin our investigation, we assume that the source term is a global function over Ω rather than point sources or sinks. The latter case will be considered in the next section. For simplicity let us assume that Ω is a region which is decomposable into rectangular subregions. The assumption is not essential, but does greatly simplify the discussion.

Let us first consider the straightforward approach. We shall refer to it as the **fine grid strategy**. If from geological data one is able to determine a length scale l such that over most squares of area l^2 the transmissivity is approximately constant, then one may partition Ω into squares of area l^2 and small rectangles along the boundaries. We may then compute T on each rectangular element using (3). If l is sufficiently small, this procedure will yield a reasonably accurate representation of the variation of transmissivity over Ω. In fact, if the test function W defined on a given square is constructed so that it converges to the Dirac delta function associated with the center of the square as l tends to zero, then it can be shown that the estimated transmissivity for the square tends to the true transmissivity value at the center as the square shrinks.

Clearly the implementation of this method is straightforward. Once the transmissivity values are computed, they may be sorted into ascending order and partitioned into a certain number of groups. To each group a colour type or gray scale is then assigned. By displaying the transmissivity values in a graphical form, the zonation structure may easily be identified and regions that require further analysis may also be pinpointed. Note however that this procedure may also have some potential disadvantages. The requirement that l be small implies not only a large storage requirement, but also a substantial amount of computing time since we must determine the appropriate form of the test function W and evaluate (3) over a large number of cells as well as post-process the computed transmissivity values so as to identify the nature of the zonation.

An alternative procedure, which we will call the **adaptive grid approach**, is to begin with a coarse mesh, and then adaptively refine that mesh, using the computed values of T as the control, in order to determine the nature of the zonation.

This approach will not only minimize the storage requirement and the amount of computational work required, but its implementation will also be facilitated by the existence of data structures developed for various finite element mesh refinement schemes (Bank *et al.*, [3], Rheinboldt and Mesztenyi [11]). Moreover, the procedure is easily parallelizable, since the estimation for each subregion is independent of the others.

The basic algorithm for this adaptive grid approach will have the following structure. First decompose Ω into a suitably small number of rectangular subdomains Ω_i, $i = 1, \cdots, N$. For each i, we compute the corresponding transmissivity T_i with (3) after setting $\Lambda = \Omega_i$. Each of the Ω_i's is then subdivided into four rectangular regions $\{\Omega_{ij}\}_{j=1}^{4}$, and the transmissivity T_{ij} of Ω_{ij} is evaluated from (3) with $\Lambda = \Omega_{ij}$. We then compare the T_{ij} with T_i. If they agree to a preassigned tolerance, we take the transmissivity of Ω_i to be T_i and assume that no refinement of Ω_i is required. On the other hand, if the comparison of the values of T_i and T_{ij} fails the specified criterion, the procedure described above is applied adaptively to each of the subregions Ω_{ij} with each Ω_{ij} now playing the rôle of Ω_i. This refinement procedure may be carried on until the length scale of the smallest subregion is small enough for us to determine the interfaces of regions with different transmissivity. This mesh has, not surprisingly, a striking resemblance with finite element meshes generated by adaptive refinement procedures (Demkowicz and Oden [9]). Thus, in implementing this algorithm, it is possible to utilize any one of the many well-tested data structure routines designed for handling this type of mesh structure. Another important consideration is that since this procedure automatically places more grid points near the zonation interfaces, the resulting mesh is also well suited for any subsequent solution of forward problems involving (1). Because the solution of the forward problem is generally less accurate on regular meshs near discontinuities, the aggregation of mesh points in such regions will help to improve the accuracy.

We note in passing that triangles could have been chosen to subdivide the domain. The basic algorithm remains essentially the same.

We now discuss a third approach, which will be called the **hybrid approach** because it may be regarded as a compromise between the two methods described above. Beginning with a suitably coarse mesh, the corresponding transmissivities T_i and T_{ij} are computed for each of the subregions. Over subregions where the transmissivities fail the specified criterion, a *uniformly refined* local mesh is imposed. Over the refined region, this process is continued until the nature of the zonation is clarified.

On first sight, it may appear that the approach is not as computationally efficient as the adaptive grid idea described earlier. However, due to the *uniform* refinement characteristics of the hybrid approach, it drastically decreases the bookkeeping necessary to keep track of the adaptive mesh strategy. Moreover, it also permits the application of the method in a parallel computing environment and implementation is considerably easier than the adaptive approach.

It is important to note that the linear functional strategy (3) will fail if $\Theta = ((\text{ grad } \phi, \text{ grad } W))$ is equal to zero. Thus, to minimize error, W should be chosen so that $|\Theta|$ is as large as possible. This fact has already been noted by Anderssen and Dietrich [2]. Consequently, if ϕ is constant in some subregion Λ, T cannot be determined and new measurements with a source term q that generates a non-constant ϕ in Λ must be employed to identify T in that region.

3.2 Point Source Term.

If the source term is composed of a global piecewise continuous part and a discrete set of point sources and sinks, we may proceed in exactly the same manner as described in the previous section. Thus as long as we have available a global source term, the presence of point sources and sinks poses no additional problem.

In the situation where the source term q corresponds only to a series of wells (modelled as Dirac delta functions) inside the aquifer, the problem of determining T becomes more difficult, especially when the number of wells is small. Consider for example an aquifer with a single well and no other source term. If T is known to be constant over a subregion Λ not containing the well, then either T cannot be uniquely determined on Λ (when ϕ is harmonic), or T is estimated to be zero on Λ (when ϕ is not harmonic). Both situations are unsatisfactory and are only resolved if additional information is supplied by way of drilling a well inside Λ.

If T is constant over some region Λ containing a well, we must contend with the fact that the piezometric head contains a logarithmic singularity. Also the optimal way of determining Λ is not so clear. A natural approach to determine Λ is to begin with a small subregion R_1 containing the well, compute the transmissivity T_{R_1} using (3) and then comparing it with the transmissivity T_{R_2} computed for an enlarged region R_2 containing R_1. If $T_{R_1} = T_{R_2}$ then the process may be repeated by enlarging R_2. Eventually we will encounter some region R_N such that $T_{R_1} \neq T_{R_N}$. We may then take R_{N-1} as an approximation of Λ. This is a reasonable approach but one must bear in mind that there is no guarantee that R_{N-1} is "close" to Λ since the best way to select R_{j+1} as an enlargement of R_j is not at all clear.

Consequently, in view of the fact that point sources provide only very limited and localized information, the determination of the aquifer parameter and thus the implementation of the linear functional strategy may be difficult. In fact, in the absence of additional information, the problem may not be well-posed. To see this, let us suppose that the aquifer consists of one single source inside the subregion Λ and that over this region the transmissivity is constant. By subtracting out the singularity using the fundamental solution Γ corresponding to the point source, we have

$$- \text{div} (\text{ grad } (T\phi - \Gamma)) = 0 \qquad \text{in } \Lambda \ .$$

Thus $T\phi - \Gamma$ is harmonic over Λ. As there are infinitely many harmonic functions that may be defined over Λ, we may obtained many different values of the transmissivity T without any further information. This potential source of non-uniqueness indicates that the parameter identification problem may be very ill-posed.

One possible solution is to approximate the local source terms by global source terms. For example, we may use the Gaussian distribution function to approximate the Dirac delta function and then apply the methods described in the previous section. This tactic may also be coupled with an interactive approach on a graphics workstation. Through sensible human intervention the process of determining transmissivity and zonation may be optimized. Hopefully, the strategies proposed above will provide us with valuable insight into the nature of the zonation in any aquifer of interest.

4. The Linear Functional Strategy.

4.1 Sensitivity and Stability.

In this section we examine the sensitivity of the estimated transmissivity with respect to different choices of test functions and describe some stability results. In the next section, we discuss the optimal piecewise constant approximation in relation to the linear functional strategy.

For a given approximate piezometric head ϕ_h and an approximate source term q_h, let T_1 and T_2 be the values of the estimated transmissivity over the subregion Λ corresponding to the test functions W_1 and W_2, respectively. Thus

$$T_1((\text{ grad } \phi_h, \text{ grad } W_1)) = ((q_h, W_1))$$

and

$$T_2((\text{ grad } \phi_h, \text{ grad } W_2)) = ((q_h, W_2))$$

So, applying integration by parts and noting that W_1, W_2, *and* $W_2 - W_1$ vanishes on $\partial\Lambda$, we obtain

$$\begin{aligned}
&(T_1 - T_2)((\text{ grad } \phi_h, \text{ grad } W_1))((\text{ grad } \phi_h, \text{ grad } W_2)) \\
&\quad = ((q_h, W_1))((\text{ grad } \phi_h, \text{ grad } W_2)) - ((q_h, W_2))((\text{ grad } \phi_h, \text{ grad } W_1)) \\
&\quad = ((q_h, W_1 - W_2))((\text{ grad } \phi_h, \text{ grad } W_2)) + ((q_h, W_2))((\Delta\phi_h, W_1 - W_2)),
\end{aligned}$$

where $\Delta\phi_h$ denotes the Laplacian of ϕ_h. Dividing through by $((\text{ grad } \phi_h, \text{ grad } W_2))$ which is assumed to be nonzero, we find that

$$|T_1 - T_2| = \frac{|((q_h + T_2\Delta\phi_h, W_1 - W_2))|}{|((\text{ grad } \phi_h, \text{ grad } W_1))|}$$

$$\leq \frac{\|q_h + T_2\Delta\phi_h\|_{L^2(\Lambda)}}{|((\text{ grad } \phi_h, \text{ grad } W_1))|}\|W_1 - W_2\|_{L^2(\Lambda)}. \tag{4}$$

This result signifies that, in order to have a transmissivity value which is relatively insensitive to changes in the test functions, these functions should be chosen so that the inner products $((\text{ grad } \phi_h, \text{ grad } W_i))$, $i = 1, 2$, are maximized, and the actual transmissivity should be approximately constant over the subregion Λ so that the $\|q_h + T_i\Delta\phi_h\|_{L^2(\Lambda)}$, $i = 1, 2$, are reasonably small.

It is easy to check that, when the approximate source term q_h and test function W are fixed, the sensitivity of the transmissivity with respect to different choices of approximate piezometric head functions ϕ_h^1 and ϕ_h^2 is given by the inequality

$$\left|T^1 - T^2\right| \leq \frac{|T^1| \ \|\text{ grad } W\|_{L^2(\Lambda)}}{|((\text{ grad } \phi_h^1, \text{ grad } W))|}\|\text{ grad } (\phi_h^1 - \phi_h^2)\|_{L^2(\Lambda)},$$

where T^i denotes the transmissivity corresponding to ϕ_h^i for $i = 1, 2$.

Turning now to consider the issue of stability of the linear functional strategy, let us recall that a basic assumption in applying the strategy is that the transmissivity $T(x)$ is a piecewise constant function over the region Λ. In reality, it is more likely that the transmissi slowly varying and is only approximately piecewise constant. It is therefore of interest to examine the stability of the linear functional strategy under small perturbation.

To model the slow variation of the transmissivity over a subregion Λ, let us assume that

$$T(x) = T_0 + \epsilon(x),$$

where T_0 is a constant and ϵ is a small perturbation. In [5] it was shown that whenever the piezometric head ϕ satisfies certain boundary condition on $\partial\Lambda$, the transmissivity $\tilde{T}$ obtained through the application of the linear functional strategy to (1) over Λ satisfies the stability inequality

$$\left|T_0 - \tilde{T}\right| \leq \|\epsilon\|_{L^\infty(\Lambda)}. \tag{5}$$

Moreover, in cases where the piezometric head and the source term are only available in approximate forms ϕ_h and q_h, the approximate transmissivity $\tilde{T}_h$, which is obtained by applying the linear functional strategy to (1) while utilizing ϕ_h and

q_h in place of ϕ and q respectively, may be shown (see [5]) to satisfy the stability inequality

$$\begin{aligned} \left|T_0 - \tilde{T}_h\right| &\leq \tfrac{1}{c_0}\|T\|_{L^\infty(\Lambda)}\|\text{ grad }(\phi - \phi_h)\|_{L^2(\Lambda)} + \|\epsilon\|_{L^\infty(\Lambda)} \\ &+ \tfrac{C}{c_0}\|q - q_h\|_{L^2(\Lambda)} \end{aligned} \tag{6}$$

provided that

$$\|\text{ grad } \phi_h\|_{L^2(\Lambda)} \geq c_0$$

for some positive constant c_0.

These results clearly show that the linear functional strategy yields a stable reconstruction of the transmissivity. As long as the perturbation remains small and the approximate data are sufficiently accurate, the strategy recovers the representative parameter value over the subregion Λ.

4.2 Optimal Approximation.

We now turn to consider the optimality of the piecewise constant approximation. Suppose the aquifer is partitioned by n subregions Λ_j, $j = 1, \ldots, n$ and that over each Λ_j, the transmissivity T_j is approximately constant. In order to compute the optimal constant approximation $\hat{T}_j$ to T_j, we solve the minimization problem

$$\sum_{j=1}^{n} \min_{\hat{T}_j \in \mathbb{R}} \|q + \text{ div } (\hat{T}_j \text{ grad } \phi)\|^2_{L^2(\Lambda_j)}.$$

Differentiating with respect to $\hat{T}_i$, setting the resulting expression to zero, and solving for $\hat{T}_i$, we obtain

$$\hat{T}_i = \frac{\int_{\Lambda_i} q\Delta\phi dx}{\int_{\Lambda_i}(\Delta\phi)^2 dx}, \qquad i = 1, \ldots, n. \tag{7}$$

The values $\hat{T}_i$ thus obtained will therefore minimize the residual (or equation error), and the optimal piecewise constant approximation to the transmissivity is given by

$$\hat{T}(x) = \hat{T}_i \qquad \text{for } x \in \Lambda_i.$$

It is easy to check that, if $\Delta\phi = 0$ on $\partial\Lambda_i$, then the transmissivity $\hat{T}_i$ computed from (7) is identical to the transmissivity $\tilde{T}_i$ obtained from the linear functional strategy using $W = \Delta\phi$ as a test function over Λ_i.

At first sight it appears that, since the $\hat{T}_i$ are easily evaluated using (7) and since the resulting piecewise constant approximation $\hat{T}$ of (7) is optimal in the sense of minimizing equation error, there is no need to apply the linear functional strategy if we are only seeking a piecewise constant representation of the transmissivity.

This is indeed the case if we have an exact expression for $\Delta\phi$ and have a clear idea of the zonation structure of the aquifer. However, in practice the piezometric head is only available in an approximate form ϕ_h and more likely than not the zonation structure is by and large unknown.

In fact, the optimal piecewise constant representation approach is quite unsatisfactory in applications. The major defect is its inability to detect zonation structure within an aquifer. This is illustrated in an example in the last section. The insensitivity to the presence of discontinuous parameter value within a subregion simply implies that the approach is not of great value compared with the linear functional approach.

Moreover, even if the zonation structure is clearly defined, the transmissivity values evaluated from the expression

$$\hat{T}_j = \frac{\int_{\Lambda_j} q_h \Delta\phi_h dx}{\int_{\Lambda_j} (\Delta\phi_h)^2 dx}, \qquad j = 1, \ldots, n \tag{8}$$

cannot be assumed to be close to those obtained from the optimal constant approximation (7) even if ϕ_h approximates ϕ well, since any attempt to numerically differentiate ϕ_h to find the second derivative term $\Delta\phi_h$ will introduce a large and perhaps unacceptable error.

On the other hand, since only the first derivative terms grad ϕ_h of the approximate piezometric head are needed in the linear functional strategy and since test functions may be chosen to minimize any error introduced in the process of numerical differentiation, the estimated transmissivity will be less susceptible to instability and thus provide a more reliable value. These facts clearly make the linear function strategy more favourable.

Consequently, the idea of combining the two strategies, by first determining the zonation structure using the linear functional strategy and then evaluating the optimal piecewise constant approximation to the transmissivity via (8), is not as attractive as it may seem. Proper use of the linear functional strategy alone will lead us to the desired result.

5. A Petrov-Galerkin Generalization.

In the event that the amount of data permits a more accurate representation of the parameter, it is of interest to consider generalization of the linear functional straegy to yield piecewise linear or quadratic approximation to the transmissivity. In [5], it was shown that using a Petrov-Galerkin interpretation of the linear functional strategy, it is possible to develop techniques that simultaeously determine the zonation structure of the aquifer under study and an accurate piecewise higher order polynomial representation of the transmissivity.

Essentially, for the piecewise constant approximation that we are considering here, a proper choice of trial and test spaces corresponding to the linear functional strategy leads to the construction of a Gram matrix A that is in fact diagonal. This special structure allows us to explain the total decoupling of the calculation, leading to the localized and the highly parallelizable nature of the methodology, and the insensitivity to the absence of boundary information.

Consequently, we see that by choosing the trial space and the triangulation carefully, and by constructing the test functions judiciously, the linear functional strategy may be extended to compute piecewise linear or higher order approximation to the transmissivity while ensuring that no boundary information is required in the computation and that the solution is parallelizable. A proper construction would lead to a Gram matrix A with a block diagonal structure, with the block boundaries providing a clear picture of the zonation structure of the aquifer.

6. Numerical Examples.

To illustrate the numerical performance of the linear functional strategy, let us consider the following examples (using the notation of Section 4).

Example 1.

(a) Let $\phi_h = \frac{1}{4}x^4$, $q_h = -qx^2$ and $T \equiv 3$ over $\Lambda = (-1,1)$.
For this example, ϕ_h', ϕ_h'' and ϕ_h''' all vanish at $x = 0$ inside Λ. However, this has no effect on the transmissivity value $\tilde{T}$ estimated from (3) using the test function $W = 1 - x^2$, as $\tilde{T} = 3$.

(b) If $\phi_h = \frac{1}{4}x^4(1+\delta)$, $q_h = -9x^2$ and $T = 3(1+\epsilon e^x)$ over $\Lambda = (0,1)$, then with $W = x(1-x)$, we have $\tilde{T} = \frac{3}{1+\delta}$ which gives a reasonable approximation for $|\delta| < 1$. Note that since no information concerning the ϵ term in T occurs in ϕ_h and q_h, it is not surprising to find that $\tilde{T}$ does not reflect such structure of T. This example also show how perturbation in the piezometric head is reflected in the estimated transmissivity.

Example 2.

Consider $\phi_h = \frac{1}{3}x^3 + x + \delta$, $q_h = -6x$ and $T = 3(1+\epsilon e^x)$ over $\Lambda = (-1,1)$. The linear functional strategy breaks down when we pick the natural choice of test function $W = 1 - x^2$, since we now have $((\phi_h', W')) = 0$. On the other hand, the optimal piecewise constant approximation will give the value $\hat{T} \equiv 3$. However, if we had selected the test function as $W = (1-x^2)(1+\sigma(x+1))$. Then $((\phi_h', W')) = -\frac{8}{15}\sigma$ and $((q, W)) = -\frac{8}{5}\sigma$, thus for $\sigma \neq 0$, we recover the transmissivity value $\tilde{T} = 3$. This result is interesting as it demonstrates the robustness of the linear functional strategy: even with a test function that is a small perturbation from a bad test function, we are still able to estimate the transmissivity value reasonably accurately.

With the test function $W = x^2 \sin \pi x$, which is an odd function like q_h, we have

$$((\phi_h', W')) = -\frac{4}{\pi^3}(\pi^2 - 6),$$

and

$$((q, W)) = -\frac{12}{\pi^3}(\pi^2 - 6),$$

thus again we have $\tilde{T} = 3$.

Example 3.

Consider now a problem with $\phi_h = \frac{1}{3}x^3 + (1+\delta)x$,

$$q_h = \begin{cases} -6x & x > 0, \\ -2x & x < 0, \end{cases}$$

and

$$T_1 = \begin{cases} 3 & x > 0 \\ 1 & x > 0 \end{cases}$$

Note that q_h is continuous at $x = 0$. It is possible to construct example with q_h as smooth as we like.

Over the interval $(0,1)$ with $W = x - x^2$ and the interval $(\frac{1}{2}, 1)$ with test function $W = (2x-1)(x-1)$, it is easy to check that the transmissivity has a value 3 over these intervals. Likewise, using $W = (2x+1)(x+1)$ over $(-1, -\frac{1}{2})$ and $W = x + x^2$ over $(-1, 0)$, we recover the transmissivity value 1.

Over the interval $(-a, a)$ where the transmissivity has a discontinuity at $x = 0$, the test function $W = a^2 - x^2$ gives a zero value for $((\phi_h', W'))$ and so the linear functional strategy does not apply. The optimal piecewise constant approximation gives a value $\hat{T} = 2$ for all $a > 0$. Such $\hat{T}$ is obviously the average of the values of the transmissivity on both sides of the discontinuous point $x = 0$. However, the independence of $\hat{T}$ on a is clearly a problem if such method is to be used to detect zonation structure. Such method will simply provide false information concerning the zone over which the transmissivity is constant.

On the other hand, by selecting a test function that is a small perturbation of the inadmissible test function $W = a^2 - x^2$, we are able to discover the zonation information contained in the data. More specifically, consider $W = (a^2 - x^2)(1 + \sigma x)$. We have

$$((\phi_h', W')) = -\frac{8}{15}a^5\sigma, \text{ and}$$
$$((q_h, W)) = -a^4 - \frac{16}{15}a^5\sigma.$$

Thus

$$\tilde{T} = 2 + \frac{15}{2\sigma a}. \quad (9)$$

Thus if we are considering the transmissivity in successively smaller neighbourhood of the origin, the linear functional strategy will yield increasing values of $\tilde{T}$, thus indicating the rapid changing nature of the transmissivity near the origin.

Another interesting feature is the rôle of the perturbation parameter played in (28). The smaller the value of σ, the more prominent will the change of $\tilde{T}$ be as we vary a. Thus, it appears that even if we are making use of test functions that are close to those that are inadmissible, we may still be in a fairly good position to determine the zonation structure of the transmissivity.

7. Conclusion.

In summary, we have shown in this paper that through a careful and systematic way of employing the linear functional strategy, it is possible to arrive at an efficient, robust, accurate, stable, adaptive and parallelizable approach to the problem of determination of the parameter values of the transmissivity of an aquifer. The implementation leads not only to a piecewise constant representation of the transmissivity but also to a clear picture of the zonation structure of the aquifer in question. With this approach, it seems possible to extract the maximal amount of information that is carried by the differential equation (1) and the (approximate) piezometric head and source data.

Acknowledgement. The advice of Drs. Garry Newsam and Claude Dietrich is gratefully appreciated.

References.

1. Anderssen, R.S., The linear functional strategy for improperly posed problems, in *Inverse Problems*, J.R. Cannon and U. Hornung, editors, Birkhäuser Verlag, Basel, 1986, pp. 11-30.

2. Anderssen, R.S. and C.R. Dietrich, The inverse problem of aquifer transmissivity identification, *Research Report CMA-R01-87*, Centre for Mathematical Analysis, Australian National University, Canberra, Australia, January 1987.

3. Bank, R.E., A.H. Sherman, and A. Weiser, Refinement algorithms and data structures for regular local mesh refinement, in *Scientific Computing*, R. Stepleman, *et al.*, editors, IMACS, North Holland, 1983.

4. Bear, J., *Dynamics of Fluids in Porous Media*, Elsevier, New York, 1972.

5. Chow, S.-S. and R.S. Anderssen, Determination of the transmissivity zonation using a linear functional strategy, *Inverse Problems* 7 (1991), 841-851.

6. Coats, K.H., J.R. Dempsey, and J.H. Henderson, A new technique for determining reservoir description from field performance data, *Soc. Petroleum Engin. Journal*, 10 (1), 1970, pp. 66-74.

7. Cooley, R.L., A method of estimating parameters and accessing reliability for models of steady state groundwater flow 1, *Water Resour. Res.*, 13 (2), 1977, pp. 318-324.

8. Cooley, R.L., A method for estimating parameters and accessing reliability for models of steady state groundwater flow 2, *Water Resour. Res.*, 15 (3), 1979, pp. 603-617.

9. Demkowicz, L. and J.T. Oden, A review of local mesh refinement techniques and corresponding data structures in h-type adaptive finite element methods, *TICOM Report 88-02*, Texas Institute for Computational Mechanics, University of Texas at Austin, Austin, Texas.

10. Emsellem, Y. and G. de Marsily, An automatic solution for the inverse problem, *Water Resour. Res.*, 7 (5), 1971, pp. 1264-1283.

11. Rheinboldt, W.C. and Ch. K. Mesztenyi, On a data structure for adaptive finite element mesh refinements, *ACM Trans. on Math. Software*, Vol. 6, No. 2, 1980, pp. 166-187.

12. Sun, N.Z. and W.W.-G. Yeh, Identification of parameter structure in groundwater inverse problems, *Water Resour. Res.*, 21 (6), 1985, pp. 869-883.

13. Yeh, W.W.-G., Review of parameter identification procedures in groundwater hydrology: The inverse problem, *Water Resour. Res.*, 22 (2), 1986, pp. 95-108.

14. Yeh, W.W.-G. and Y.S. Yoon, A systematic optimization procedure for the identification of inhomogeneous aquifer parameters, in *Advances in Groundwater Hydrology*, Z.A. Saleen, editor, American Water Resources Association, Minnesota, 1976, pp. 72-82.

R. S. Anderssen,
CSIRO Division of Mathematics and Statistics,
and Centre for Mathematical Analysis, ANU,
Canberra, ACT 2601, Australia

and

S.-S. Chow,
Department of Mathematics,
University of Wyoming,
Laramie, Wyoming 82071, USA