COUPLED DERIVATIVE/MIXED FINITE ELEMENT APPROACH TO VISUAL RECONSTRUCTION

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1 INTRODUCTION

We are primarily concerned in this paper with the reconstruction of quantities from an image. We include in this class image reconstruction (where the reconstructed quantity is a restored or enhanced version of the image) or visual reconstruction (where the reconstructed quantity can be the distance to objects within the scene or some other "real world" quantity of interest).

It has become popular in these areas to seek a regularized solution since one can consider the problem as being an inverse problem. Typically, one may characterise the regularised problem as seeking a function Ψ that minimises some functional

(1) $$E(\Psi) = D(\Psi; g) + S(\Psi)$$

where g is the given data, and the functionals D and S are chosen to encourage solutions that are faithfull to the data and are smooth (respectively).

A common choice for the smoothness functional is a (possibly weighted) sum of the squares of various orders of derivative of the function Ψ (usually this becomes a Sobolev norm or semi-norm). This paper proposes an approach where various orders of the derivative are explicitly reconstructed simultaneously with the function Ψ itself. Such an approach creates a series of coupled sub-problems. We also suggest a novel analog (neural network-like) implementation for the solution of these coupled sub-problems.

Our proposed scheme is similar in philosophy to the Harris Coupled Depth-Slope model of visual reconstruction [Har87], and also to methods of solution of partial differential equations that are known as mixed finite element methods.

Section 2 outlines the variational theory behind our approach. We then (section 3) introduce our mixed finite element formulation and a novel mathematical programming approach to solve these formulations. We also illustrate with some examples.

2 VARIATIONAL FORMULATION

The abstract minimization problem is formulated for a subset U of a normed vector space V, a continuous bilinear form $a(\cdot,\cdot) : V \times V \mapsto R$, and a continuous linear form $f : V \mapsto R$. This problem is to select the element $u \in U$ such that:

$$J(u) = \inf_{v \in U} J(v) \tag{2}$$

$$J(v) = \frac{1}{2}a(v,v) - f(v).$$

It can be shown (e.g. [Cia78] pp. 3-8) that the solution u of the abstract minimization problem 2 satisfies:

$$a(u,v) = f(v) \;, \forall v \in U \tag{3}$$

if U is a closed subspace. Moreover, 3, can have relevance whether $a(\cdot,\cdot)$ is symmetric (in which case J is an energy functional) or not.

Proofs of well-posedness within this variational setting generally rely upon the generalized Lax-Milgram theorem [CO83].

Terzopoulos [Ter82] used the type of theory just outlined to formulate and to prove uniqueness and existence of a solution to his thin plate visual surface reconstruction formulation: we will need to elaborate on this theory to provide a basis for our mixed finite element reformulation. In essence, our approach depends upon Augmented Lagrangian formulations. These formulations can be considered to be the generalization (or blending) of the Lagrangian (see section 2.1) or Penalty (see section 2.2) approaches to constrained minimization.

2.1 ABSTRACT LAGRANGE MULTIPLIER FORMULATION

Suppose we wish to find a minimizer $u \in U$ of 2 subject to the condition:

$$b(u,q) = g(q), \tag{4}$$

for $q \in Q$ and "data" g.

We now form the Lagrangian $L : U \times Q \mapsto R$:

$$L(v,q) = J(v) + b(v,q) - g(q) \tag{5}$$

and seek the saddle point u, λ:

$$L(u,q) \leq L(u,\lambda) \leq L(v,\lambda), \ \forall v \in U, \ q \in Q. \tag{6}$$

We denote by P_{Lag} the problem of finding the solution u, λ to the Lagrangian formulation. Furthermore, where appropriate, we denote by P^h_{Lag} the corresponding finite dimensional problem using a discretization characterized by the parameter h.

It is easy to show that the saddle point must satisfy the necessary conditions:

$$\text{(7a)} \qquad a(u,v) + b(v,\lambda) = f(v) \ , \forall v \in U$$

$$\text{(7b)} \qquad b(u,\lambda) = q(\lambda) \ , \forall q \in Q$$

Existence and uniqueness of a solution can be ensured using either the theory outlined by Babuska [BA73] or Brezzi [Bre74].

2.2 PENALTY METHODS

Let $P : U \mapsto R$ be a penalty functional ([CO83] p.149, Theorem 2.7 [Hag85]), the sequence of solutions $\{u_\rho\}_{\rho \mapsto \infty}$ defined as the minimum u of:

$$\text{(8)} \qquad J_\rho(v) = J(v) + \rho P(v) \ \rho > 0, \ v \in U$$

converges under appropriate conditions on P, to the required solution of the constrained problem. We denote the penalty problem of minimizing 8 by $Ppen_\rho$ and the corresponding finite dimensional problem obtained under discretization $Ppen_\rho^h$. However, a naive discretization of the $Ppen_\rho$ does not generally work in practice ([CO83] p. 153) in that the approximate penalty term may fail to vanish when the constraints are satisfied.

2.3 AUGMENTED LAGRANGIAN

It is well known (e.g. [Ber76] [Ber82]), at least in a finite dimensional context, that the Lagrangian methods can be greatly enhanced in terms of stability by adopting an Augmented Lagrangian approach.

We denote by P_{Aug} the problem formulated using the Augmented Lagrangian approach. The Augmented Lagrangian itself is defined as:

$$L_A = L(v,p) + \rho P(v). \tag{9}$$

where ρ is some positive constant, and $P(\cdot)$ is defined as in section 2.2. In our case we propose to use $P(v) = \frac{1}{2} \parallel Bv - g \parallel_Q^2$:

$$L_A = J(v) + [q, Bv - g] + \rho\frac{1}{2}(Bv - g, Bv - g). \tag{10}$$

It is a straightforward task to derive the abstract Euler-Lagrange equations for the Augmented form: the last term, after expansion, simply contributes $[Bu - g, Bv]$ to the expression. As such, the augmenting term does not affect the existence (and nature) of the solutions: it does add stability to many algorithms that solve Lagrangian formulations. For simplicity, we will generally omit the Augmenting term, but it is generally to be understood, for implementation, we will use the Augmented Lagrangian form.

2.4 REPRESENTING THE MULTIPLIERS

One common objection to the use of primal-dual approaches such as Lagrangian or Augmented Lagrangian techniques is that it would appear to be very costly (at least in terms of storage to represent the additional multiplier variables - if not also in terms of computation) to have to expand the dimension of the problem by introducing the Lagrange multiplier space. It turns out, however, that in our intended applications, the Lagrange multipliers have a particular interpretation that makes them useful quantities to estimate. Specifically, the problems of interest to us here are of the type:

$$\text{(11)} \qquad \text{minimize} \underset{\{\mathrm{u} : \mathrm{Bu} = \mathrm{q}\}}{J(u)} = F(Bu) + G(u) \ ,$$

where $F(x) = \frac{1}{2}(x, x)$, and B is a linear operator; and we are interested in seeking approximations to both u and q. Naturally we transform this into the unconstrained problem of seeking a saddle point of:

$$\text{(12)} \qquad L(u, q, \lambda) = F(q) + G(u) + < \lambda, (Bu - q) > .$$

The Euler-Lagrange equations then follow:

$$\text{(13a)} \qquad \frac{\delta L(u, q, \lambda)}{\delta q} = (q, q') - (\lambda, q') = 0, \ \forall \ admissible \ q'$$

$$\text{(13b)} \qquad \frac{\delta L(u, q, \lambda)}{\delta \lambda} = (\lambda', (Bu - q)) = 0, \ \forall \ admissible \ \lambda'.$$

Equation 13a thus implies that $q = \lambda$[1] and, in turn, equation 13b implies that $q = Bu$. Thus we do not need to explicitly represent and calculate q and λ separately. Furthermore, even in cases where no useful interpretation of these variables can be found, it can be unnecessary to specifically compute these variables if one adopts a multiplier method approach to the solution of the Augmented Lagrangian [Ber76].

2.5 STANDARD MIXED METHODS

Our coupled "sub-problem/mixed finite element" approach (section 3) borrows heavily from, and was partly motivated by, standard mixed methods for solving elliptic boundary value problems. We will outline these methods before defining our approach.

The particular mixed methods we consider here fall under the general term "Decomposition-Coordination" approaches ([FG83] Chapter III). In its more restricted forms, our formulation is very close to that of standard mixed methods for boundary value problems. In this more simple form, our approach can be directly applied to decompose the thin-plate visual reconstruction approach of Terzopoulos [Ter82] into two coupled membrane problems. Such an approach not only has consequent computational advantages, but also is capable of delivering independent approximations to various bending moments (which could be used, for example, in a process that seeks likely candidates for edge points in segmentation).

[1] More precisely, they differ only by a quantity that is orthogonal to the admissible functions.

We will illustrate the standard methods in this section, for the 4^{th} order problems.

4^{th} order Biharmonic Equation

Example 2.1 *This 4^{th} order problem is particularly interesting since it is closely related to the thin-plate visual reconstruction formulation of Terzopoulos [Ter82].*

It is well known that minimizing the quadratic variation gives rise to the biharmonic equation for the Euler-Lagrange equations:

$$\Delta^2 w = f. \tag{14}$$

In attempts to solve the biharmonic equation, several authors have used a two stage decomposition:

$$u + \Delta w = 0 \tag{15a}$$

$$-\Delta u = f \tag{15b}$$

(e.g. [OC83] page 144). In addition to the computational advantages, such methods have been used to obtain simultaneously the stream function and vorticity in hydrodynamics (e.g. [CR74]), or to obtain the plate displacement and second order moments for the thin plate problem (e.g. [BR77]). Indeed, if one views the

4^{th} *order problem as the minimization of the functional:*

$$J(\Psi, u) = \frac{1}{2}\int_{\Omega} u^2 \,, \tag{16}$$

subject to the constraint $\Delta\Psi = u$*, then one naturally approximates the two scalar fields* Ψ, u*. If, instead* [2]*, one chooses to minimize the quadratic variation by seeking to minimize:*

$$J(\Psi, p, q, r) = \frac{1}{2}\int_{\Omega} p^2 + 2q^2 + r^2 \,, \tag{17}$$

subject to $\frac{\partial\Psi}{\partial x^2} = p, \frac{\partial\Psi}{\partial xy} = q, \frac{\partial\Psi}{\partial y^2} = r$*, one approximates the moments as well as the function itself. We have already mentioned that the moments may prove useful for segmentation purposes.*

Thus these mixed methods are immediately applicable to the problem of reconstruction of depth values in stereopsis if one maintains the standard formulation of [Ter82]. This allows one to take advantage of the computational advantages that accrue from the decomposition process. In particular, they allow the use of much more simple elements. We can thus avoid the complex non-conforming element of Terzopoulos [Ter82][3].

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[2] Terzopoulos [Ter82] formulated, in principle, his problem as the convex combination of the Laplacian and the quadratic variation (i.e. both of these alternatives); although he later specialized to the quadratic variation

[3] It is well known (see [CO83] p. 35, for example) that if the solution actually belongs to the Sobolev space $H^r(\Omega)$, the variational statement contains elements from the space $H^m(\Omega)$, and the finite element basis functions contain complete polynomials up to degree k; then the error in

3 MIXED FINITE ELEMENT VISUAL RECONSTRUCTION

In this section we introduce an approach[4] to visual reconstruction that makes explicit every order of derivative (i.e. the reconstructed function, the first derivatives, the second derivatives, and so on) up to the highest order[5] included in the regularization.

The general idea of our mixed finite element approach is to decompose higher order smoothness constraints into a cascade of lower order smoothness constraints by independently approximating each order of derivative by a separate function. We then introduce a series of constraints connecting these approximations to the appropriate order of derivative of the function we wish to

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the interpolant u_h (measured in the H^s norm) satisfies the following error estimate as $h \mapsto 0$:

$$\| u - u_h \|_s \leq Ch^{k+1-s} \| u \|_r, \; 0 \leq s \leq m \,, \tag{18}$$

provided that $r \geq k+1 > m$. In the above, C is a constant, and the inequality essentially states that the error is $O(h^{k+1-s})$. For second order problems $m = 1$ and thus $k > 0$. Whilst for fourth order problems $m = 2$ and so $k > 1$. Therefore, if we decompose a fourth order problem into 2 second order problems, we can use finite elements constructed from linear elements rather than quadratic elements. It would seem that we pay a price for this simplicity by a poorer reconstruction of the function between nodal points: this can be alleviated somewhat by employing postprocessing techniques such as those discussed in [BX89], for example.

[4] We call this approach by several names. This reflects the fact that, what we propose, in total, includes contributions at all stages (formulation, discretization, algorithms, and implementations). Thus we will sometimes refer to our "coupled sub-problem approach" (the general decomposition idea), our "mixed finite element approach" (discretization), or our "neural network/mathematical programming approach" (our implementation and algorithms). We will also use "coupled depth-slope" or "explicit depth slope" to describe our approach. These emphasize the similarity with the analog network proposed by Harris [Har87].

[5] Although our complete model calls for the independent approximation of all derivatives up to the highest order included in the formulation, we often find it sufficient to approximate only some of these. Furthermore, we may sometimes wish to approximate a *combination* of some partial derivatives, rather than each one separately. Thus our approach is the natural generalization of the mixed methods outlined previously.

reconstruct.

This idea is actually independent of the discretization/approximation stage (i.e. the finite element part), and provides a very general mathematical model that is an abstract generalization of the "Coupled Depth Slope" structural model of of analog computation in visual reconstruction [Har87]. It should be emphasized that the model we propose is not only more abstract and mathematical, but we make it more general by identifying two different methods for ensuring consistency between the function and the approximations to its derivatives: one is through penalty terms, and the other is through Lagrange multipliers (leading directly to the mixed finite element formulation). Indeed, our proposed Augmented Lagrangian approach combines the two formulations and hence can be specialised to either.

Being able to represent and approximate all orders of derivatives simultaneously has some potential advantages: for example in shape from shading problems the data directly constrains the slopes but, ultimately, we may wish to calculate the depth. In general surface approximation the second derivatives are useful for detecting "roof" edges, the first derivatives for step edges; so the recovery of all derivatives up to order 2 would aid simultaneous reconstruction and segmentation approaches. Various combinations of second derivatives could be used as feature detectors [ZB90] and may be useful in optical flow calculations [VGT90].

A complete formulation of this, for arbitrary order and arbitrary dimension of the problem, is possible but may involve excessively burdensome notation. We note that, in general, a second order Sobolev norm is the highest order generally used in practice, so we restrict our illustrations to a that order. We also restrict our attention to functions in one dimension: generalization to higher dimensions are straightforward.

3.1 SURFACE RECONSTRUCTION

The reconstruction problem is formulated as one of determining a saddle point of a Lagrangian $L(\psi, u, p)$. The Lagrange multiplier terms enforce the correct relationship between the functions ψ, u, p (i.e. they are zero'th, first, and second order derivatives of Ψ!).

We than discretize this functional to obtain a finite dimensional problem involving the nodal values ψ_i, u_i, p_i: this corresponds to a mixed finite element formulation. In order to solve this discrete saddle point problem we use the scheme of Platt [PB87] and Snyman [Sny88] where one performs gradient *descent* on the primary variables ψ_i and *ascent* on the dual variables u_i, p_i. This leads to a set of differential equations that must be intergrated to find the stationary point of the network/dynamical system. In these simple examples we use a straight forward[6] Euler integration routine.

Example 3.2 *We begin by construction our functional. In this example we con-*

[6]Pun intended!

sider using first order and second order smoothness terms:

(19a) $$\frac{1}{2}\int_{\Omega}(\frac{d\psi}{dx})^2\,dx$$

(19b) $$\frac{1}{2}\int_{\Omega}(\frac{d^2\psi}{dx^2})^2\,dx\;,$$

and the "spring" data compatibility term [Ter82]:

(20) $$\sum_{i\in C}\frac{\beta}{2}(\psi_i-d_i)^2\;,$$

where C is some set of constraint points.

Thus, our objective is to find ψ that minimizes:

(21) $$J(\psi)=\frac{1}{2}\int_{\Omega}(\frac{d\psi}{dx})^2\,dx+\frac{1}{2}\int_{\Omega}(\frac{d^2\psi}{dx^2})^2\,dx+\sum_{i\in C}\frac{\beta}{2}(\psi_i-d_i)^2.$$

Following our coupled approach, we introduce independent representations for the first two derivatives and thus reformulate as the constrained minimization problem of finding ψ,u,p that minimizes:

(22a) $$J(\psi,u,p)=\tfrac{1}{2}\textstyle\int_{\Omega}u^2\,dx+\tfrac{1}{2}\int_{\Omega}p^2\,dx+\sum_{i\in C}\tfrac{\beta}{2}(\psi_i-d_i)^2$$

(22b) *subject to:* $$u=\tfrac{d\psi}{dx}$$

(22c) *and* $$p=\tfrac{d^2\psi}{dx^2}.$$

We can then use Lagrange multipliers λ_1 and λ_2 to turn the problem into one of *seeking a saddle point of:*

$$L(\psi, u, p, \lambda_1, \lambda_2) = \tfrac{1}{2}\int_\Omega u^2\,dx + \tfrac{1}{2}\int_\Omega p^2\,dx + \sum_{i\in C}\tfrac{\beta}{2}(\psi_i - d_i)^2 + \int_\Omega \lambda_1(\tfrac{d\psi}{dx} - u)\,dx + \int_\Omega \lambda_2(\tfrac{d^2\psi}{dx^2} - p)\,dx. \tag{23}$$

Furthermore, it can easily be shown that the following holds (section 2.4):

$$\lambda_1 = u \tag{24a}$$

$$\lambda_2 = p\,, \tag{24b}$$

so we can simplify 23 to:

$$L(\psi, u, p) = -\frac{1}{2}\int_\Omega u^2\,dx - \frac{1}{2}\int_\Omega p^2\,dx + \sum_{i\in C}\frac{\beta}{2}(\psi_i - d_i)^2 + \int_\Omega u\frac{d\psi}{dx}\,dx + \int_\Omega p\frac{d^2\psi}{dx^2}\,dx. \tag{25}$$

Finally, we transform the last term to its weak constraint equivalent

$$L(\psi, u, p) = -\frac{1}{2}\int_\Omega u^2\,dx - \frac{1}{2}\int_\Omega p^2\,dx + \sum_{i\in C}\frac{\beta}{2}(\psi_i - d_i)^2 + \int_\Omega u\frac{d\psi}{dx}\,dx + \int_\Omega \frac{dp}{dx}\frac{d\psi}{dx}\,dx. \tag{26}$$

We now have to discretize 26. We choose to use the most simple finite

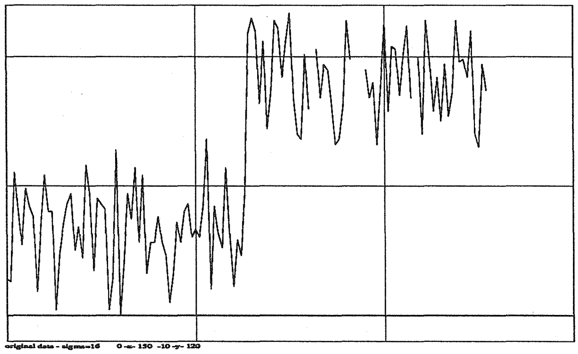

Figure 1: Step edge corrupted with guassian noise $\sigma = 16$
Original data of 128 samples. The step edge has been corrupted by additive gaussian noise of standard deviation $sigma = 16$.

elements possible - linear elements for $\psi, u,$ and p; over domains that are equal size intervals partitioning Ω into n parts. We label the nodal points $i : i = 0 \ldots n$.

We now have a finite dimensional saddle point problem. To solve this problem, we adopt the mathematical programming approach of Platt [PB87] and Snyman [Sny88]. In this approach, one performs gradient descent on the primary variables ψ_i, and ascent *on the dual variables u_i, p_i. This leads naturally to a dynamical system or neural network analog scheme:*

$$\frac{d\psi_i}{dt} = -\frac{\partial L}{\partial \psi_i} \tag{27a}$$

$$\frac{du_i}{dt} = \frac{\partial L}{\partial u_i} \tag{27b}$$

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial p_i}. \tag{27c}$$

These equations were integrated for the 128 data samples taken from Blake

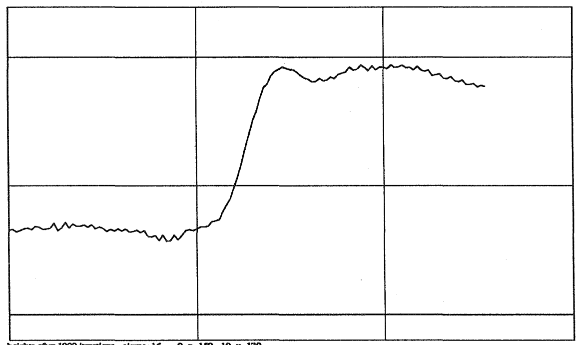

Figure 2: Reconstructed Step Edge - 1000 iterations
After 1000 iterations of the integration procedure, the function has been considerably smoothed.

[Bla89] with standard deviation of the noise $\sigma = 16$ *(figure 1). It is well known that the optimal value of* β*, in a bayesian sense, is* $\frac{1}{2\sigma^2}$*: this value was chosen in all simulations. The original values of the function were set to the data and the initial values of all derivatives are zero. A stepsize of* 0.01 *was used in the integration. The results are displayed after 1000 iterations (in figures 2, 3 and 4), and after 10000 iterations (in figures 5, 6 and 7). The results clearly demonstrate that the method can effectively reconstruct the function and its first and second derivatives simultaneously.*

3.2 DISCONTINUOUS REGULARIZATION

In this section, we consider whether the proposed scheme can be adapted for discontinuous regularization: i.e. to be able to reconstruct a piecewise continuous function. We have already demonstrated how the reconstruction process simul-

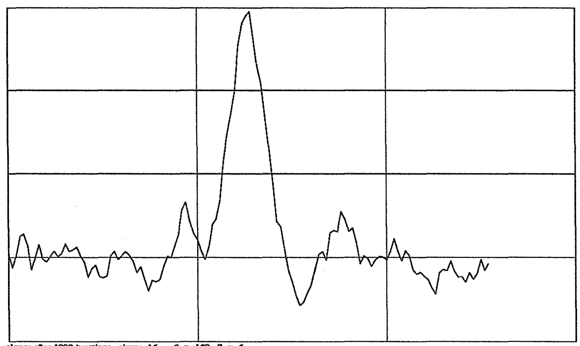

Figure 3: Reconstructed 1^{st} Derivative - 1000 iterations
After 1000 iterations, the smoothness in the reconstructed function is reflected by the present estimate of slopes. The slope of the original step edge clearly dominates.

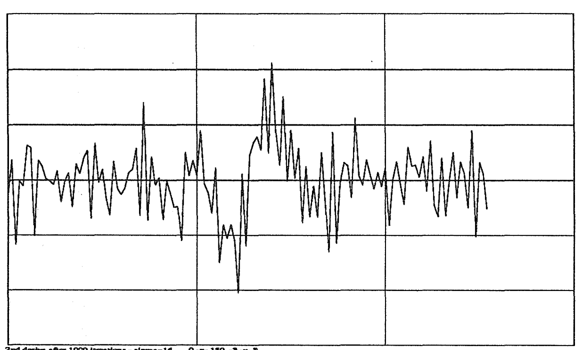

Figure 4: Reconstructed 2^{nd} Derivative - 1000 iterations
After 1000 iterations, there is a clear zero crossing of the second derivative at the edge location.

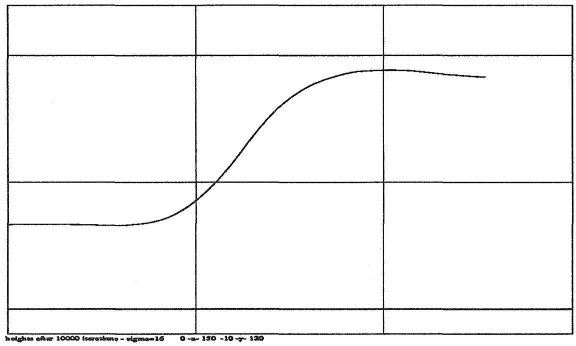

Figure 5: Reconstructed Step Edge - 10000 iterations
After 10000 iterations of the integration procedure, the function has been smoothed to a point where no apparent noise remains. As expected, the edge is over smoothed.

taneously yields estimates of the derivatives of the reconstructed function: these estimates could be used to locate edge candidates during the reconstruction. It is then possible to imagine a type of feedback where the output begins to inhibit the smoothness constraint in regions where there are high derivatives. In other words, the output of the derivative estimator "neurons" may inhibit certain actions of the smoothness "neurons". For simplicity, we illustrate this with first order "membrane" smoothness only.

Example 3.3 *We proceed in a similar manner to the previous example. Firstly, our objective is to minimize the functional:*

$$J(\psi) = \frac{1}{2}\int_{\Omega}(\frac{d\psi}{dx})^2\,dx + \sum_{i\in C}\frac{\beta}{2}(\psi_i - d_i)^2. \tag{28}$$

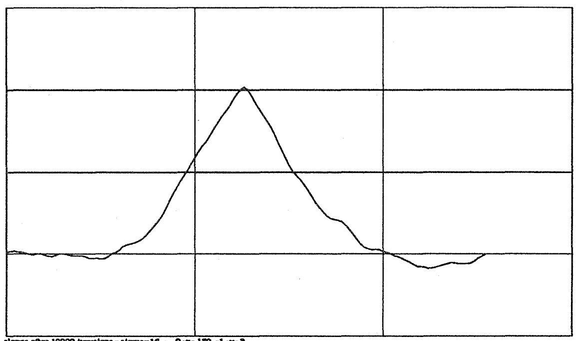

Figure 6: Reconstructed 1^{st} Derivative - 10000 iterations
After 10000 iterations, the step edge is clearly visible in the dominant peak of the first derivative. Smoothing has considerably broadened this peak, however.

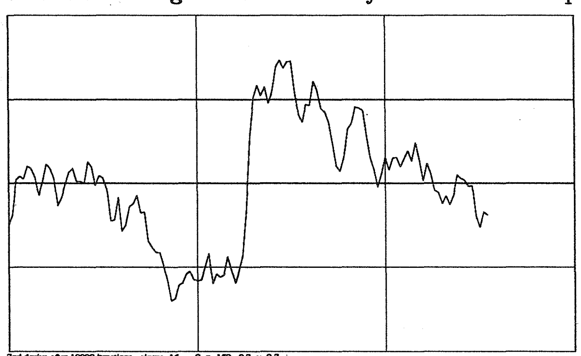

Figure 7: Reconstructed 2^{nd} Derivative - 10000 iterations
After 10000 iterations, there is a dominant zero crossing of the second derivative at the edge location. Note: however, the smoothing has progressed to such an extent that the second derivative is much smaller everywhere (the vertical scale is now much smaller than in figure 4) and, as a result, the noise is more apparent.

After introducing the auxiliary variable u to represent the slope, and reformulating as a saddle point problem, we obtain the Lagrangian:

$$L(\psi, u) = -\frac{1}{2}\int_\Omega u^2\,dx + \sum_{i\in C}\frac{\beta}{2}(\psi_i - d_i)^2 + \int_\Omega u\frac{d\psi}{dx}\,dx. \tag{29}$$

In order to discretize this functional, we choose piecewise linear elements for ψ (as before) and, now, piecewise constant elements for u. This can be viewed as associating a single node at the mid-point of each domain to denote the slope of the function over that interval. Such a choice is sensible as we require no higher order derivatives. So we have the discrete Lagrangian

$$L(\psi, u) = \sum_{i=0}^{n-1} -\frac{1}{2}u_i^2 + \sum_{i\in C}\frac{\beta}{2}(\psi_i - d_i)^2 + \sum_{i=0}^{n-1} u_i(\psi_{i+1} - \psi_i). \tag{30}$$

Again, we have arbitrarily set the nodal spacing to one unit.

We now perform gradient descent on this Lagrangian for the primary variable ψ, and ascent for the dual variable u.

$$\frac{d\psi_i}{dt} = \begin{cases} -\beta(\psi_i - d_i) - u_{i-1} + u_i & i = 1\ldots n \\ -\beta(\psi_0 - d_0) + u_0 & i = 0 \end{cases} \tag{31a}$$

$$\frac{du_i}{dt} = \Big\{ -u_i + (\psi_{i+1} - \psi_i) \quad i = 0\ldots n-1. \tag{31b}$$

We again can integrate these equations, and as expected, we obtain smoothed versions of the function and its derivative.

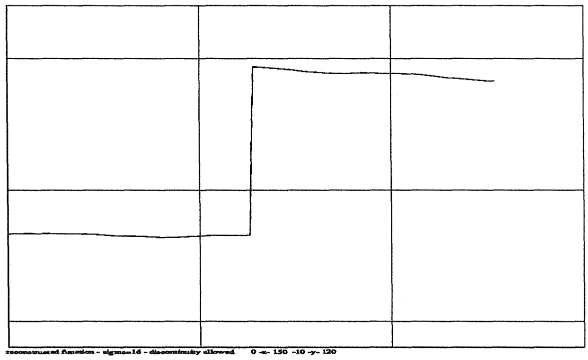

Figure 8: Reconstructed Step Edge - Discontinuity Allowed

However, in this example, it is immediately transparent how to allow a discontinuity (known to exist between data items 63 and 64). We already have, in our previous formulation, two discontinuities: one at the start and one at the end of our sample. It is immediately apparent that we only need to modify our update for ψ_{64} *by deleting the reference to* ψ_{63}*. A similar analysis, although more tedious, can be made for more complicated functionals. For our purposes we simply demonstrate the results with the present functional (see 8 and 9). Our previous example has shown how, during the reconstruction process, the first derivative is reliably reconstructed. Therefore, in "real life" applications it is possible that a thresholded first derivative may be used to decide at which points to turn off the smoothness constraint.*

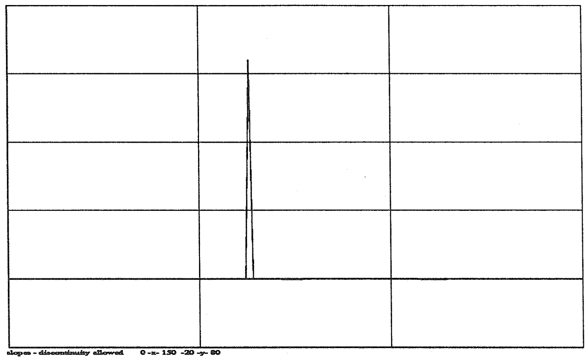

Figure 9: Reconstructed 1^{st} Derivative - Discontinuity Allowed

4 Conclusion

In addition to having an appealing theoretical form, our approach also has many potential advantages in terms of computational simplicity, ability to incorporate constraints upon the derivatives, natural analog implementation, and the ability to simultaneously reconstruct various derivatives (perhaps for segmentation or for feature extraction).

It is interesting that the approach outlined here seems to provide the necessary generalization and framework for the analog network approaches to visual reconstruction of Harris [Har87] [Har89] and of the recent digital approaches to shape from shading [Hor89] (the latter being motivated more on computational stability grounds). A more complete discussion of these issues can be found in [Sut90].

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