Chapter 1

Introduction

In the last ten years or so a considerable amount of work has been done to transform general relativity into a mathematically rigorous discipline. With the work of Christodoulou and Klainerman [29] on stability of Minkowski space–time, the work of Schoen and Yau [113] on the positive energy theorem, the work of Christodoulou [26] [27] on the gravitational collapse, the work of Newman [100] and others (*cf.* *e.g.* [55] and references therein) on Yau's Lorentzian splitting conjecture, the work of Bartnik [7] on maximal hypersurfaces in Lorentzian manifolds, general relativity has become a respectable field of mathematical research. For an analyst interested in differential geometry, general relativity turns out to be a rich source of various, sometimes extremely difficult, mathematical problems, encompassing all classical classes of partial differential equations — hyperbolic (*cf.* *e.g.* [49] [29] [53]), elliptic (*cf.* *e.g.* [19] [7] [9] [74] [127] [3] [2]), and even parabolic (*cf.* *e.g.* [8] [33]), as well as some difficult problems of the theory of dynamical systems (*cf.* *e.g.* [12] [121]). The aim of this paper is to present to a mathematically oriented reader one of the current research problems in general relativity — the problem of uniqueness in the large of solutions of Einstein's equations, also known under the baroque name of "strong cosmic censorship".

In this chapter we discuss some old and new results on global structure of space–time.

In Section 1.1 some classical results on the Cauchy problem in general relativity are reviewed — emphasis is put on things which are not known (and which the author would like to know[1]). In Section 1.2 the shortcomings of the classical singularity theory are pointed out. In Sections 1.3 and 1.4 the "strong cosmic censorship conjecture" and the "weak cosmic censorship conjecture" are discussed. In the Sections that follow, answers to some of the questions raised in the previous Sections are presented. In Section 1.5 results on cosmological space–times with spacelike Killing vectors are presented. In Section 1.6 some stability results are reviewed. In Section 1.7 the Robinson–Trautman space–times are discussed. We close this Chapter by discussing in Section 1.8 what restrictions one needs to impose on both Cauchy data and space–times so that generic predictability of Einstein's theory can be eventually attained. In Chapter 2, proofs of the results discussed in Section 1.5, the symmetry group $U(1) \times U(1)$ non-including, are given. The "$U(1) \times U(1)$ stability of strong cosmic censorship" in a neighbourhood of a $(p_1, p_2, p_3) = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ Kasner metric is proved in Chapter 3. Some miscellaneous results are collected in the Appendices. The motivation for this work and the results presented here are best illustrated by Tables 1.0.1 and 1.0.2 ("S.C.C." stands for "Strong Cosmic Censorship", "T.T.B.P." for "Theorem–to–be–proved", cf. Section 1.1). In table 1.0.1 all connected Lie groups which admit an effective action on a compact, connected, orientable three dimensional manifold and the appropriate manifolds are listed, following Fischer [45] [2].

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[1]Questions similar to the ones raised here can also be found in Refs. [43] [94] [6] [123].

[2]It should be pointed out that the list of manifolds which admit a $U(1)$ action given in [45] is incomplete.

Symmetry Group	${}^3\Sigma$ *Topology*	*Supplementary Conditions*	*S.C.C. (T.T.B.P.)*
$SO(4)$ $SO(3) \times SO(3)$	S^3 $\mathbb{P}^3$		no vacuum metrics[3]
$(U(1) \times SU(2))/D$	$L(p,1)$ p — odd		no[4]
$U(1) \times SO(3)$	$L(p,1)$ p — even		no[4]
	$S^1 \times S^2$		$\frac{1}{2}$ yes[5]
$SU(2)$	$L(p,1)$ p — odd		? (probably yes[6])
$SO(3)$	$L(p,1)$ p — even		? (probably yes[6])
	$S^1 \times S^2$ $\mathbb{P}^3 \# \mathbb{P}^3$		$\frac{1}{2}$ yes[5, 7]
	$\mathbb{P}^3$ S^3/Γ		no vacuum metrics[8]
$U(1) \times U(1) \times U(1)$	T^3		yes[9]
$U(1) \times U(1)$	$L(p,q)$ $S^1 \times S^2$	$g_{\mu\nu} X_1^\mu X_2^\nu = 0$	yes[10]
	T^3	$g_{\mu\nu} X_1^\mu X_2^\nu = 0$ $\epsilon_{\mu\nu\rho\sigma} X_1^\mu X_2^\nu \nabla^\rho X_a^\sigma = 0$	yes[10]
		"small" data, $\epsilon_{\mu\nu\rho\sigma} X_1^\mu X_2^\nu \nabla^\rho X_a^\sigma = 0$	$\frac{3}{4}$ yes[11]
	$L(p,q)$ $S^1 \times S^2$ T^3		?[12]
$U(1)$			?[13]

Table 1.0.1: Strong cosmic censorship, spatially compact case

Symmetry Group	${}^3\Sigma$ *Topology*	*Supplementary Conditions*	*S.C.C. (T.T.B.P.)*
$U(1) \times \mathbb{R}$	$\mathbb{R}^3$		yes[14]
$\{1\}$	$\mathbb{R}^3$	"small", smoothly conformally compact[15] "hyperboloidal" data	$\frac{1}{2}$ yes (yes to the future[16], ? to the past)
		"small", asymptotically flat data	yes[17]
	$\mathbb{R} \times S^2$	Robinson–Trautman space–times	$m > 0$: no[18] $m < 0$: $\frac{1}{2}$ yes

Table 1.0.2: Strong cosmic censorship in vacuum "minisuperspaces", spatially open case.

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[3] *cf.* page 25; $\mathbb{P}^3$ denotes the three dimensional real projective space.

[4] D is the subgroup $\{(1,1),(-1,-1)\}$, $L(p,q)$ denotes a lens space, recall that $L(1,1) = S^3$. These are the Taub-NUT metrics, *cf.* Section 2.2.

[5] These are the Kantowski–Sachs cosmological models; locally, the metric is the Schwarzschild metric "below the Schwarzschild horizon", $r < 2m$, *cf.* Section 2.3; the space-times are extendible to, say, the past of ${}^3\Sigma$ and inextendible to its future.

[6] These are the Bianchi IX metrics, *cf.* page 26; no rigorous results on inextendability (strong cosmic censorship) have been established so far.

[7] $\#$ denotes the connected topological sum.

[8] *cf.* Section 2.3; for a list of all possible groups Γ by which S^3 is divided, *cf.* [45].

[9] These are the Kasner metrics, *cf.* Section 2.4.

[10] These are the polarized Gowdy metrics; the X_a's, $a = 1, 2$, are Killing vectors, *cf.* Theorem 1.5.1.

[11] Yes to the, say, past, probably yes to the future, *cf.* Theorem 1.5.2.

[12] Partial results on global existence are available, *cf.* Section 1.5, but no inextendibility results are known.

[13] *cf.* [45] for a list of possible manifolds and actions; no rigorous results on s.c.c. are known. To the list of manifolds with a $U(1)$ action given in [45] one should add the Eilenberg–Mac Lane spaces $K(\pi, 1)$ whose fundamental group has an infinite cyclic center (A.Fischer, private communication).

[14] This is an unpublished result of Christodoulou, *cf.* Theorem 1.5.3.

[15] By this we mean that $({}^3\Sigma, g)$ is conformally isometric to a compact manifold with boundary, with a

1.1 The spacelike Cauchy problem in General Relativity

Let us start with recalling some basic facts about the Cauchy problem in general relativity. The Cauchy data for vacuum Einstein equations consist of a triple (Σ, g, K) where (Σ, g) is a three dimensional Riemannian manifold[19] with a metric of appropriate differentiability class (which we shall discuss later), K is a symmetric tensor on Σ (representing, roughly speaking, the time derivative of g), and g, K are assumed to satisfy a system of four coupled "constraint equations":

$$^3R = K^{ij}K_{ij} - (g^{ij}K_{ij})^2,$$

$$K^{ij}{}_{|j} - g^{kl}g^{ij}K_{kl|j} = 0,$$

where ^3R is the Ricci curvature scalar of g, $|$ denotes the Levi–Civita covariant derivative defined by the metric g, which arise as a consequence of the Gauss–Codazzi equations. A triple (M, γ, i), where (M, γ) is Lorentzian manifold and $i : \Sigma \to M$ is an embedding of a three dimensional manifold Σ in M, is called a *development* of (Σ, g, K) if γ satisfies the vacuum Einstein equations:

$$R_{\mu\nu} = 0 \ ,$$

where $R_{\mu\nu}$ is the Ricci tensor of γ, and if

$$i^*\gamma = g, \qquad i^*\tilde{K} = K \ ,$$

metric which can be extended in a sufficiently differentiable way to a metric on a slightly larger manifold, *cf.* Section 1.6 and Appendix A for details.

[16] This result is due to Friedrich; no symmetry conditions are assumed, *cf.* Theorem 1.6.1.

[17] This result is due to Christodoulou and Klainerman; no symmetry conditions are assumed, *cf.* Theorem 1.6.2.

[18] $m > 0$: no, *cf.* Theorem 1.7.2; $m < 0$: no to the future, yes to the past, *cf.* Theorem 1.7.1.

[19] Unless specified otherwise, "manifold" is used here to denote a connected, σ–compact (*cf. e.g.* [78]) manifold without boundary, the degree of differentiability of which can be in principle inferred from the context. We will not assume at the outset Hausdorffness of space–times, because of the existence of non-Hausdorff extensions of the globally hyperbolic region of the Taub–NUT space–time, *cf.* [64]. On the other hand we shall be conservative and always assume that the Riemannian manifold Σ on which Cauchy data (g, K) are defined is *always* Hausdorff.

where $\tilde{K}$ is the extrinsic curvature tensor of $i(\Sigma)$ in M. We shall often say that (M,γ) is a development, whenever no need arises to make i explicit. Some authors add the requirement of global hyperbolicity in the definition of development, it should be stressed that we are *not* adding this restriction. From the invariance of Einstein equations under diffeomorphisms it follows that for every diffeomorphism $\Phi : N \to M$, if (M,γ) is a development of (Σ, g, K), so is $(N, \Phi^*\gamma)$ (the appropriate embedding is $\Phi^{-1} \circ i$). A development will be called *maximal* if "there exists no larger development"; technically, if $\Phi : M \to M'$ is an isometric diffeomorphism[20] from M to $\Phi(M)$ and if the metric γ' on M' satisfies vacuum Einstein equations, then $M' = \Phi(M)$. (For completeness we prove the existence of maximal developments in Appendix C.1.) (M,γ) will be said *strongly maximal* if there exists no larger Lorentzian manifold (not necessarily satisfying the field equations) in which (M,γ) can be isometrically embedded). It should be stressed that *some maximal developements may fail to be strongly maximal*, because it could happen that there exist non–vacuum space–times $(\tilde{M},\tilde{\gamma})$ extending a given space–time, while no vacuum extensions of (M,γ) exist. We shall say that (M,γ) is *strongly maximal to the future*, respectively *to the past*, if in any extension $(\tilde{M},\tilde{\gamma})$ of (M,γ) the set of points $p \in \tilde{M}$, $p \notin M$ such that p lies to the future of M, respectively p lies to the past of M, is empty.

The theorem one would like to prove in general relativity is the following:

Theorem–to–be–proved 1 (TTBP) *For every three dimensional manifold Σ there exists a space of Cauchy data $X(\Sigma)$ and a class of four dimensional Lorentzian manifolds $\mathcal{M}(\Sigma)$ such that:*

1. *for all Cauchy data in a dense set of "generic Cauchy data" $Y(\Sigma) \subset X(\Sigma)$ the*

[20] By an abuse of language a locally Lipschitz continuous bijection with locally Lipschitz continuous inverse will also be considered to be a diffeomorphism, if useful in the context. The usefulness of such a weak notion of diffeomorphism relies in the fact, that such transformations are "as regular +1" as the least regular of g and Φ^*g: more precisely, if both g and Φ^*g are in a Sobolev class $H_s^{\rm loc}$, then Φ must be in H_{s+1}^{loc} (similar properties hold in C^k_{loc} or $C^{k,\alpha}_{loc}$ spaces). Therefore once the class of differentiability of the metrics one speaks about is fixed, one needs not to worry about the differentiability class of such "diffeomorphisms".

Cauchy problem admits a unique maximal *solution* (M, γ) *in* $\mathcal{M}(\Sigma)$,

2. *with* (M, γ) — strongly maximal, *moreover*

3. *all smooth pairs* (g, K) *satisfying the constraint equations and satisfying some completeness and/or fall–off conditions are in* $X(\Sigma)$.

Condition *1* is the requirement of *generic predictability* of any theory. As we shall see later, there exist smooth initial data in general relativity for which it seems that neither uniqueness nor strong maximality of developments holds. The undefined condition of genericity is also meant to capture the fact, that Cauchy data for which uniqueness in the large holds are stable, and a precise statement should of course be implemented by a definition of "genericity" — a tentative possibility could be that $Y(\Sigma)$ is open and dense. Condition *2* — *strong maximality* — is a necessary condition to be able to predict the evolution of "things" as long as they can "potentially exist": the existence of an extension (M', γ') of a maximal development (M, γ) would mean that there are observers which leave M to end up in M' in which, by definition of maximal development, the metric cannot satisfy the field equations. Condition *3* is a requirement for the applicability of the theory to all possible physical situations[21]; the proviso of some completeness and/or fall–off restrictions is necessary, as will be discussed in Section 1.8. At this stage the reader may wish to assume that if Σ is compact, then all smooth solutions of the constraint equations should be in $X(\Sigma)$; if Σ is not compact, than $X(\Sigma)$ could be required to contain *e.g.* all smooth solutions of the constraint equations with g_{ij} — asymptotically flat (*e.g.* in the sense of Theorem 1.6.2) in the "ends" of Σ. If the theorem–to–be–proved holds, then the Cauchy data for which uniqueness and/or strong maximality fail are unstable, and therefore their existence, though an unpleasant feature, does not seem to be a very serious threat to predictability of the theory. There are three possible attitudes to a theory in which theorem-to-be-proved does not hold:

[21] One could also adopt the point of view, that since there is only one universe accessible to our observations, namely the one we live in, it would actually suffice to have $Y(\Sigma) = X(\Sigma) =$ one point — the Cauchy data for the universe we live in. Such an approach does not seem to be very fruitful.

1. accept that the theory has poor predictability power, and cannot thus be considered as a fundamental one, or

2. isolate the set of Cauchy data, say $Z(\Sigma)$, for which uniqueness and strong maximality hold, and admit it as a prediction of the theory that no other Cauchy data than the ones in $Z(\Sigma)$ are admissible, or

3. renounce the idea, that unique predictability is a basic requirement for any satisfactory (nonquantum) physical theory.

Since none of these seems attractive to the author, he will hope for the best, namely that the theorem-to-be-proved can be proved indeed.

Let us recall some of the known results concerning the Cauchy problem for Einstein equations. The following theorem, due to Hughes, Kato and Marsden [72], is an improvement (as far as the differentiability hypotheses are concerned) of the pioneering work by Choquet-Bruhat [49] [17] (followed by some subsequent results by Hawking and Ellis [66], and Fischer and Marsden [46]; cf. [24] for an extensive review of this and other topics treated here):

Theorem 1.1.1 (Local existence) *Let (Σ, g, K) be an initial data set for vacuum Einstein equations. Suppose that Σ can be (locally finitely) covered by coordinate charts U_α, C^1 related to each other, such that $(g, K) \in H^{loc}_s(U_\alpha) \times H^{loc}_{s-1}(U_\alpha)$, $s > 5/2$. Then there exists a globally hyperbolic development (M, γ) of (Σ, g, K), for which Σ is a Cauchy surface, with γ determined uniquely (up to isometry) by (g, K)*[22]*. If Σ is Hausdorff, M can be chosen to be Hausdorff.*

This theorem guarantees the existence of some development in which the metric is uniquely determined by the data. It also tells us that the space–time constructed in the proof will be globally hyperbolic. Recall that (M, γ) is said to be globally hyperbolic if in

[22]Recall that $f \in H^{\rm loc}_s(U)$ if for every conditionally compact subset K of U $\| f \|_{H_s(K)} < \infty$, where H_s are the standard Sobolev spaces, $s \in I\!R$. s can be taken greater than or equal to 3 in the local existence theorem if one prefers integer values of s.

M there exists a hypersurface Σ with the property that every inextendible causal curve meets Σ once and only once. The hypersurface Σ of this definition is called a Cauchy surface. The fact that (M, γ) so constructed is globally hyperbolic is welcome since it means that M is not "smaller than globally hyperbolic", *i.e.* "no causal holes" are "dug" in M; on the other hand the method of proof does not allow one to construct "larger than globally hyperbolic" space–times, *i.e.* space–times in which Cauchy horizons occur. (The notion of global hyperbolicity was first introduced by Leray in a slightly different form[23], resulting in his famous theorem that the Cauchy problem for linear hyperbolic systems with Cauchy data on a spacelike submanifold of a globally hyperbolic space–time is well posed for, say, smooth data, i.e. smooth data evolve to a unique globally defined solution on M. Let us also mention that it follows from the theorems of Geroch [57] and Seifert [114] that the notion of global hyperbolicity of a space–time coincides with the condition of "strict hyperbolicity" of the associated d'Alembert operator, *cf.* [71][Volume 3, Chapter 23].) As far as existence and uniqueness of *maximal globally hyperbolic* developments is concerned we have the following result[24] due to Choquet-Bruhat and Geroch [21], with some improvements on the differentiability conditions due to Hawking and Ellis [66]:

Theorem 1.1.2 (Uniqueness of maximal globally hyperbolic Hausdorff developments) *Let Σ be a three dimensional Hausdorff manifold. For every $(g, K) \in H_4^{loc}(\Sigma) \times H_3^{loc}(\Sigma)$ there exists a Hausdorff manifold M with a $C^{1,1}_{loc}$ metric[25] γ, which is maximal in the space of globally hyperbolic Hausdorff developments of (g, K). M is diffeomorphic to $\Sigma \times \mathbb{R}$ and (M, γ) is unique (up to diffeomorphisms) in the class of globally hyperbolic Hausdorff manifolds (satisfying appropriate differentiability conditions).*

[23] When discussing equivalence of various definitions of global hyperbolicity one should be careful about the differentiability conditions needed to establish them: the usual causality theory, as presented e.g. in [66], requires $C^{1,1}_{loc}$ metrics (first derivatives of the metric are Lipschitz continuous on conditionally compact subsets of coordinate charts); several (and maybe all) essential results can be recovered under the differentiability conditions considered in [41]. If the parameter s of the local existence theorem is larger than $5/2$ the metric will be $C^{1,\alpha}_{loc}$ with an appropriate $0 < \alpha < 1$, but not necessarily $C^{1,1}_{loc}$ so that the classical results require reexamination, and in fact several results seem not to go through due to technical problems. In the case of theorem 1.1.2 the manifold M can certainly be chosen small enough to be globally hyperbolic both in the original Leray sense and in the sense described above.

[24] This theorem holds in considerably more general situations than vacuum Einstein equations, *cf.* [21].

[25] The metric will actually have more regularity then $C^{1,1}_{loc}$, *cf. e.g.* [20] for details.

$\tilde{I\!R}$:

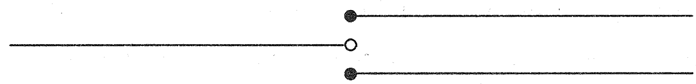

Figure 1.1.1: The bifurcating real axis.

Theorem 1.1.2 is unfortunately much weaker than what needs to be proved for "theorem" 1 to hold: uniqueness and maximality are guaranteed only within the class of Hausdorff globally hyperbolic developments, and nothing is said about *strong maximality.* It is easy to see that if one wishes to maintain uniqueness of M, some kind of requirement of Hausdorffness–type cannot be avoided: if one drops Hausdorffness altogether then if ${}^3\Sigma \times I\!R$ is a development, the manifold ${}^3\Sigma \times \tilde{I\!R}$ will also be a development, where $\tilde{I\!R}$ is the standard non–Hausdorff "bifurcating real axis", Figure 1.1.1, and one can continue to produce manifolds with at least a countable infinity of different topologies by adding new bifurcation branches and removing some. On the other hand allowing some kind of weak violation of Hausdorffness — *e.g.*, as proposed by Hajiček, requiring non–existence of bifurcate causal curves in the space–time — may restore uniqueness of maximal developments of the Taub—Newman–Unti–Tambourino (Taub–NUT) space–time[26] (cf. e.g. [66]). Let us discuss this space–time in some more detail. The Taub–NUT space–time is a highly symmetric model for which space–sections are "initially" diffeomorphic to S^3, "initially" meaning "in a neighborhood of some spacelike $(S^3)_o$" on which Cauchy data may be given, and these Cauchy data evolve to a space–time the global structure of which is well visualized by Figure 1.1.2 [85] [36]. The shaded region in Figure 1.1.2 is globally hyperbolic, and for any spacelike S^3 lying in this region it turns out to be the maximal globally hyperbolic Hausdorff development, as guaranteed by Choquet-Bruhat and Geroch. The unshaded region contains closed timelike geodesics. It can be shown [36] that there exist at least two different smooth ways (in fact even analytic) of at-

[26]If it were possible to prove uniqueness of all solutions at the price of some form of violation of Hausdorffness of the topology of the resulting maximal developments, it could consider this as a prediction of the theory, namely that in some special, maybe highly unstable (*cf.* the discussion of Section 4) situations, violation of Hausdorffness of space–time occurs. It must be pointed out that it is not known, whether Hajiček's non–Hausdorff extension of Taub–NUT space–time [64] is unique.

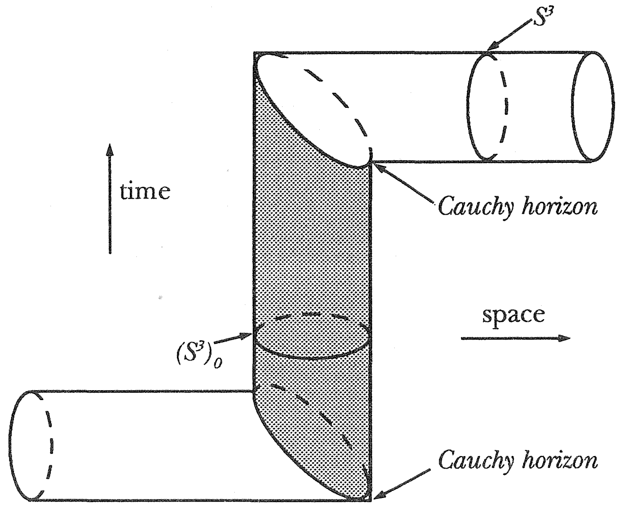

Figure 1.1.2: Taub–NUT space–time.

taching the acausal NUT region to the causal Taub region, one obtains in this way two *non–isometric developments* of the Taub region[27]. The "corners" in Figure 1.1.2 at the junction of the causal and acausal regions are misleading because both the manifold and the metric are smooth at the "junction". They are however correctly illustrating the fact that this junction acts as a singular surface for geodesics, the reader is referred to [85] [66] for more details. Let us note for further use that the hypersurface separating the globally hyperbolic region from the rest of the universe is called a *Cauchy horizon.* A similar behaviour is exhibited by the Misner space–time [84] (cf. also [66]), the globally hyperbolic region of which is diffeomorphic to $I\!R \times T^3$ ($T^3 = S^1 \times S^1 \times S^1 =$ the three dimensional torus), and in a large class of space–times constructed by Moncrief [88] [89], in polarized Gowdy space–times which we discuss in more detail in Section 1.5; some flat space–times with Cauchy horizons are also presented in Appendix D. All these examples necessitate the proviso of genericity in TTBP.

The global hyperbolicity condition plays an essential role in the proof of the Choquet-

[27]It would be of great interest to prove or disprove that the two known extensions are the only possibilities with, say, compact horizons.

Bruhat–Geroch theorem, because no systematic method of solving Einstein's equations beyond Cauchy horizons is known (*cf.* however [89] [89] in the analytic case). Let us also note a discrepancy of differentiability conditions between Theorems 1.1.2 and 1.1.1 — it is not clear whether it can be removed. One may wonder whether one should really worry about this since one often encounters the point of view (cf. e.g. [66]) that we can assume without any loss of physical information that the metric is smooth. The argument is, that since we are not able to measure whether a quantity is either smooth or C^2 or, say, continuous, then, quantum considerations put apart, one can as well assume smoothness of physical fields. It seems that a logical conclusion of such a reasoning should be exactly the opposite, namely that in a physical theory one should assume only the minimal differentiability conditions under which the theory makes sense; let us illustrate this by the following considerations: by a well known result by Whitney [128], (*cf.* also [129]) every C^k manifold, $k \geq 1$, is C^k diffeomorphic to an analytic one, which implies that every field on a manifold can be approximated, locally to arbitrary accuracy, by analytic fields. Continuing the previously described line of thought, one could thus as well assume that all fields are analytic. Such a hypothesis leads immediately to at least one erroneous conclusion, namely that the solutions of the field equations are uniquely defined everywhere by their value on some open subset: the unique continuation property of analytic fields completely obscures the fundamental property of hyperbolic equations, that uniqueness of solutions holds within domains of dependence only. Another striking example in which completely wrong conclusions are drawn by assuming analyticity of fields is given by the Robinson–Trautman space–times (*cf.* Section 1.7): there exists a large family of analytic initial data for a Robinson–Trautman space–time which can be evolved both to the future and to the past of the initial surface. On the other hand for all data which are smooth but fail to be analytic, a Robinson–Trautman solution of the vacuum Einstein equations can exist only either to the future or to the past of the initial hypersurface, depending upon the sign of m (*cf.* Section 1.7). This clearly demonstrates that restricting oneself to analytic fields may be rather misleading. Analycity put apart, we wish moreover to point out that *the assumption of global smoothness of solutions may* *be simply inconsistent with vacuum Einstein equations, or, more generally, with Einstein equations coupled with matter fields the energy–momentum tensor of which satisfies some positivity conditions.* This is due to the possibility of breakdown of differentiability of the metric in the course of evolution[28]. In order to discuss this in more detail, let us first turn to the basic question, what are the most general conditions under which Einstein equations make sense. The Ricci tensor may be written in the form

$$R_{\mu\nu} = \partial_\lambda A^\lambda_{\mu\nu} + B_{\mu\nu}$$

where A and B are expressions of the form

$$A^\lambda_{\mu\nu} = \sum g^{-1}\partial g, \quad B_{\mu\nu} = \sum g^{-1}\partial g g^{-1}\partial g,$$

where g^{-1} stands for the tensor $g^{\mu\nu}$ inverse to the metric tensor $g_{\mu\nu}$. If[29] $g \in H_1^{loc}(M)$ and $g^{-1} \in L^\infty_{loc}(M)$, the "distributional equation"

$$\forall \phi^{\mu\nu} \quad \int_M \{\partial_\lambda \phi^{\mu\nu} A^\lambda_{\mu\nu} - \phi^{\mu\nu} B_{\mu\nu}\} = 0$$

is well defined, with $\phi^{\mu\nu}$ — symmetric C^1 tensor density of compact support (cf. [41] and [60] for a discussion of some properties of such metrics). This class of metrics is much larger than the one considered in the existence theorem. As long as the metric does not leave this class during evolution, one can interpret Einstein equations in the above sense[30]. In the theory of partial differential equations it is often much easier to prove existence of some weak generalized solutions than to prove existence of solutions with high regularity and therefore existence of such weakly differentiable metrics seems highly plausible[31]. The problem with this kind of solutions is their potential non–uniqueness and it is likely that with such weak conditions the theorem-to-be-proved may not hold.

[28] cf. [116] for an example of evolution equations in which smooth initial data cease to be differentiable during evolution, while globally defined weak solutions exist (the weak solutions fail, however, to be unique [117]). The example studied by Shatah may be of some relevance to our discussion, because harmonic maps are related to Einstein equations, cf. e.g [90] [87].

[29] It is worthwile noting that similar conditions arise naturally when studying the problem of weakest possible conditions for finite and well defined ADM mass, *cf. e.g.* [5].

[30] One cannot exclude the possibility of being able in the future to formulate Einstein equations in an even larger class of metrics.

[31] A standard approximation argument should show the existence part of theorem 2.1 holds with (g, K) in $H_s^{\rm loc} \times H_{s-1}^{loc}, s > 3/2$ (the resulting space–time metric will actually be still more regular than the requirements of well posedness of distributional Einstein equations pointed out above).

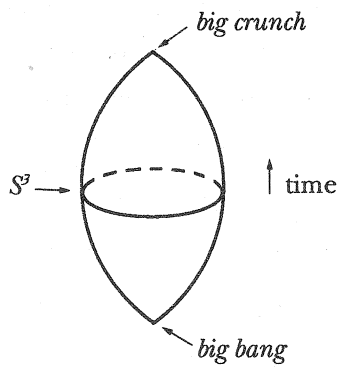

Figure 1.2.1: Friedman–Robertson–Walker space–time.

1.2 Singularities?

Although of fundamental importance, the Choquet-Bruhat–Geroch theorem gives only very poor information (global hyperbolicity, Hausdorffness) about the structure of the space–time whose existence and uniqueness it guarantees. An especially unpleasant feature of some maximal solutions of Einstein equations is the potential failure of strong maximality, i.e. *existence of extensions to space–times on which there is no way of ensuring satisfaction of field equations.* The possibility of existence of such extensions leads us to the question, what global properties generic solutions possess, and therefore to the problem of singularities. Soon after Einstein's theory had been formulated it was realized that Einstein's equations led to formation of singularites (cf. [123] for a review of the history of the problem). A standard example of such an occurence is given by the Friedman–Robertson–Walker metrics on $I\!R \times S^3$ evolving from the "big bang" singularity to the "big crunch" singularity, cf. Figure 1.2. In this crude cosmological model Einstein theory describes very precisely what will happen to the inhabitants of the universe: in finite proper time they will be crushed to zero volume by infinite gravitational forces. One can rightly raise the objection, that the assumed dust model fails to describe physical reality near the singularities — to avoid such discussions we shall from now on restrict our attention to vacuum Einstein equations. The question whether singularities generically occur in vacuum Einstein gravity has intrigued physicists for years, the general belief being that this problem has been satisfactorily settled by Penrose and Hawking in their celebrated singularity theorems. Let us present[32] one of those [67] [66]:

Theorem 1.2.1 (Hawking–Penrose Singularity Theorem) *Suppose that Σ is a compact three dimensional manifold, let (g, K) be Cauchy data which admit $C^{1,1}_{loc}$ developments, let (M, γ) be any development in the $C^{1,1}_{loc}$ differentiability class and suppose that (M, γ) satisfies a genericity condition (for details,* cf. *[67] [66]). Then either*

1. *(M, γ) contains a Cauchy horizon beyond which closed timelike curves occur, or*

2. *(M, γ) is geodesically incomplete.*

We shall not discuss in detail what "generic data" means in this context, let us just mention that there is hope (and some evidence) that the hypothesis of genericity can be removed by adding a third possibility, namely that (M, γ) splits isometrically as $\mathbb{R} \times \Sigma^3$, with Σ^3 — flat (*cf.* [100] [55] and references therein). 1) seems to be a very interesting prediction of general relativity — we have seen an example of such a behaviour in the Taub-NUT space–time. We have chosen the somewhat artificial dynamical formulation of the above theorem to emphasize that there is no causality violation in a neighborhood of the hypersurface Σ, the closed timelike curves "develop" beyond Cauchy horizons, in particular the timelike loops which develop in Taub–NUT space–time beyond, say, a future Cauchy horizon, do not allow for any interference with the past of the observers on the other side of the horizon. It is widely believed that generically the second part of the alternative occurs, namely existence of incomplete geodesics. The problem with point 2) is, as already pointed out by Hawking and Penrose [67], that though it leaves one with the feeling that something goes wrong, it doesn't say what actually is going wrong. In view of the weak differentiability conditions for existence of solutions of Einstein equations,

[32] One would expect that the conditon of genericity of (M, γ) required below might hold for generic Cauchy data, but this remains to be proved.

not to mention well posedness of the distributional equations, this theorem completely fails to indicate whether we are really going to face a situation in which either Einstein equations will fail or some other unwanted physical phenomena will occur. A breakdown of $C^{1,1}_{loc}$ character of the metric may lead to a blow up of the Riemann tensor[33], but even this need not be accompanied by any extremely unpleasant physical effects. A simple way of analyzing the effects of curvature on matter is via tidal forces induced by gravity, the effect of which is given by an integral of some components of the Riemann tensor along world lines. If the blow up preserves finiteness of such integrals no extraordinary physical effects will be observed. Rather than interpreting this theorem as a breakdown of general relativity, one can think of it as evidence for the need of considering weakly differentiable metrics. There is little doubt that some singularities that occur in general relativity cannot be removed by the introduction of coordinate systems in which the metric is only weakly differentiable, and it is important to isolate those from the other kind. Rather than having a theorem telling us that *something* goes wrong, one would like to have a result which tells us exactly *what* goes wrong, e.g.:

Theorem–to–be–proved 2 (TTBP–2) *Suppose that Σ is a three dimensional Riemannian manifold, let $(g, K) \in X(\Sigma)$, $X(\Sigma)$ as in TTBP and let (M, γ) be a maximal development of (Σ, g, K). For all/most/some Γ's, where Γ is either*

1. *a past/future incomplete inextendible timelike geodesic, or*
2. *a past/future incomplete inextendible null geodesic*[34]*, or*
3. *a finite total acceleration future/past inextendible timelike curve of finite proper length, i.e. a curve for which (s — proper time along Γ)*

$$\int_\Gamma g_{\mu\nu}\frac{D^2x^\mu}{ds^2}\frac{D^2x^\nu}{ds^2}ds < \infty.$$

4. *a past/future inextendible timelike curve of finite proper length,*

[33]Some results of this kind have been established by Clarke [41] [123] under supplementary hypotheses.
[34]If the metric is not C^1, a geodesic may eventually be defined as a curve possessing some extremal properties in appropriate classes of curves.

it holds that

???

The "finite total acceleration" curves are interesting, because they may be thought to represent observers which have at their disposition a finite amount of fuel for their rocket's engine. It is not to be excluded that in generic situations "a lot of" geodesics will be complete either to the future, or to the past, or both.

Let us emphasize that the analysis of what goes wrong along incomplete geodesics is relevant to TTBP, at least in the $C^{1,1}_{loc}$ differentiability class of metrics, because of the following inextendability criterion:

A Maximality Criterion: *Let γ be a $C^{1,1}_{loc}$ Lorentzian metric on M. If on every incomplete timelike geodesic in M some curvature scalar blows up, then (M,γ) is* strongly maximal *in the class of Lorentzian space–times with $C^{1,1}_{loc}$ metrics.*

It follows that a proof of curvature blow–up is a useful step in the proof of TTBP. It should be noted that the above criterion also shows that a geodesically complete space–time is necessarily maximal in the class of space–times with $C^{1,1}_{loc}$ metrics. The lack of criteria for inextendibility for Lorentzian metrics in weak differentiability classes is at the origin of our inability to exclude extensions which are not $C^{1,1}_{loc}$. For instance, it is not known whether *e.g.* Minkowski space–time can or cannot be extended to a larger manifold with a continuous Lorentzian metric, though such a possibility seems rather unlikely to the author. This, and some other inextendability criteria, are proved in Appendix C.2.

1.3 Strong Cosmic Censorship?

One often encounters the opinion, that an important issue in general relativity is to establish the validity of the *strong cosmic censorship*, a hypothesis formulated by Penrose [103] (*cf.* also [70][Section 5.4]), which can tentatively be formulated as follows:

Strong Cosmic Censorship Conjecture (SCCC): *Every maximal Hausdorff development of a generic Cauchy data set (Σ, g, K), with (Σ, g) — compact or asymptotically flat, is globally hyperbolic*[35].

This conjecture is often formulated in the $C^{1,1}_{loc}$ context, and a breakdown of $C^{1,1}_{loc}$ differentiability class of the metric on a globally hyperbolic manifold is considered as a breakdown of validity of the conjecture. Note that by the Choquet–Bruhat—Geroch theorem there is only one globally hyperbolic development, thus SCCC implies uniqueness in generic situations. The supposed importance of this conjecture is motivated by the belief that

1. a failure of $C^{1,1}_{loc}$ differentiability of the metric implies necessarily a breakdown of predictability of Einstein theory (non–uniqueness of solutions in the large), and

2. non–global hyperbolicity means existence of Cauchy horizons, and it is believed that there is no physically reasonable class of space–times in which one can uniquely solve Einstein equations beyond the Cauchy horizon.

As we have emphasized in the previous sections, we have no convincing evidence that 1) is justified. We have purposefully stated the SCCC without making the differentiability conditions explicit to simply disregard 1) and concentrate on 2). Let us recall some facts about *compact* Cauchy horizons: Isenberg and Moncrief have shown [75], under some technical conditions which can probably be removed, that an analytic space–time with a compact Cauchy horizon must admit a non–trivial isometry group. Moncrief [88] [89] has studied the analytic Cauchy problem posed on an analytic U(1) symmetric horizon with S^3 and T^3 topology and has shown that this problem is well posed: more precisely, by proving a singular version of the Cauchy–Kowalevskaya theorem, he has shown that there exists a unique analytic development of appropriate analytic data on the horizon both into the causal and, what interests us more here, into the acausal region. It is not known whether this result is a pure accident of analyticity, or reflects the possibility

[35] It should be stressed that we *do not* include the notion of global hyperbolicity in the definition of developement, *cf.* the beginning of Section 1.1.

that the problem of evolving say, smooth data into the acausal region is well posed. Let us just point out at this stage that solvability of the analytic Cauchy problem is usually an excellent starting point for the proof of solvability of the smooth or the H^s Cauchy problem — what is needed is some kind of *a priori* estimates. *If one could establish generic unique solvability beyond those Cauchy horizons which can occur in space–times containing compact or asymptotically flat space–like hypersurfaces, it seems that this formulation of SCCC would cease to be a fundamental issue.* To the author's knowledge, no conclusive study of this problem has been undertaken yet. It must be noted that it follows from the proofs of the Penrose–Hawking theorems [67] [66] that causality violations will occur beyond Cauchy horizons. As emphasized previously, in no way does this lead to causality problems in the globally hyperbolic regions, and ending in an acausal region does not seem to be a worse fate for a universe than ending in a big crunch. It is often argued, quite convincingly (for references cf. e.g. [103]), that Cauchy horizons will be unstable (cf. also [91]). Whatever the status of generic solvability of Einstein equations beyond Cauchy horizons, it would be of interest to isolate the set of Cauchy data which lead to formation of Cauchy horizons:

Theorem–to–be–proved 3 (TTBP–3) *Let $\Sigma, X(\Sigma), Y(\Sigma), \mathcal{M}(\Sigma)$ be as in TTBP. Let $X_H(\Sigma)$ be the set of Cauchy data for which Cauchy horizons occur. Then*

1. *$X(\Sigma) \setminus X_H(\Sigma)$ is dense and open ???,*
2. *The intersection $X_H(\Sigma) \cap Y(\Sigma)$ is empty ??? is dense in $X_H(\Sigma)$??? is equal to $X_H(\Sigma)$ for appropriatedly chosen $\mathcal{M}(\Sigma)$???*

We would like to emphasize once again that

- if it were convincingly demonstrated that there is no sense in which one can have generic unique solvability of Einstein equations beyond the kind of Cauchy horizons which can occur by evolution from asymptotically flat or spatially compact initial data, then SCCC would be of fundamental importance to Einstein's theory of gravitation. Its failure would imply that TTBP does not hold.

- If generic unique solvability beyond Cauchy horizons as above can be established, SCCC becomes an interesting but not fundamental problem.

This last possibility seems to be highly unlikely to most authors (*cf.*, however, [50] and [95]).

1.4 (Weak) Cosmic Censorship?

Another famous conjecture due to Penrose [105] is the so–called *(weak) cosmic censorship hypothesis* (w.c.c.), which expresses the hope that for *generic collapsing isolated gravitational systems the singularities that might develop will be hidden beyond a smooth event horizon* (*cf.* also [70] [123] [66] for discussion, and for various technical formulations of the problem). Since 1976 the following examples of naked singularity formation have been found, we list them in chronological order of appearence:

1. Yodzis, Seifert and Müller zum Hagen have shown that "naked singularities" may occur during the collapse of spherically symmetric dust [131] ($T_{\mu\nu} = \rho u_\mu u_\nu$, with $\gamma_{\mu\nu}u^\mu u^\nu = -1$, where $T_{\mu\nu}$ is the energy–momentum tensor), or of spherically symmetric perfect fluids [131] [97] ($T_{\mu\nu} = (\rho+p)u_\mu u_\nu + p\gamma_{\mu\nu}$, $\gamma_{\mu\nu}u^\mu u^\nu = -1$), for a large class of equations of state $p = p(\rho)$. These are the so–called "shell–crossing" singularities, which arise because of crossing of shells of matter, and their occurrence is stable under spherically symmetric perturbations [97].

2. Steinmüller, King and Lasota [122] have noted that naked singularities can form during a radiative collapse of a star emitting "null[36] dust" ($T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\gamma_{\mu\nu} + \epsilon k_\mu k_\nu$, $\gamma_{\mu\nu}u^\mu u^\nu = -1$, $\gamma_{\mu\nu}k^\mu k^\nu = 0$), in a family of metrics considered by Demiański and Lasota [42].

3. Hiscock, Williams and Eardley have shown [68] that the implosion of null dust ($T_{\mu\nu} = \epsilon k_\mu k_\nu$, $\gamma_{\mu\nu}k^\mu k^\nu = 0$) can lead to the formation of so-called "shell–focusing"

[36] A "null dust" can be obtained from *e.g.* a geometric optics approximation to a massless scalar field.

naked singularities[37] (the same result has been noticed independently by Papapetrou in [102], *cf.* also [125] and references therein for more information about these singularities).

4. Christodoulou has proved that the collapse of spherically symmetric dust (p=0) can also lead to the formation of a central naked singularity [25] of "shell–focusing" type; numerical evidence suggesting that this might occur has been previously obtained by Eardley and Smarr [43] (*cf.* also [98] for more information about these singularities). Numerical results of Ori and Piran [101] suggest that central naked singularities may also form for perfect fluids with equation of state $p = \kappa\rho$, κ =const.

5. Shapiro and Teukolsky [115] have presented numerical evidence suggesting that axially symmetric collapse of the Einstein–Vlasov system may lead to formation of naked singularities, and that this behaviour is stable under axially symmetric perturbations.

6. In his recent analysis of a self–gravitating massless scalar field, Christodoulou has shown [27] that *generic initial data* (of BMO class, prescribed on an outgoing light cone, *cf.* [27] for details), do not lead to the formation of naked singularities (thus both w.c.c. and also TTBP hold for the spherically symmetric Einstein–massless scalar field problem). He has also shown that there exists a set of codimension 1 of initial data for which naked singularities will occur.

It must be stressed that all the examples above suffer from the drawback of not being realistic, and from all of them Christodoulou's scalar field seems to be the most acceptable one. The deficiencies of the numerical results are 1) the problem of numerical uncertainties arising when approaching the singularity, and 2) the difficulty of deciding, on the basis of a numerical solution of finite space extent and obtained only up to some finite

[37] There are some reasons to believe that the central "shell–focusing" singularities are "worse" than the shell crossing ones, *cf.* [99] for details; the status of these assertions does not seem to be clear.

value of time, whether or not the solution in question has an event horizon[38] (recall that an event horizon is defined in terms of asymptotic structure [66], and thus in terms of the behaviour of the solution as both the space and time variables tend to infinity). The only result above which does not assume spherical symmetry is the one due to Shapiro and Teukolsky, and it has been pointed out by Rendall [108] that the presumed occurrence of a naked singularity might be an artefact of the singular character of the initial data[39] assumed for the Vlasov field in [115]. To give support to his suggestion Rendall draws attention to the recent results of Pfaffelmoser [106], who has shown that the Poisson–Vlasov system (which is the Newtonian equivalent of the Einstein–Vlasov system) has global solutions for smooth data, while the numerical results of Shapiro and Teukolsky show that some solutions of the Poisson–Vlasov system with Dirac–δ–type initial data blow up in finite time.

The original motivation for the formulation of the w.c.c conjecture seems to be the expectation, that formation of singularities may lead to unpredictability, *i.e.* non–uniqueness of solutions. If w.c.c. holds, then the potential regions of non–uniqueness will be hidden beyond event horizons, and predictability in the exterior world will be saved. This attitude is similar in spirit to that of TTBP, with the difference however that TTBP insists on predictability throughout the space–time. Clearly if one adopts the attitude that the main point of w.c.c. is predictability, then a proof of TTBP will make w.c.c. irrelevant. One might of course consider that a significant part of w.c.c. is the requirement of smoothness of the metric on the event horizon, or maybe even in a neighbourhood of the event horizon, in which case w.c.c. and TTBP may contain only partially overlapping information.

It should be noted that a rather different "curvature strength" approach to the w.c.c. problem has been initiated by Królak and Newman, *cf.* [80] [99] [110].

[38]In spherically symmetric situations with matter with a sharp imploding boundary this problem does not arise because by Birkhoff's theorem the metric is the Schwarzschild one in the vacuum region; no such information is available if spherical symmetry is not assumed.

[39]Shapiro and Teukolsky model the Vlasov field by a swarm of particles, which corresponds to initial data for the Vlasov function f which are a sum of Dirac δ functions.

1.5 Some answers, in spatially compact space–times with two or more spacelike Killing vector fields.

TTBP, TTBP2 and TTBP3 seem to be a formidable challenge to analysts, and it must be said that the possibility of establishing them in the near future seems to be rather remote. When facing a difficult problem it is usually believed that insight can be gained by studying it under certain restrictive assumptions, and the obvious idea is to consider space–times with Cauchy data invariant under a smooth action of a group. The ideal would be to assume the smallest possible isometry group — $U(1)$ or $\mathbb{R}$. Although several remarkable simplifications occur in such space–times (cf. [90] and [16]) this problem also seems to be out of reach at the time of writing this review. Let us thus take the reverse approach, and start with the largest possible isometry group. In the remainder of this Section we shall consider the cosmological vacuum space–times only, unless explicitely specified otherwise, i.e. space-times which develop out of a compact, connected, orientable Cauchy surface. All possible groups acting on such manifolds have been listed by Fischer ([45], p. 334), *cf.* Table 1.0.1, p. 5, let us examine Fischer's list one by one:

1. $G = SO(4)$ or $SO(3) \times SO(3)$: it follows from the equations on p. 471 of [126] that no such vacuum metrics exist;

2. $G = (U(1) \times SU(2))/D, D = \{(1,1), (-1,-1)\}$, or $G = U(1) \times SO(3)$, acting on lens spaces (recall that S^3 is a lense space: $S^3 = L(1,1)$): as shown in Section 2.2, the Taub-NUT metrics exhaust the space of metrics with this symmetry group, thus TTBP certainly does not hold in this "minisuperspace" of metrics: all the maximal globally hyperbolic space-times in this class are non–uniquely extendible both to the future and to the past of the Cauchy surface (*cf.* [36] for a detailed discussion);

3. $G = U(1) \times SO(3)$ acting on $S^1 \times S^2$: by a generalization of the Birkhoff theorem (*cf.* Section 2.3) all such metrics are, locally, isometric to the "$r, 2m$" Schwarzschild metric. It follows that the maximal globally hyperbolic development is inextendible

either to the future or to the past of the Cauchy surface, because of the $r = 0$ Schwarzschild singularity, and extendible either to the past or to the future of the Cauchy surface, because of the Schwarzschild event horizon $r = 2m$; thus TTBP "half–holds" in this space of metrics;

4. $G = SU(2)$ or $SO(3)$ with three dimensional principal orbits: these are the so-called Bianchi IX space-times. In spite of the fact that these space–times have been studied by several authors (cf. [12] and references therein; cf. also [81] for some recent results) no information of the kind looked for in TTBP — TTBP2 — TTBP3 seems to be available. The general belief seems to be that all such space-times except for the Taub-NUT family are curvature singular[40] both to the future and to the past of the Cauchy surface;

5. $G = SO(3)$ with two dimensional principal orbits: as discussed in Section 2.3, the generalized Birkhoff theorem shows that these metrics are, locally, the interior Schwarzschild metric[41];

6. $G = U(1) \times U(1) \times U(1)$ acting on T^3 — these are the so–called Bianchi I space-times, which we discuss in detail in the Section 2.4. The picture that emerges is the following: generic maximal space–times in this class are globally hyperbolic and contain incomplete causal geodesics, however on those some scalars constructed out of curvature blow up to infinity, which shows that generic space–times of this class are inextendible in the $C^{1,1}_{loc}$ class of metrics, and which gives a version of TTBP2. Only a "zero measure set" of Bianchi I space–times contains horizons, therefore TTBP holds (in the class of Hausdorff Lorentzian manifolds with $C^{1,1}_{loc}$

[40]There is an ongoing discussion whether the Bianchi IX dynamical system is chaotic or not; and it has even been claimed [12] that the orbits of the Bianchi IX system asymptotically tend to orbits of a dynamical system which contains a strange attractor. These assertions about chaos in Bianchi IX metrics do not seem to be sufficiently justified. Let us also mention that the positive Lyapunov exponents criterion of chaos, which is sometimes claimed to be fulfilled in Bianchi IX space–times [10] [15] (*cf.* however [69] for an opposite point of view) seems to be irrelevant for dynamical systems on non–compact manifolds (which is the case for Bianchi IX space–times). It would be of great interest to give a mathematically rigorous analysis of whether Bianchi IX cosmologies are chaotic or not.

[41]This excludes the S^3 and P^3 spatial topologies listed by Fischer, *cf.* Section 2.3.

metrics). Nothing is known about the possibility (or the impossibility) of extending the metrics as solutions of field equations beyond the "curvature singularities", whenever they occur, in some weaker differentiability class of space–times. Since all the Bianchi I metrics are analytic they can be analytically extended as vacuum metrics beyond the Cauchy horizons (which can always be assumed to be compact), whenever they occur; the extensions fail to be unique, as in the Taub-NUT case.

Thus, as we see, vacuum spatially compact space–times for which the dimension of the isometry group is larger than or equal to three are fairly well understood, the only case in which our knowledge is unsatisfactory being the general Bianchi IX metrics. There is no doubt that this last case deserves a careful analysis, from a dynamical point of view it seems, however, more interesting to study space–times in which the dimension of the maximal isometry group is less than or equal to two: in all the cases listed above the dynamics reduces to ODE's rather than PDE's, and one is left with the feeling of missing something fundamental by turning wave phenomena off. One can therefore expect that more insight can be gained by studying those space-times for which the isometry group is the next group on Fischer's list, namely $G_2 = U(1) \times U(1)$[42]. In the remainder of this section we shall consider space–times which evolve from a Cauchy data set (Σ, g, K) symmetric under a G_2 action, namely:

1. Σ is a manifold which admits a differentiable action of a G_2, and
2. (g, K) are invariant under G_2.

It follows that in any development of (Σ, g, K) there is a neighborhood of Σ on which G_2 acts by isometries, and in which the orbits of G_2 are spacelike. Such space–times have played an important role in the development of general relativity: the cylindrically symmetric Einstein–Rosen space–times [79] provided the first non-perturbative arena to discuss gravitational radiation, the "boost–rotation" symmetric space–times were the first example of asymptotically flat radiating space–times (cf. [11] and references therein),

[42]The only remaining possibility on the list given in [45] is $G = U(1)$.

the $I\!R \times I\!R$ symmetric plane waves of Kahn and Penrose [77] gave the first example of singularity formation without spherical symmetry, finally Gowdy [61] used $U(1) \times U(1)$ symmetric metrics to exhibit radiation phenomena in spatially closed space–times. There has been recently a renewal of interest in these space–times, because of many interesting properties and because they have not been studied with sufficient detail and rigour in the original papers. One should mention a careful reexamination of the radiative properties of the boost–rotation symmetric space–times by Bičák and Schmidt [11], a study of plane waves by Ernst and Hauser [44], and an exhaustive study of polarized Gowdy metrics by Isenberg and Moncrief [76] which we shall discuss in more detail below. It is also worth mentioning some studies of polarized Gowdy metrics in the quantum gravity context, e.g. some recent work by Husain and Smolin [73].

Before presenting in detail the results on polarized Gowdy metrics derived in Refs. [76], [37] and [35], let us recall a few facts about G_2 isometric space–times. If Σ is a compact manifold on which a topological group acts effectively, then G must be a compact Lie group. It may be shown that if G_2 is the maximal isometry group of the metric, then $G_2 = U(1) \times U(1)$ (otherwise the real isometry group of the metric would be larger). It is also a standard result that if a compact connected orientable three dimensional manifold is acted upon by $G = U(1) \times U(1)$, then $\Sigma = T^3$ or S^3 or a lens space $L(p,q)$ or $S^2 \times S^1$, and the action of G is unique up to diffeomorphisms. Metrics on $L(p,q)$ can be identified with metrics on S^3 and need not be discussed separately in the applications we shall be concerned with here. The "polarized Gowdy space–times" form a small but non-trivial subset of the set of metrics on which $U(1) \times U(1)$ acts by isometries on spacelike Cauchy surfaces. By definition for these space–times a coordinate system exists such that the metric takes the form

$$ds^2 = f(t,\theta)(-dt^2 + d\theta^2) + g_a d\theta dx^a + g(t,\theta)(dx^1)^2 + h(t,\theta)(dx^2)^2 \;, \tag{1.5.1}$$

$$\partial_\mu g_a = 0 \;.$$

In the S^3 or $S^2 \times S^1$ case these metrics can be characterized as all $U(1) \times U(1)$ symmetric metrics for which

$$g_{\mu\nu} X_1^\mu X_2^\nu = 0, \tag{1.5.2}$$

where $X_a, a = 1, 2$ are appropriate Killing vectors; in the T^3 case they can be characterized by (1.5.2) and the requirement $c_a = 0$, where the constants c_a are defined in (1.5.3). We have the following result [37]:

Theorem 1.5.1 (Global structure of maximally developed polarized Gowdy spacetimes) *Let $X(\Sigma) = \{(g, K) \in C^\infty(\Sigma) \times C^\infty(\Sigma)\}$[43], with (Σ, g, K) — Cauchy data for a polarized Gowdy metric,.*

- *Let $\Sigma \approx T^3$. There exists an open dense subset $Y(\Sigma)$ of $X(\Sigma)$ such that, for every (g, K) in $Y(\Sigma)$, there exists a globally hyperbolic Hausdorff development (M, γ), $M \approx I\!R \times T^3$, and a time orientation of (M, γ) with the following properties:*

  1. *on every past directed inextendible timelike curve the scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ uniformly tends to infinity in finite proper time (in particular (M, γ) is past inextendible[44] in the set of Hausdorff manifolds with metrics of $C^{1,1}_{loc}$ differentiability class),*

  2. *(M, γ) is future geodesically complete (in particular (M, γ) is future inextendible in the set of Hausdorff manifolds with metrics of $C^{1,1}_{loc}$ differentiability class).*

- *Let $\Sigma \approx S^3$ or $\Sigma \approx S^2 \times S^1$. There exists an open dense subset $Y(\Sigma)$ of $X(\Sigma)$, with the property that for all Cauchy data in $Y(\Sigma)$ there exists a globally hyperbolic Hausdorff development (M, γ) for which on every past directed inextendible timelike*

[43] Several claims of this theorem hold under much weaker differentiability conditions, which we shall not discuss here because the main issue — the proof of inextendability — requires $C^{1,1}_{loc}$ differentiability, and as far as Theorem 1.5.1 is concerned no new phenomena arise when assuming some Sobolev conditions on the initial data consistent with $C^{1,1}_{loc}$ differentiability of the metric instead of C^∞ data.

[44] By this we mean that in any extension M' of M the implication $(p \in M'$ and $p \in J^-(\Sigma)) \Rightarrow (p \in M)$ holds.

curve the Riemann tensor blows to infinity both in the future and in the past in finite proper time (in particular (M,γ) is inextendible in the set of Hausdorff manifolds with metrics of $C^{1,1}_{loc}$ differentiability class).

It should be emphasized that this theorem ensures generic maximality and uniqueness within the $C^{1,1}_{loc}$ class, not only the globally hyperbolic $C^{1,1}_{loc}$ class — the global hyperbolicity of the maximal extensions follows from the theorem. Inextendibility in a space of metrics of weaker differentiability class is not known. This theorem is "TTBP" for this class of metrics and gives some information of the kind looked for in TTBP2. It seems to be the most precise *singularity theorem* known for a reasonably large class of metrics, especially in view of the sharp estimates on the blow up rates of the Riemann tensor one can obtain from the results proved in [76]. Theorem 1.5.1 shows, that in the S^3 or $S^2 \times S^1$ case the vacuum $U(1) \times U(1)$ symmetric picture is, generically, essentially the same as in the Friedman–Robertson–Walker dust–filled cosmological model, i.e. the "big bang to big crunch" scenario holds.

In the polarized Gowdy class one can obtain [37] [35] almost exhaustive information about the extendible metrics. Let us consider the T^3 case first. It should be noted that causal future geodesic completeness (and thus future inextendability) actually holds for all polarized Gowdy spacetimes with T^3 spatial sections, so that nothing remarkable happens for large t. On the other hand several interesting phenomena occur at the boundary $t = 0$. For all Cauchy data (g, K) in a certain nonempty closed subspace of $X(\Sigma) \setminus Y(\Sigma)$ the maximal globally hyperbolic development has the property that $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is bounded on M. For such (M,γ) there exist at least two nonisometric manifolds $(M_a, \gamma_a), a = 1, 2$, with C^∞ metrics in which M can be isometrically embedded in such a way that $\overline{M}_a \setminus M$ is a Cauchy horizon[45]. In this case a *necessary* condition for existence of vacuum extensions (M_a, γ_a) in the polarized Gowdy class is analycity of (M,γ): thus if smooth Cauchy data out of which a compact Cauchy horizon develops are not analytic, there will be no vacuum polarized Gowdy extension. This result is

[45] $\overline{M}_a$ denotes the closure of M in M_a.

not as strong as one would wish since it does not exclude vacuum extensions which are not in the polarized Gowdy class. One can also show existence of Cauchy data out of which non–compact Cauchy horizons develop: a particularly interesting class of metrics can be constructed in which the Cauchy horizon has a countable infinity of connected components, across every one of which the metric can be extended as a solution of the vacuum Einstein equations in two inequivalent ways. One obtains in this way maximal globally hyperbolic spacetimes which admit a *countable infinity* of *inequivalent* vacuum extensions [36].

In the $S^2 \times S^1$ and S^3 very similar results hold, the only essential difference is, that on S^3 no compact Cauchy horizons are possible in this class of metrics. On $S^2 \times S^1$ polarized Gowdy space–times with a compact Cauchy horizon can be shown to exist, in such a case the Cauchy horizon must lie to one side of the Cauchy surface only, i.e. if the Cauchy horizon has a compact, non–empty, connected component to, say, the future of ${}^3\Sigma$, then $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ tends to infinity uniformly on all past incomplete causal curves. Again a necessary and sufficient condition for vacuum extendability is analycity of the Cauchy data when a compact Cauchy horizon is present. Non–compact Cauchy horizons are possible both in the S^3 and $S^2 \times S^1$ case; for both topologies there exist space–times for which the Cauchy horizon has connected components both to the future and to the past of the Cauchy surface; for both topologies the Cauchy horizon may have an infinite number of connected components, and, finally, an infinite number of inequivalent vacuum extensions is possible for a large family of polarized Gowdy space–times with S^3 or $S^2 \times S^1$ space sections.

Since polarized Gowdy metrics are of "zero measure" in the space of all Gowdy metrics, a natural question to ask is what happens if one relaxes the polarization condition. Let us recall some general features of vacuum metrics with two commuting Killing vectors $X_a, a = 1, 2$. In such a space–time one can form two scalar functions

$$c_a = \epsilon_{\alpha\beta\gamma\delta} X_1^\alpha X_2^\beta \nabla^\gamma X_a^\delta, \tag{1.5.3}$$

and one shows that in vacuum [58] the c_a's must be constant. It follows immediately from this and the definition of the c_a's that if there is a symmetry axis (i.e. a subset on which one of the Killing vectors vanishes) then the c_a's must both vanish (since they are constant and since the right hand side of (1.5.3) vanishes on the axis, they must vanish when axes occur) if the space–time is connected (they may, of course, also vanish if no symmetry axes exist). A space–time in which the c_a's vanish will be called a *Gowdy space–time.* In this case Einstein equations simplify considerably and it can be shown that they reduce to a system of equations which can either be interpreted as a harmonic map from a three dimensional space–time to the two–dimensional hyperboloid H_2 of constant curvature [32] [87], or a harmonic map from a two–dimensional space–time to the same target space with a "source term". (A map $\phi : M \to N$ between two Riemannian (or pseudo–Riemannian) manifolds $(M,g),(N,h)$ is called harmonic if it is a formal stationary point of the "energy" integral:

$$E(\phi) = \int_M g^{\mu\nu}(x) h_{AB}(\phi(x)) \partial_\mu \phi^A \partial_\nu \phi^B d\mu_g(x).)$$

Once the harmonic map problem is solved, the components of the metric are obtained either by algebraic manipulations or by line integrals. In the T^3 problem one is left to study a simple generalization of the harmonic map equations from $I\!R \times S^1$ to H_2 when both c_a's vanish, and we have the following results (the first part of this theorem is due to Moncrief [87]; the second is proved in Chapter 3):

Theorem 1.5.2 **($U(1) \times U(1)$ stability of the singularity of the $(p_1,p_2,p_3) = (\frac{2}{3},\frac{2}{3},-\frac{1}{3})$ Kasner metric)** *Suppose that $\Sigma \approx T^3$, set $X(\Sigma) = \{(g,K) \in H_2(\Sigma) \times H_1(\Sigma)$, with (Σ,g,K) — Cauchy data for a Gowdy space–time$\}$. One can choose a time orientation on Σ in such a way that:*

1. *there exists a globally hyperbolic Hausdorff future development (M,γ) of (Σ,g,K) which can be covered by a family of Cauchy hypersurfaces $\Sigma_\tau, \tau \in I\!R^+$, such that $\lim_{\tau\to\infty} d(\Sigma_\tau,\Sigma) = \infty$, where d is the Lorentzian distance between sets,*

2. there exists $\epsilon > 0$ such that for all initial data satisfying $\| (g - g_o, K - K_o) \|_X < \epsilon$ where (g_o, K_o) are Cauchy data for a Kasner metric with exponents $(p_1, p_2, p_3) = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$, there exists a globally hyperbolic Hausdorff development of (g, K) for which on every past directed past inextendibile timelike curve the scalar $|R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}|$ tends to infinity in finite proper time (thus (M, γ) is future inextendible in the class of Hausdorff manifolds with $C^{1,1}_{loc}$ metrics).

This theorem is almost the equivalent of theorem 1.5.1 for general Gowdy metrics on T^3 with small initial data[46] — future inextendability is missing, though strongly suggested by *1* above. When $\Sigma = S^3$ or $S^2 \times S^1$ additional difficulties occur because the space of orbits of the isometry group is not a manifold. These have been recently overcome in a related model by Christodoulou [28], namely for non–linear Einstein–Rosen waves[47]:

Theorem 1.5.3 (Geodesic completeness of (non–linear) Einstein–Rosen space–times) *Let $\Sigma \approx I\!R^3$. For all cylindrically symmetric $(g, K) \in X(\Sigma) = \{(g, K) \in H^{loc}_3(\Sigma) \times H^{loc}_2(\Sigma), (g, K)$ satisfying certain decay conditions[48] for large $r\}$ there exists a unique globally hyperbolic, timelike and null geodesically complete (and therefore strongly maximal), Hausdorff development.*

One of the steps of the proof of Theorem 1.5.3 is to show that the constraint equations imply non–existence of trapped surfaces in Σ (*cf.* [32][Corollary 5.2]). This is a necessary condition for null geodesic completeness, as follows from a well–known singularity theorem due to Penrose [66]. Using the methods of proof of Theorem 1.5.2 and the results proved by Christodoulou one can via a "patching method" establish some properties of the maximally developped Gowdy space–times with topology $I\!R \times S^3$ and $I\!R \times S^1 \times S^2$, *cf.* [32]. The up to date known results are not as sharp as theorems 1.5.1–1.5.2.

[46] One can also establish "probably generic" existence of singularities in the past without restriction on the size of the data [38], these results do not however have such a simple formulation as the ones presented here for small data.

[47] It should be stressed that theorem 1.5.3 is concerned with the *nonlinear* Einstein–Rosen metrics, and *not* the "polarized" ones in which the evolution equations reduce to a linear equation.

1.6 Spatially open space–times: some small data results.

In cosmological considerations one often assumes the existence of compact spacelike hypersurfaces. However, when studying isolated gravitational systems, Σ cannot be assumed compact. On non–compact manifolds a natural class of metrics which describe isolated gravitating systems are the asymptotically flat metrics — in a sense to be made precise in Theorems 1.6.1 and 1.6.2 — and under the hypothesis of asymptotic flatness the only known results concerning global properties of "big classes" of solutions of Einstein equations are

1. the results on Robinson–Trautman space–times discussed in Section 1.7,
2. some results on analytic "boost–rotation" symmetric metrics due to Bičák and Schmidt [11], and
3. some "small data" results, which will be discussed in detail below

(recall that the Einstein–Rosen metrics discussed in Theorem 1.5.3, although "asymptotically flat" in a 2+1 dimensional sense, are not asymptotically flat in a 3+1 dimensional sense, because of translation invariance along the z–axis). The idea of "small data" results is to fix some space–time (M, γ_o), maximal globally hyperbolic developement of some data (Σ, g_o, K_o), and try to prove that for (g, K) sufficiently close to (g_o, K_o) in some norm the global properties of maximal developments (M, γ) of (Σ, g, K) will mimic those of (M, γ_o).The first class of results of this type has been proved some five years ago by Friedrich [52], with or without a cosmological constant, and also for the Einstein–Yang–Mills system [53]; here we shall consider the vacuum case with zero cosmological constant only. Friedrich's approach takes advantage of the fact that conformal transformations can map infinite domains into finite ones, reducing in this way the global in time stability problem to a much simpler short time stability problem for conformally rescaled fields. Suppose thus that a (spatially non–compact) space–time (M, γ) can be conformally mapped into a spatially compact space–time $(\tilde{M},\tilde{\gamma})$ (this can be done *e.g.* for the Minkowski space–time, with $(\tilde{M},\tilde{\gamma})$ — the "Einstein cylinder", $\tilde{M}\approx I\!R\times S^3$, *cf. e.g.* [66] or [52]). After such an infinite compression the conformal factor which relates the physical metric γ to the "unphysical" metric $\tilde{\gamma}$, $\gamma_{\mu\nu}=\Omega^{-2}\tilde{\gamma}_{\mu\nu}$, will vanish on the boundary $\partial M\equiv\mathcal{I}$ of M in $\tilde{M}$, which introduces singular terms when one naively rewrites Einstein equations for γ in terms of $\tilde{\gamma}$ and Ω. Friedrich has observed that one can write a well posed system of equations for the conformally rescaled fields which is regular even at points at which Ω vanishes, and which is equivalent to vacuum Einstein equations on the set $\Omega>0$ (*cf.* also [22] [23] for a different "conformally regular" system of equations), which leads to the following [52] [54]:

Theorem 1.6.1 (Future stability of the "hiperboloidal initial value problem") *Let (g_o,K_o) be the data induced on the unit hyperboloid $\Sigma=\{(t,x,y,z)\in I\!R^4 : t=\sqrt{1+x^2+y^2+z^2}\}\approx I\!R^3$ from the flat metric of Minkowski space-time. Consider the space X of Cauchy data (g,K) such that*

1. *(g,K) are smoothly conformally compactifiable*[49], i.e. *there exist a smooth compact Riemannian manifold $(\tilde{\Sigma},\tilde{g})$ with boundary, with $Int(\tilde{\Sigma})\approx\Sigma$, where $Int(\cdot)$ is the interior of $\cdot$, and a smooth (up to boundary) non-negative function Ω on $\tilde{\Sigma}$, vanishing only on $\partial\tilde{\Sigma}$, with $d\Omega(p)\neq 0$ for $p\in\partial\tilde{\Sigma}$, such that we have*

$$g_{ij}=\Omega^{-2}\tilde{g}_{ij}\ ,$$

and the fields

$$\tilde{L}^{ij}\equiv\Omega^{-3}(g^{ik}g^{j\ell}K_{k\ell}-\frac{1}{3}g^{\ell m}K_{\ell m}g^{ij})\ ,\qquad \tilde{K}\equiv\Omega g^{ij}K_{ij}$$

are smooth (up to boundary) on $\tilde{\Sigma}$;

[49]In [52] one assumes, roughly speaking, that $(\Omega,\tilde{g},\tilde{L},\tilde{K})\in H_k(\tilde{\Sigma})\oplus H_k(\tilde{\Sigma})\oplus H_{k-1}(\tilde{\Sigma})\oplus H_{k-1}(\tilde{\Sigma})$, $(\Omega_n,d^{\alpha}_{\beta\gamma\delta},f_{\alpha\beta})\in H_{k-1}(\tilde{\Sigma})\oplus H_{k-2}(\tilde{\Sigma})\oplus H_{k-2}(\tilde{\Sigma})$, $k\geq 6$; it is rather clear that by not too difficult technical improvements of the existence theorems used in [52] this threshold can be relaxed to $k\geq 5$ and probably even to $k\geq 4$.

2. *the Weyl tensor $C^\alpha{}_{\beta\gamma\delta}$ of the four–dimensional metric, formally calculated from (g, K) using vacuum Einstein equations, vanishes at the conformal boundary $\partial\tilde\Sigma$;*

3. *there exist fields Ω_n and Ω_{nn}, smooth (up to boundary) on $\tilde\Sigma$, which we identify with tetrad components in the directions normal to $\tilde\Sigma$ of the gradient, respectively the Hessian, of Ω, such that*

$$(\Omega_n^2 - \tilde g^{ij}\Omega_i\Omega_j)|_{\partial\tilde\Sigma} = 0 \ ,$$

and the tensor field

$$e_{\alpha\beta} = \nabla_\alpha\nabla_\beta\Omega - \frac{1}{4}\tilde\gamma^{\mu\nu}\nabla_\mu\nabla_\nu\Omega\,\tilde\gamma_{\alpha\beta}$$

vanishes at $\partial\tilde\Sigma$.

Set $d^\alpha{}_{\beta\gamma\delta} \equiv \Omega^{-1}C^\alpha{}_{\beta\gamma\delta}$, $f_{\alpha\beta} = \Omega^{-1}e_{\alpha\beta}$. There exists $\epsilon > 0$ such that for all $(g, K) \in X$ satisfying

$$\| \Omega - \Omega_o \|_{H_6(\tilde\Sigma)} + \| \tilde g_{ij} - \tilde g_{o\,ij} \|_{H_6(\tilde\Sigma)} + \| \tilde L_{ij} - \tilde L_{o\,ij} \|_{H_5(\tilde\Sigma)} + \| \tilde K - \tilde K_o \|_{H_5(\tilde\Sigma)}$$

$$+ \| d^\alpha{}_{\beta\gamma\delta} \|_{H_4(\tilde\Sigma)} + \| \Omega_n - \Omega_{o\,n} \|_{H_5(\tilde\Sigma)} + \| f_{\alpha\beta} - f_{o\,\alpha\beta} \|_{H_4(\tilde\Sigma)} < \epsilon$$

(where $H_\ell(\tilde\Sigma)$ is the Sobolev space of tensors on $\tilde\Sigma$ which are square integrable together with all the derivatives up to order ℓ on $\tilde\Sigma$ with respect to the Riemannian measure $d\mu_{\tilde g}$ of the metric $\tilde g$, and Ω_o, $\tilde g_{o\,ij}$, etc., denote the corresponding quantities for Minkowski space–time), the maximal globally hyperbolic development (M, γ) is future null and timelike geodesically complete, hence (M, γ) is strongly maximal *to the future.*

By the very nature of Friedrich's construction (*cf.* the discussion of Example 4 in Section 1.8) the above theorem gives only "50 %" of TTBP – it guarantees global uniqueness to the future of Σ only, nothing is known about the possibility of supplementing the Cauchy data on Σ by Cauchy data on the part of $\mathcal{I}$ which lies to the past of Σ to obtain global uniqueness to the past. Moreover, the following features of this Theorem deserve further investigation: 1) the rather high differentiability conditions needed for stability, 2) the hypothesis of the vanishing of the Weyl curvature on the conformal boundary $\partial\tilde\Sigma$

— the so called Weyl tensor condition, 3) the hypothesis of the vanishing of $e_{\alpha\beta}$ at the conformal boundary, 4) the independence of the various hypotheses above. Let us recall that the Weyl tensor condition has been shown by Penrose [104] to be necessary for C^k, $k \geq 3$, differentiability of the conformally rescaled fields at the conformal boundary of a space–time, but some new results of Christodoulou and Klainerman [29] suggest that the Weyl tensor condition needs not to hold in generic space–times obtained by evolution from asymptotically flat (at spacelike infinity) initial data (thus generic $\mathcal{I}$'s obtained in this way will probably *not* be C^3). As discussed in detail in Appendix A, the Cauchy data sets satisfying the Weyl tensor condition also turn out to be *non generic* in the space of solutions of the constraint equations which can be constructed by the conformal method. These drawbacks of Friedrich's theorem are more than compensated by the (relative) simplicity of the method. It has recently been shown in [3] that the vanishing of the space components e_{ij} of $e_{\alpha\beta}$ at $\partial\tilde{\Sigma}$ and smoothness of $\tilde{g}_{ij}$ imply the vanishing of the Weyl tensor at $\partial\tilde{\Sigma}$, under the supplementary hypotheses that the extrinsic curvature of $\tilde{\Sigma}$ is pure trace on $\partial\tilde{\Sigma}$ ($\tilde{L}_{ij}|_{\partial\tilde{\Sigma}} = 0$), and the Cauchy surface Σ has constant extrinsic curvature ($g^{ij}K_{ij}$ =const).

Friedrich's construction uses Cauchy data which are "asymptotically flat at null infinity". A different stability result, with Cauchy data "asymptotically flat at spatial infinity" has been proved recently [29] [30] by Christodoulou and Klainerman:

Theorem 1.6.2 (Nonlinear stability of Minkowski space–time) *Let $p \in \Sigma \approx I\!R^3$, $a > 0$, consider the quantity*

$$Q(a,p) = a^{-1} \int_\Sigma \{\sum_{\ell=0}^{1}(d_p^2 + a^2)^{\ell+1}|\nabla^\ell Ric|^2 + \sum_{\ell=0}^{2}(d_p^2 + a^2)^{\ell}|\nabla^\ell K|^2\} d\mu_g \ , \qquad (1.6.1)$$

where d_p is the geodesic distance function from p, Ric is the Ricci tensor of the metric g, $d\mu_g$ is the Riemannian measure of the metric g and ∇ is the Riemannian connection of g. Let

$$Q_* = \inf_{a>0,\, p\in\Sigma} Q(a,p) \ .$$

There is an $\epsilon > 0$ such that if $Q_* < \epsilon$, then the maximal globally hyperbolic development of (Σ, g, K) is geodesically complete, thus strongly maximal.

Christodoulou and Klainerman supplement this important and extremely difficult theorem by detailed information on the asymptotic behaviour of the gravitational field in various regimes, under the hypothesis, however, of stronger than required above fall–off of the initial data. Due to lack of space, and because these results are only loosely related to the problem of uniqueness in the large, we shall not discuss them here, the reader is referred to [29] for more details. It is worthwile noting that the condition of finiteness of $Q(a,p)$ defined in (1.6.1) will be satisfied if *e.g.* there exists a coordinate system covering the complement of a compact set such that

$$g_{ij} - \delta_{ij} = O(r^{-1/2-\epsilon}), \quad \partial_{i_1} g_{ij} = O(r^{-3/2-\epsilon}), \quad \ldots, \quad \partial_{i_1} \cdots \partial_{i_3} g_{ij} = O(r^{-7/2-\epsilon}) \ ,$$

$$K_{ij} = O(r^{-3/2-\epsilon}), \quad \ldots, \quad \partial_{i_1}\partial_{i_2} K_{ij} = O(r^{-7/2-\epsilon}), \quad \epsilon > 0 \ .$$

In terms of rates of decay of the metric to the flat one, these are the well known conditions for a finite and well defined ADM mass [5] [31] of the initial Cauchy data set. Under the condition that $Q(a,p)$ is finite, Christodoulou and Klainerman have also been able to establish the important fact, that the maximal globally hyperbolic development of (Σ, g, K) contains "a neighbourhood of i^o", *cf.* [29] for more details.

1.7 Beyond the spacelike Cauchy problem: Robinson–Trautman space–times.

In the standard dynamical formulation of general relativity one considers space-times which develop from Cauchy data prescribed on a spacelike hypersurface. This situation seems natural for addressing questions such as existence and uniqueness of solutions, but there may be other settings in which such questions make sense, since space-times can be constructed by various other methods, e.g. some solution generating techniques. A natural extension of the spacelike Cauchy problem is the characteristic initial value problem, in which the initial data are prescribed on a null rather than spacelike hypersurface. In this context the only general result available (in the vacuum) is a local existence theorem for data prescribed on two null transversally intersecting hypersurfaces [96] [107]. The analytic initial value problem with Cauchy data given on a Cauchy horizon has been shown to be well posed in the vacuum by V. Moncrief [88] [89]. In the non-vacuum case, the characteristic initial value problem for a spherically symmetric self-gravitating scalar field has been studied by D. Christodoulou [26], the initial hypersurface being the light cone of a point. In this section we shall discuss an interesting class of metrics which evolve from singular data prescribed on a null hypersurface — the Robinson–Trautman (RT) space-times. There are several interesting features exhibited by the RT metrics: the evolution of the metric is unique in spite of a "naked singularity"; suprisingly, Einstein equations reduce to a single parabolic fourth order equation in this class of metrics. From a physical point of view the RT metrics can be thought of as representing an isolated gravitationally radiating system — in fact these metrics were the first ones to be found, describing such a situation [109]. By definition the Robinson–Trautman space–times can be foliated by a null hypersurface orthogonal shear free geodesic congruence. It has been shown by I. Robinson and A. Trautman that in such a space–time there always exists a coordinate system in which the metric takes the form

$$ds^2 = -\Phi\, du^2 - 2du\, dr + r^2 e^{2\lambda} \mathring{g}_{ab}\, dx^a\, dx^b, \quad x^a \in {}^2M, \quad \lambda = \lambda(u, x^a), \tag{1.7.1}$$

$$\mathring{g}_{ab} = \mathring{g}_{ab}(x^a), \quad \Phi = \frac{R}{2} + \frac{r}{12m}\Delta_g R - \frac{2m}{r}, \quad R = R(g_{ab}) \equiv R(e^{2\lambda}\mathring{g}_{ab}),$$

m is a constant which is related to the total Bondi mass of the metric, R is the Ricci scalar of the metric $g_{ab} \equiv e^{2\lambda}\mathring{g}_{ab}$, and $({}^2M, \mathring{g}_{ab})$ is a smooth Riemannian manifold which we shall assume to be a two dimensional sphere (other topologies are considered in [34]). The Cauchy data for an RT metric consist of $\lambda_0(x^a) \equiv \lambda(u = u_0, x^a)$, which is equivalent to prescribing the metric $g_{\mu\nu}$ of the form (1.7.1) on the null hypersurface $\{u = u_0, x^a \in {}^2M,\ r \in (0, \infty)\}$, which extends up to a curvature singularity at $r = 0$ (the scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverges at $r = 0$ as r^{-6}).

The global structure of RT space-times turns out to be different, depending upon the

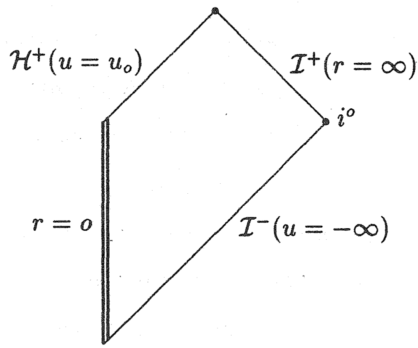

Figure 1.7.1: $m < 0$

sign of m: let us discuss the negative m case first. Recall that for $m < 0$ it is natural to consider a backwards (in u) initial value problem rather than a forwards initial value problem; alternatively one could think of u as being a retarded null coordinate ($u \sim$ "$r + t$") rather than an advanced one ($u \sim$ "$r - t$"). The results of B. Schmidt [112] (cf. also [48]) and of reference [33] imply:

Theorem 1.7.1 (The global structure of negative-mass RT space-times) *For any $\lambda_0 \in C^\infty({}^2M)$ there exists a unique RT space-time with a complete i^0 (in the sense of [4]), a complete $\mathcal{I}^-$ and "a piece of $\mathcal{I}^+$", as shown in Figure 1.7.1, moreover*

1. *${}^4\mathcal{M}$ is smoothly extendible through $\mathcal{H}^+$,*

2. *If λ_0 is* not *analytic there exist* no *vacuum RT extension through $\mathcal{H}^+$.*

There are several interesting features of this result. Because of the singularity $r = 0$ in the initial data one could wonder whether any solutions of the Einstein equations would exist at all: it turns out that solutions exist either in the backwards or in the forward direction in u, depending upon the sign of m, moreover they are unique in the Robinson–Trautman class[50]. Let us recall that the "weak cosmic censorship" conjecture, discussed in Section

[50]It may be possible that there exist vacuum solutions with the same data at $u = u_o$ which are not in the Robinson–Trautman class.

1.4, can be formulated as the statement that the past of $\mathcal{I}^+$ is determined uniquely by the initial data. In the negative m case we are solving a backwards[51] initial value problem, *i.e.* a *"final"* value problem, so in this context the cosmic censorship hypothesis should be reformulated as the requirement that the future of $\mathcal{I}^-$ be determined uniquely by the "final" data. Theorem 1.7.1 establishes such a fact (in the RT class of vacuum metrics) for RT space-times with negative mass[52]. The information about the structure at i^o is also rather interesting, though it must be said that its interest is somewhat diminished by the negativity of the ADM mass there. (It may be of some relevance to note that the limit as $u \to -\infty$ of the Bondi mass coincides with the ADM mass, as expected).

The generic non-extendability of the metric through $\mathcal{H}^+$ in the vacuum RT class is rather suprising, and seems to be related to a similar non-extendability result for compact non–analytic Cauchy horizons in the polarized Gowdy class, *cf.* [37]. Since it may well be possible that there exist vacuum extensions which are not in the RT class, this result does not convicingly demonstrate a failure of Einstein equations to propagate generic data forwards in u in such a situation; however, it certainly shows that the forward evolution of the metric via Einstein equations breaks down in the class of RT metrics. One can think of this as a "50%" failure of TTBP in the space of negative mass RT metrics: generic (in the sense: non–analytic initial data) RT metrics are extendible but *not RT vacuum extendible* beyond $\mathcal{H}^+$, and thus in the terminology of Section 1.1 the space–time of Figure 1.7.1 is *maximal* but not *strongly maximal* in the RT class.

In the positive mass case the results of [112] [124] [33] [34] [39] show the following:

Theorem 1.7.2 (The global structure of positive-mass RT space-times) *For any $\lambda_0 \in C^\infty(S^2)$ there exists a Robinson–Trautman space-time $({}^4M, g)$ with a "half-complete"*

[51]One can of course put everything upside down, changing u to $-u$, which then becomes an "advanced null coordinate" rather than a retarded one, and what was a backwards initial value problem becomes a standard one.

[52]In this formulation of weak cosmic censorship (w.c.c.) no mention is made of horizons. One should recall that in the usual form of w.c.c. one "hides" singularities under a horizon to "hide" the regions of potential non-uniqueness of solutions. In the RT case uniqueness holds regardless of "nakedness" of the singularity $r = 0$.

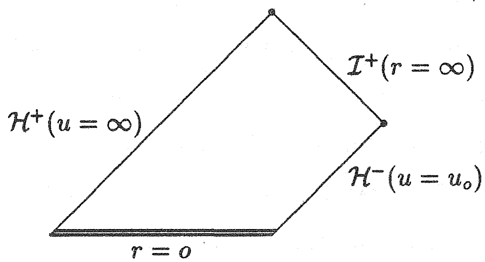

Figure 1.7.2: $m > 0$.

$\mathcal{I}^+$, the global structure of which is shown in Figure 1.7.2, moreover

1. *$(^4\mathcal{M}, g)$ is smoothly extendible to the past through $\mathcal{H}^-$, if however λ_0 is* not *analytic* no *vacuum Robinson–Trautman extensions through $\mathcal{H}^-$ exist.*

2. *There exist an infinite number of non-isometric vacuum Robinson–Trautman C^5 extensions*[53] *of $(^4\mathcal{M}, g)$ through $\mathcal{H}^+$, which are obtained by gluing to $(^4\mathcal{M}, g)$ any other positive mass Robinson–Trautman spacetime, as shown in Figure 1.7.3.*

3. *There exist an infinite number of C^{117} vacuum RT extensions of $(^4\mathcal{M}, g)$ through $\mathcal{H}^+$ — one such extension can be obtained by gluing a copy of $(^4\mathcal{M}, g)$ to itself, as shown in Figure 1.7.3.*

4. *For any $6 \leq k \leq \infty$ there exists an open set $\mathcal{O}_k$ of Robinson–Trautman space-times (in a C^k topology on the set of RT Cauchy data on a hypersurface $u_o = 0$) for which no C^{123} extensions beyond $\mathcal{H}^+$ exist, vacuum or otherwise. For any u_o there exists an open ball $\mathcal{B}_k$ around the Cauchy data for the Schwarzschild metric such that $\mathcal{O}_k \cap \mathcal{B}_k$ is dense in $\mathcal{B}_k$.*

The picture that emerges from Theorem 1.7.2 is the following: generic initial data lead to a space–time which has no RT vacuum extension to the past of the initial surface, even

[53]By this we mean that the metric can be C^5 extended beyond $\mathcal{H}^+$; the extension can actually be chosen to be of $C^{5,\alpha}$ differentiability class, for any $\alpha < 1$.

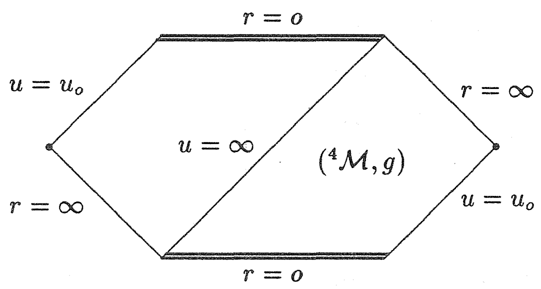

Figure 1.7.3: Vacuum RT extensions beyond $\mathcal{H}^+$

though the metric can be smoothly extended (in the non–vacuum class); and generic "small[54] initial data" lead to a space–time for which no smooth vacuum RT extensions exist beyond $\mathcal{H}^+$. This shows that considering smooth extensions across $\mathcal{H}^+$ leads to non–existence, and giving up the requirement of smoothness of extensions beyond $\mathcal{H}^+$ leads to non–uniqueness. It follows that TTBP completely fails in the class of positive mass Robinson–Trautman metrics. It should be recognized that this might be thought of as demonstrating only some pathological aspects of the Robinson–Trautman conditions, rather than some real features of the theory.

1.8 Necessary conditions for uniqueness in the large.

In this section we shall discuss, by means of examples, some natural restrictions which one may wish to impose on the spaces $X(\Sigma)$ and $\mathcal{M}(\Sigma)$ introduced in TTBP.

Example 1: Let (Σ, g, K) be Cauchy data for a cylindrically symmetric polarized metric:

$$ds^2 = e^{2(U-A)}(-dt^2 + dr^2) + e^{-2U} r^2 d\phi^2 + e^{2U} dz^2 \tag{1.8.1}$$

$$U = U(t,r)\,, \qquad A = A(t,r)\,,$$

$$\Sigma = \{t = 0\} \approx I\!R^3\,.$$

[54] It is rather clear from the results of [39] that generic RT space–times will not be smoothly extendible across $\mathcal{H}^+$, without any restrictions on the "size" of the initial data; but no rigorous proof is available.

For metrics of the form (1.8.1) vacuum Einstein equations essentially reduce to a single linear wave equation (in the flat Minkowski metric) for U,

$$\Box U = \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right] U = 0\,, \tag{1.8.2}$$

$$x = r\cos\phi\,, \qquad y = \sin\phi\,,$$

and the Cauchy data reduce to Cauchy data for $U : \varphi = \varphi(r) = U(0,r)$, $\psi = \psi(r) = \frac{\partial U}{\partial t}(0,r)$. (Given any solution U of the wave equation (1.8.2), the function A appearing in the metric can be found by elementary integrations, *cf. e.g.* Section 3.5.).

Let $\rho = \rho(t,r) \in \mathcal{D}'(\mathbb{R}^3)$ be a distribution on $\mathbb{R}^3$, such that $\operatorname{supp}\rho \cap \tilde{\Sigma} = \phi$, where $\tilde{\Sigma} = \{(t,x,y) \in \mathbb{R}^3 : t = 0\}$, define U_ρ as the unique solution of the problem

$$\Box\, U_\rho = \rho\,, \tag{1.8.3}$$

$$U_\rho(t=0) = \varphi\,(= U(t=0))\,, \quad \frac{\partial U_\rho}{\partial t}(t=0) = \psi = \left(\frac{\partial U}{\partial t}(t=0)\right)\,.$$

(U_ρ will exist if e.g. $\rho \in H_m(\mathbb{R}^3)$, $(\varphi,\psi) \in H_k(\tilde{\Sigma}) \oplus H_{k-1}(\tilde{\Sigma})$, for any $m,k \in \mathbb{R}$ (in particular if $f \in C^0(\mathbb{R})$, then the distribution ρ given by

$$\rho = f(t)\,\delta_0 \in H_m(\mathbb{R}^3)\,, \qquad m < -3/2 \tag{1.8.4}$$

is an allowed distribution.) Let M_ρ be the interior of $\mathbb{R}^4 \setminus \{(t,x,y,z) : \rho(t,x,y) \neq 0\}$, let γ_ρ be the metric (1.8.1) with $U = U_\rho$, and an appropriate A — the family (M_ρ, γ_ρ) is thus a family of vacuum space-times parametrized by the set of functions $\rho \in \mathcal{D}'(\mathbb{R}^3)$ subject to the restrictions above, each member (M_ρ, γ_ρ) being a vacuum development of (Σ, g, K), and it is easy to see that for different ρ's one will in general obtain non–isometric space–times.

Obviously the non-uniqueness described here arises from the fact that we have put some "matter" ρ in our space-time, and pretended "it is not there" by removing from our manifold the regions where matter was present. This example shows that in order to achieve any kind of uniqueness it is natural to consider these developments (M,γ) only, which contain as a subset the maximal globally hyperbolic development (M_0,γ_0) of the data, in other words, that there exists an isometric embedding of (M_0, γ_0) into (M, γ). Such a restriction, when imposed in the example above, would exclude all the M_ρ's except for the space-time obtained by solving (1.8.3) with $\rho = 0$.

The space-times (M_ρ, γ_ρ) obtained from ρ of the form (1.8.4) provide a family of examples which suggest that non-uniqueness of solutions might arise in space-times with naked singularities. As is shown in Appendix E, given a smooth $f(t)$ such that $0 \notin \operatorname{supp} f$, and smooth φ and ψ, there exists a unique solution U_ρ of (1.8.3) with ρ given by (1.8.4) which is smooth on $I\!R^3 \setminus \operatorname{supp} \rho$. If we set $\operatorname{supp} f$ to be, say, the interval $[1,2]$, and we vary f, we will obtain an infinite dimensional family of non-isometric space-times with a "naked singularity sitting on the set" $\{t \in [1,2], x = y = 0\}$. The arbitrariness of f represents an arbitrariness introduced by the singularity, thus f can be thought of as a "boundary condition at the singularity". All these space-times are of course excluded by the criterion that we consider only these developments which are "at least as large as the maximal globally hyperbolic development", they seem however to indicate that the occurrence of real singularities might lead to behaviour which is difficult to control.

Example 2: Let (Σ, g, K) be the initial data for Minkowski space-time on an open unit ball: $\Sigma = B(1) \subset I\!R^3$, $g_{ij} = \delta_{ij}$, $K_{ij} = 0$. As has been shown by Bartnik [8] (Σ, g, K) may be extended in an infinite number of ways to a Cauchy data set[55] $(\tilde{\Sigma}, \tilde{g}, \tilde{K})$, $\tilde{\Sigma} = I\!R^3$. The maximal globally hyperbolic development (M, γ) of (Σ, g, K) is the set $\{-1 < t < 1, 0 \le r < 1 - |t|\} \subset I\!R^4$ with the Minkowski metric, and any globally hyperbolic development $(\tilde{M}, \tilde{\gamma})$ of $(\tilde{\Sigma}, \tilde{g}, \tilde{K})$ will provide an extension of (M, γ). This example shows that in TTBP it is natural to restrict our attention to *inextendible* Cauchy data sets (Σ, g, K) — such a condition would exclude the behaviour described there.

There are at least two ways for a Cauchy data set (Σ, g, K) to be inextendible: one is to assume that (Σ, g) is complete, another possibility is the occurrence of a singularity at "what would have been $\partial\Sigma$", let us consider the latter first:

[55] $(\tilde{\Sigma}, \tilde{g}, \tilde{K})$ can even be chosen to be asymptotically flat.

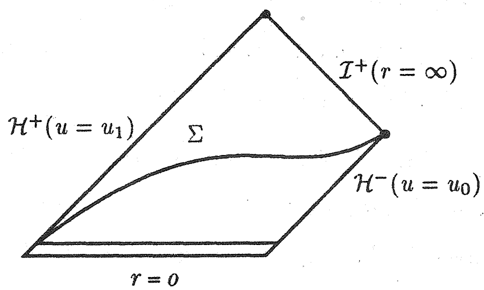

Figure 1.8.1: ${}^2M \approx S^2$, $m > 0$.

Example 3: Fix $u_0 < u_1 \leq \infty$, consider some smoth space-like hypersurface $\Sigma \subset [u_0, u_1) \times I\!R \times S^2$, as shown in Figure 1.8.1, let (g, K) be the data induced on Σ for a Robinson–Trautman space-time obtained by prescribing some smooth function $\mathring{\lambda} \in C^\infty(S^2)$ at $u = u_0$ (*cf.* Section 1.7). The hypersurface Σ is inextendible through its "left corner", Figure 1.8.1 because of the singularity at $r = 0$ of Robinson–Trautman metrics. Theorem 1.7.2 shows that there exist an infinite number of C^{117} extensions to the future of Σ of the maximal globally hyperbolic development of (Σ, g, K), which is, in Robinson–Trautman coordinates (u, r, θ, φ), the set $(u_0, u_1) \times I\!R \times S^2$ (note that if $u_1 < \infty$ then there exists a neighbourhood of the hypersurface $\{u = u_1\}$ in which the metric is uniquely defined in the vacuum Robinson–Trautman class by (Σ, g, K), this fails, however, at the horizon $\mathcal{H}^+ = \{$"$u = \infty$"$\}$). This example suggests that in TTBP it may not be possible to allow for Cauchy data (Σ, g, K) which are inextendible through "$\partial\Sigma$" because of "singularities sitting on $\partial\Sigma$". It should be, however, pointed out that although this behaviour is generic in the class of Robinson–Trautman space-times, the Robinson–Trautman space-times themselves are not generic, and it cannot be excluded that this kind of non-uniqueness might disappear in generic situations.

Example 4: Let (Σ, g, K) be "hyperboloidal initial data", as described in Appendix A, in particular (Σ, g) is a complete Riemannian manifold; suppose moreover that (Σ, g, K) is "smoothly conformally compactifiable" and that the hypotheses of Theorem 1.6.1 hold

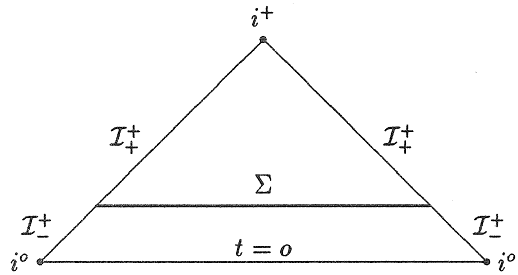

Figure 1.8.2: A "hyperboloidal" initial data surface.

(*cf.* *e.g.* [3], or Section 1.6 and Appendix A for more details). We can choose the time orientation in such a way that the maximal vacuum globally hyperbolic development (M, γ) of (Σ, g, K) contains at least "a piece" $\mathcal{I}^+_+$ of $\mathcal{I}^+$, where $\mathcal{I}^+_+$ is the part of $\mathcal{I}^+$ to the future of Σ, *cf.* Figure 1.8.2 (*cf.* Theorem 1.6.1 and [52]). Using *e.g.* the techniques of Ref. [107] one can show[56] that supplementing (Σ, g, K) by appropriate smooth data on $\mathcal{I}^+_-$ — the part of $\mathcal{I}^+$ in the past of Σ — one can find a vacuum metric on a neighbourhood $\mathcal{O}$ of $\Sigma \cup \mathcal{I}^+_-$. There is arbitrariness in the choice of the "missing data on $\mathcal{I}^+_-$", and different data[57] will lead to non-isometric extension of (M, γ) to the past of Σ. This example shows that even the requirement of completeness of (Σ, g) is not sufficient in TTBP.

Let us close this section by emphasizing that it seems natural to require that

- The space $\mathcal{M}(\Sigma)$ of Lorentzian manifolds introduced in TTBP should contain only those developments (M, γ) of (Σ, g, K) which contain the unique maximal globally hyperbolic Hausdorff development of (Σ, g, K);
- if Σ is compact, then all the metrics in $X(\Sigma)$ should be complete;
- for non-compact Σ, all metrics in $X(\Sigma)$ should be complete and *e.g.* asymptotically

[56] H. Friedrich, private communication.

[57] By choosing (Σ, g, K) to be the data induced on the standard hyperboloid in Minkowski space-time one can by this method construct a curious space-time which is the Minkowski space-time to the future of a hyperboloid, and not-Minkowski to its past.

flat in the sense of Theorem 1.6.2.