DEPOSITION FROM A CURVED SHALLOW FLOW TREATED AS A MOVING BOUNDARY OF CONSTANT FORM

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Dams and terraces deposited from supersaturated flows or supercooled flows are known to occur in nature. Examples of the former include the travertine deposits at Mammoth Springs, Yellowstone National Park, U.S.A., the calcareous deposits of Pammukale, Turkey, and the siliceous deposits forming the Pink and White Terraces which existed in New Zealand prior to the Tarawera volcanic eruption of 1886 [9,10]. Examples of the second type include an ice-sand dam, formed from pressure melt water, which has been known to develop at the toe of the Mueller Glacier in the Southern Alps of New Zealand [3].

These formations are quite large (of order 1 m in height and of considerable horizontal extent), suggesting that the observed repetitive (wavelike) solid profiles are final forms which have developed at large times. The fluid flows giving rise to deposition are likely to be quite shallow, and may range from thin laminar to turbulent flow regimes. Evidently the scale of flow depth is very much smaller than the scale of the depositional features.

One mechanism whereby these deposits might be produced has been described recently [9,10], and the present short article is intended to review some aspects of the problem and provide a few additional comments. The notation previously used will be retained, and a definition diagram (Figure 4 in [9,10]) is also useful.

2. PROBLEM DESCRIPTION

If a shallow turbulent flow is assumed, which has been justified *a posteriori* in the former work, it is convenient to use Dressler's approximate hydraulic equations [2,6], including a Chézy term to represent flow resistance. From these equations, the length scales of both the fluid flow and the solid profile are fixed by flow depth, which is determined by the volume flow rate. If these quantities can be taken constant while the surface grows, then for self similarity the profile moves in uniform translation [1]. This leads to a geometrical condition on the profile of the form

$$n_t = v \sin (\varphi - \theta) \qquad (1a)$$

where n_t is the rate of accretion normal to the surface, v is the translational velocity, φ the angle of profile growth and θ the slope of the solid profile. A similar geometrical condition applies to the Saffman–Taylor problem of finger shape in a Hele–Shaw cell [4,5]. However, in the present case the direction of profile growth is not known a priori and is not necessarily horizontal.

It is assumed that deposition is controlled principally by the diffusive resistance of the laminar sublayer, the thickness of which is almost inversely proportional to the flow velocity. Lag effects are neglected on the assumption of large lengthscale/depth ratios. It follows that the minimum depth of flow should be found at a point where the solid profile is normal to the direction of profile growth. That point may not actually exist within the range of the profile. However, the corresponding depth can still be defined, and here it will be taken as the length scale of the system. If the assumed deposition law is combined with (1a), the Saffman–Taylor condition can be written as

$$1/H = \sin (\varphi - \theta) \qquad (1b)$$

where H, the dimensionless flow depth, will be taken as a function of distance S along the profile.

The expected large scale of the solid features relative to flow depth has several important consequences. Over at least part of the slope range the curvature terms in the Dressler equations can be neglected, leading to a differential equation for the solid profile which is analogous to the Bresse profile equation of hydraulics [7],

$$\frac{dH}{dS} = \frac{-H \sin \theta - \Gamma F_m/H^2}{H \cos \theta - F_m/H^2} \tag{2a}$$

involving two parameters -- a Froude number F_m and a Chézy coefficient Γ. Then the curvature K of the solid profile can be defined from (1b) and (2a) as

$$K = d\theta/dS = \frac{dH/dS}{H(H^2 - 1)^{\frac{1}{2}}} \tag{2b}$$

Since the curvature is small, it follows that dH/dS is small; the values of Γ and F_m are likely to be such that the numerator on the right hand side of (2) is small for a significant range in (H,θ). This property is related to the location of the zeros of the numerator. However, as the point of maximum growth rate is approached ($H \to 1$) the curvature becomes very large unless dH/dS also vanishes simultaneously. From observation, it appears that this condition is not necessarily satisfied, and some natural dams appear to be "perched", perhaps developing curtains of stalactites.

3. FLOW WITH A TRANSITION

The case where a transition occurs from an upstream subcritical flow to supercritical flow downstream has been described in detail elsewhere [9,10]; this leads to solutions for growing dams. Usually the critical point occurs near the top of the dam where the flow becomes shallow, and a supercritical flow covers most of the outer dam wall. At the critical point the numerator and denominator on the right hand side of (2) vanish simultaneously. The remaining roots of the denominator appear not to be of direct physical interest. However, the two available roots of the numerator are of considerable significance in determining the shape of the solid profile. In two cases theoretical profiles have been fitted to photographs of natural dams [3,9,10] with apparent success, and realistic values of the Froude number, drag coefficient and other flow parameters deduced. In both examples the roots of the numerator in (2) are found to be complex but nearly coincident, i.e. the imaginary parts are small, and the numerator is small but non-vanishing. (Note that, if these roots were real, then the slope would be constant over at least part of the solid profile.)

The condition for coincident roots provides a boundary between the space of complex roots, in which relevant solutions exist, and the space of real roots. Evidently the parameter values for realistic solutions lie close to this boundary, which is given by the following equations [10]. Let D_c be a parameter defined by

$$H_c = (1 + D_c^2)^{\frac{1}{2}} \tag{3}$$

where H_c denotes the critical depth. Then

$$\tan \varphi = 1/(2H_c - D_c) \tag{4}$$

$$\Gamma = (1 - D_c \tan \varphi)/(D_c + \tan \varphi) \tag{5}$$

$$F_m = H_c^3/(1 + \Gamma^2)^{\frac{1}{2}} \tag{6}$$

These parametric equations follow immediately from the derivations given in

[10], equation (4) being the condition for coincident roots. Elimination of D_c and tan φ between (3), (4) and (5) gives a fourth-order equation

$$3(\Gamma H_c)^4 + 4(\Gamma H_c)^3 - 4\,\Gamma^2\,(1 + \Gamma^2) = 0 \qquad (7)$$

Since H_c is bounded, the appropriate solution for ΓH_c should tend to zero with Γ, and a convenient approximation is:

$$\Gamma H_c = A - \frac{1}{4}A^2 + \frac{3}{16}A^3 - \frac{35}{192}A^4 + O(A^5) \qquad (8)$$

where $A = \{\Gamma^2(1 + \Gamma^2)\}^{1/3}$ is taken to be small on the assumption that Γ is sufficiently small. Then, from (6) and (8),

$$F_m\Gamma/(1 + \Gamma^2)^{\frac{1}{2}} = 1 - \frac{3}{4}A + \frac{3}{4}A^2 + O(A^3) \qquad (9)$$

Thus the boundary defined by coincident roots of the numerator in (2) lies close to a rectangular hyperbola in (F_m,Γ) space, as would be expected from the form of the numerator in (2).

Figure 1 gives a plot of Γ vs F_m for coincident roots, from the parametric equations (3) - (6) (continuous curve), and also the approximate result (9) (dashed curve). The symbol R denotes the zone of real roots for the numerator in (2), while C indicates the region in which the numerator has two complex roots which lead to real physical solutions of the type sought. The results from tests of the theory using photographs of natural dams (mentioned above) are indicated by the points 1, a dam formed by deposition from supersaturated geothermal flow [9,10], and 2, a dam formed by freezing of supercooled glacial meltwater [3]. Evidently these indicate two stable solutions, and a continuum of stable solutions may exist close to the curve in region C of Figure 1.

4. THE SAFFMAN-TAYLOR PROBLEM

Recently Thomé et al. [8] discussed the significance of singularities in the complex plane in the selection of zero-surface-tension analytical solutions for the Saffman-Taylor finger in a Hele-Shaw cell. In particular, they showed experimentally that a very local disturbance of the viscous flow which modified the singularities in the complex plane could change the nose shape and the selection of finger width -- a major effect. In the present problem there is not a complex plane representation of the flow, but the geometrical condition (1) is common to both problems, and control by modification of singularities is applicable in both.

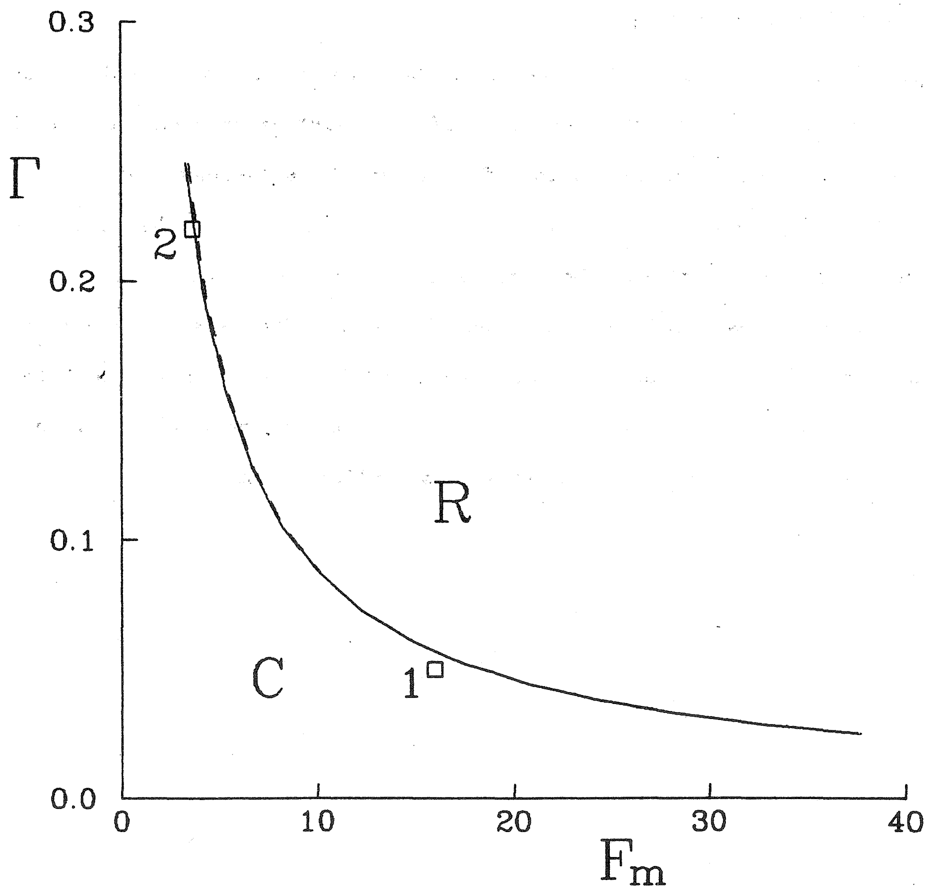

Figure 1. Curved boundary separating region of complex roots (C) from real roots (R). (For details see text.)

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