FRAGMENTABILITY OF ROTUND BANACH SPACES

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Introduction

A topological space X is said to be *fragmented by a metric* ρ if for every $\varepsilon > 0$ and every subset Y of X there exists a nonempty relatively open subset U of Y such that ρ–diam(U) < ε. The space is said to be *fragmentable* if there exists such a metric.

The concept of fragmentability was introduced by Jayne and Rogers in 1985 [1]. Further work has been done by Namioka [2], and Ribarska [3, 4]. A major application of fragmentability is in the study of differentiability of convex functions.

A continuous convex function ϕ on an open convex subset A of a Banach space X is said to be *Gâteaux differentiable* at $x \in A$ if

$$\lim_{\lambda \to 0} \frac{\phi(x+\lambda y) - \phi(x)}{\lambda}$$

exists for all $y \in X$. The function is said to be *Fréchet differentiable* at x if the above limit exists and is approached uniformly for all $y \in X$, $\| y \| = 1$. A Banach space X is said to be *Asplund* (*weak Asplund*) if every continuous convex function on an open convex domain is Fréchet (Gâteaux) differentiable on a dense G_δ subset of its domain.

In 1975 Namioka and Phelps, [5, p.739], established that a Banach space is Asplund if and only if the weak * topology on its dual is fragmented by the dual norm, although their result was not couched in these terms. Weak * fragmentability of the dual by some metric is still the most general known condition implying a Banach space is weak Asplund, [3]. Ribarska, [4], has shown that rotund Banach spaces are weakly fragmentable, and that Banach spaces with rotund dual have weak * fragmentable dual, and hence are weak Asplund. Using techniques developed by Preiss, Phelps and Namioka, [6], she has also shown that a Banach space with an equivalent norm Gâteaux differentiable away from 0 has a weak * fragmentable dual, and hence is weak Asplund. Whether or not weak Asplund spaces are characterized by having a weak * fragmentable dual remains an open question.

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The aim of this paper is to provide an alternative proof of the fragmentability of rotund spaces. Section 1 reviews Ribarska's work on a characterization of fragmentability [3]. In Section 2 we introduce the concept of a fragmenting pre–metric and show that a fragmenting metric can be generated from it. In Section 3 we define pre–metrics which fragment rotund Banach spaces, and finally in Section 4 we use the ideas developed for Banach spaces to weakly fragment a class of locally convex spaces.

1. Internal characterization of fragmentability

In 1987, Ribarska [3] produced a characterization theorem for fragmentability which relies on the concept of *relatively open partitionings*. This section outlines that part of her work which is necessary for our present purpose.

A *relatively open partitioning* of a topological space X is a well–ordered family of subsets of X, $U = \{U_\xi : 0 \le \xi < \xi_0\}$, satisfying

(i) $U_0 = \varnothing$.

(ii) $U_\xi \subset X \setminus \left(\bigcup_{\eta<\xi} U_\eta \right)$ and is relatively open in it .

(iii) $X = \bigcup_{\eta<\xi_0} U_\xi$.

A *separating σ relatively open partitioning* of a topological space X is a countable family of relatively open partitionings U^n which separates points. That is, for any $x,y \in X$, $x \neq y$, there is an $n_0 \in \mathbb{N}$ such that x and y are in different elements of the partitioning U^{n_0}.

Theorem 1.1 (Ribarska 1987)

A topological space X *admits a separating* σ *relatively open partitioning if and only if there exists a metric which fragments* X.

Proof

Let X be fragmented by the metric ρ. For every $n_0 \in \mathbb{N}$, construct a relatively open partitioning $U^n = \left\{ U^n_\xi : 0 \le \xi < \xi_n \right\}$ inductively as follows:

(i) Set $U_0^n = \varnothing$.

(ii) Suppose U_η^n have been constructed for every $\eta < \xi$. Consider the set $R = X \setminus \left(\bigcup_{\eta<\xi} U_\eta^n \right)$.

If R is empty then let $\xi_n = \xi$ and we have finished constructing U^n. If R is not empty it has a non–empty relatively open subset of ρ–diameter $< \frac{1}{n}$. This subset becomes U_ξ^n.

(iii) Since every U_ξ^n is non–empty, there exists a ξ_n such that $R = X \setminus \left(\bigcup_{\eta<\xi} U_\eta^n \right)$ is empty.

The countable family of relatively open partitionings U^n separates points, since if $x \neq y$ then $\rho(x,y) > \frac{1}{n_0}$ for some n_0, and hence x and y are in different elements of U^{n_0}. Hence $\{U^n : n \in \mathbb{N}\}$ is a separating σ relatively open partitioning of X.

Conversely let $\{U^n : n \in \mathbb{N}\}$ be a separating σ relatively open partitioning of X. Define,

$$\rho(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{n_0} & \text{if } x \neq y, \quad \text{where } n_0 \text{ is the smallest integer such that } U^{n_0} \text{ separates x and y.} \end{cases}$$

(i) $\rho(x,y) \geq 0$.

(ii) $\rho(x,y) = 0$ if and only if $x = y$.

(iii) if $\rho(x,y) = \frac{1}{n_0}$ then for $z \in X$ at least one of $\rho(x,z)$ or $\rho(y,z)$ is $\geq \frac{1}{n_0}$, so ρ satisfies the triangle inequality.

So ρ is a metric on X. To show ρ fragments X, let Y_0 be a non–empty subset of X, and $\varepsilon > 0$. Define Y_n inductively by

$$\bar{\xi}_n = \min\left\{ \xi < \xi_n : Y_{n-1} \cap U_\xi^n \neq \varnothing \right\}$$

$$Y_n = Y_{n-1} \cap \left(\bigcup_{\eta \leq \bar{\xi}_n} U_\eta^n \right).$$

Then for any $n > \frac{1}{\varepsilon}$ we have Y_n is relatively open in Y_0, and ρ–diam $(Y_n) \leq \frac{1}{n} < \varepsilon$. //

2. Fragmenting by a pre–metric

Observe that in the proof of Theorem 1.1 it was not necessary that the function ρ be a metric in order to create the separating σ relatively open partitionings. It suffices for ρ to be a non–negative, separating function which fragments X. The existence of such a function would guarantee the existence of a fragmenting metric via the second half of the proof of Ribarska's Theorem. This observation motivates the following definition.

A *fragmenting pre–metric* for a topological space X is a function $\lambda: X \times X \to \mathbb{R}$ satisfying

(i) $\lambda(x,y) \geq 0$, and $\lambda(x,y) = 0$ if and only if $x = y$,

(ii) for every $\varepsilon > 0$ and non–empty subset Y of X there exists a non–empty relatively open subset U of Y such that $\lambda\text{–diam}(U) \equiv \sup \{\lambda(x,y): x,y \in U\} < \varepsilon$.

From the above observation there is an immediate corollary to Theorem 1.1.

Corollary 2.1

A topological space X is fragmentable if and only if X has a fragmenting pre–metric.

The following Lemma will be needed in Section 3.

Lemma 2.2

A linear topological space X is fragmentable if and only if it has an absorbing subset Y which is fragmentable in the relative topology.

Proof

If X is fragmented by a metric ρ, then $\rho|_Y$ fragments any absorbing subset Y of X. Conversely, if Y is fragmented by a metric ρ, then

$$\lambda(x,y) = \begin{cases} \rho(x,y) & \text{if } \rho(x,y) < 1 \\ 1 & \text{if } \rho(x,y) \geq 1 \end{cases}$$

is a fragmenting pre–metric for Y. Define $\bar{\lambda}$ on X by

$$\bar{\lambda}(x,y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \lambda_m(x,y)$$

where $$\lambda_m(x,y) = \begin{cases} \lambda\left(\frac{x}{m}, \frac{y}{m}\right) & \text{if } x,y \in mY \\ 0 & \text{otherwise} \end{cases} .$$

Since Y is absorbing, $\bar{\lambda}$ is non–negative and separating. Also given $\varepsilon > 0$ and a non–empty subset V of X, there exists an open subset U of X such that $\lambda\text{–diam}\left(\frac{1}{m}U \cap \frac{1}{m}V \cap Y\right) < \frac{\varepsilon}{2}$ (Adopting the convention that the λ–diameter of the empty set is zero.) It follows that $\lambda_m\text{–diam}\,(U \cap V) < \frac{\varepsilon}{2}$ Define U_m inductively by setting $U_0 = V$ and letting U_m be relatively open in U_{m-1}, and hence in V, with $\lambda_m\text{–diam}\,(U_m) < \frac{\varepsilon}{2}$. Then,

$$\bar{\lambda}\text{–diam}\left(U_{m_0}\right) \le \sum_{m=1}^{m_0} \frac{1}{2^m}\frac{\varepsilon}{2} + \sum_{m=m_0+1}^{\infty} \frac{1}{2^m} \le \varepsilon$$

for $m_0 \in \mathbb{N}$ such that $\frac{1}{2^{m_0}} < \frac{\varepsilon}{2}$. Hence $\bar{\lambda}$ is a fragmenting pre–metric for X and by Corollary 2.1 there exists a metric which fragments X. //

3. Fragmenting pre–metrics for rotund Banach spaces

Recall that a Banach space X is said to be *rotund* if for all $x,y \in X$, $\| x \| = \| y \|$, $x \neq y$ we have $\| x+y \| < \| x \| + \| y \|$. Given a non–empty, bounded set C in a Banach space X, a continuous linear functional $f \in X^*$, and an $\varepsilon > 0$, the set

$$S(C, f, \varepsilon) = \{x \in C:\ f(x) > M - \varepsilon\}$$

where $M = \sup\{f(x): x \in C\}$ is called a *slice* of C of *depth* ε.

Theorem 3.1

If X is a Banach space with an equivalent rotund norm, then X is weakly fragmentable.

Proof

Consider X so renormed. It suffices from Lemma 2.2 to exhibit a pre–metric which fragments bounded sets. Consider,

$$\lambda(x,y) = \inf \left\{ \varepsilon > 0 : \exists f \in X^*, \| f \| = 1 \text{ such that } x,y \in S(B_r, f, \varepsilon) \right\}$$

where B_r is the closed ball radius $r \equiv \max \left\{ \| x \|, \| y \| \right\}$.

(i) $\lambda(x,x) = 0$ since there is an $f \in X^*$ supporting B_r at x.

If $x \neq y$ then by rotundity $\| \frac{x+y}{2} \| < r$. Since any slice of B_r containing both x and y must also contain $\frac{x+y}{2}$ we have $\lambda(x,y) \geq r - \| \frac{x+y}{2} \| > 0$.

(ii) Let Y be a non–empty bounded subset of X and let $s = \sup\{ \| x \| : x \in Y\}$, then for any $\varepsilon > 0$ there exists $f \in X^*$, $\| f \| = 1$ such that $U = Y \cap S(B_s, f, \varepsilon)$ is non–empty. U is a weakly relatively open subset of Y. For any $x,y \in U$ let $r \equiv \max \{\| x \|, \| y \|\} \leq s$ and observe that $x,y \in S(B_r, f, \varepsilon + r - s)$, so λ–diam(U) $< \varepsilon$. //

Theorem 3.2

If X is a Banach space which can be equivalently renormed to have a rotund dual norm, then X is weak * fragmentable.*

Proof

Consider X so renormed. Once again by Lemma 2.2 it suffices to exhibit a pre–metric which fragments bounded sets. In an analogous way to Theorem 3.1 we define,

$$\lambda(f,g) = \inf \left\{ \varepsilon > 0 : \exists \hat{x} \in \hat{X}, \| \hat{x} \| = 1 \text{ such that } f,g \in S(B_r^*, \hat{x}, \varepsilon) \right\}$$

where B_r^* is the closed dual norm ball radius $r \equiv \max\{\| f \|, \| g \|\}$.

The details of checking that λ is a fragmenting pre–metric are identical to those in Theorem 3.1 except in one regard. Now $\lambda(f,f) = 0$ since for any $\varepsilon > 0$,

$$f \notin B^*_{\|f\|-\varepsilon} = \bigcap \left\{ g \in X^* : \hat{x}(g) \leq \| f \| - \varepsilon, \hat{x} \in \hat{X}, \| \hat{x} \| = 1 \right\} .$$

So there is an $\hat{x} \in \hat{X}$, $\| \hat{x} \| = 1$ such that $f \in S(B_r^*, \hat{x}, \varepsilon)$. Note that to fulfil this condition it is necessary that the rotund norm on X* be a dual norm. //

Corollary 3.3

If a Banach space X *can be equivalently renormed to have a rotund dual, then* X *is weak Asplund.*

Of course this result is already known via different approaches. For an alternative proof see [7, p.249]. For a proof that weak * fragmentability of the dual implies X is weak Asplund see [3].

A Banach space with a rotund dual norm must itself have a smooth norm, although the dual of a smooth norm need not be rotund. Ribarska [4] has shown that Banach spaces with an equivalent smooth norm have weak * fragmentable dual. Unfortunately in the smooth case the pre–metric of Theorem 3.2 may fail to separate points in the dual, depending as it does on the rotundity of the dual ball, and there is no obvious generalization of this pre–metric. It therefore seems unlikely that this method can be used to simplify Ribarska's proof.

But there is another situation in which this method may be useful. Imagine a Banach space which can not be renormed to be rotund, but has a countable family of equivalent norms, p_n, with the property that given $x,y \in X$, $x \neq y$ there exists $n_0 \in \mathbb{N}$ such that $p_{n_0}(x+y) < p_{n_0}(x) + p_{n_0}(y)$. In such a case define

$$\lambda(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \lambda_n(x,y)$$

where λ_n is defined as in Theorem 3.1 for each norm p_n. Such a λ is a weakly fragmenting pre–metric for the Banach space. This observation is of use when we generalize to the setting of locally convex spaces.

4. Some locally convex spaces which are weakly fragmentable

In this section we extend the concept of a fragmenting pre–metric developed above for Banach spaces to exhibit a class of locally convex spaces which are weakly fragmentable.

Let X be a topological space and C(X) be the continuous real–valued functions on X. The *compact–open* topology on C(X) is the locally convex topology generated by the family of semi–norms of the form

$$p_K(x) = \sup\{x(t) : t \in K\}$$

for compact subsets K.

Theorem 4.1

If X *is a locally compact Hausdorff space with a countable base* $\mathbb{B}$, *and* C(X) *has the compact–open topology, then* C(X) *is weakly fragmentable.*

Proof

For any compact set K in X let $B_r(K) = \{x \in C(X) : p_K(x) \leq r\}$. For any $x,y \in B_r(K)$ define

$$\lambda^1_K(x,y) = \inf\left\{\varepsilon > 0 : \exists f \in C(X)^*,\ p^*_K(f) = 1 \text{ such that } x,y \in S(B_r(K), f, \varepsilon)\right\},$$

where $p^*_K(f) = \sup\{f(x) : p_K(x) \leq 1\}$ and $r = \max\{p_K(x), p_K(y)\}$. Note the analogy to Theorem 3.1. For each positive integer m and $x,y \in C(X)$ define

$$\lambda^m_K(x,y) = \begin{cases} \lambda^1_K\left(\frac{x}{m}, \frac{y}{m}\right) & \text{if } x,y \in B_m(K) \\ 0 & \text{otherwise .} \end{cases}$$

and

$$\lambda_K(x,y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \lambda^m_K(x,y)$$

Let $\{B_n\}$ be the elements of the countable base $\mathbb{B}$ of X, where each $\overline{B}_n$ is compact. Finally let

$$\lambda(x,y) = \sum \frac{1}{2^n} \lambda_{\overline{B}_n}(x,y) = \sum \frac{1}{2^n}\left(\sum \frac{1}{2^m} \lambda^m_{\overline{B}_n}(x,y)\right)$$

We claim that λ is a weakly fragmenting pre–metric for C(X). Note that λ is well–defined since each $\lambda^m_K \leq 1$.

(i) $\lambda(x,x) = 0$ because for any compact K there is a $k \in K$ where $x(k) = p_K(x)$. Then $\delta_k(x) = x(k)$ is in $C(X)^*$, $p^*_K(\delta_k) = 1$ and δ_k supports $B_r(K)$ at x. If $x \neq y$ then there is $k \in K$ such that $x(k) \neq y(k)$, and by continuity of x and y there exists $B \in \mathbb{B}$ such that $p_{\overline{B}}(x) \neq p_{\overline{B}}(y)$, and hence $\lambda_{\overline{B}}(x,y) \neq 0$. Therefore $\lambda(x,y) = 0$ if and only if x=y, so λ is separating.

(ii) To prove λ is fragmenting it suffices to show that any subset Y of $B_1(K)$ has weakly relatively open subsets of arbitrarily small λ^1_K–diameter, since then an inductive procedure similar to that of Lemma 2.2 allows us to achieve a weakly relatively open subset of arbitrarily small λ_K–diameter, and a further application of the procedure results in a weakly relatively open subset of arbitrarily small λ–diameter.

To construct a weakly relatively open set of small λ_K^1–diameter let $s = \sup\{p_K(y): y \in Y\}$, then for any $\varepsilon > 0$ there is a $y \in Y$ such that $p_K(y) > s-\varepsilon$, and hence there is $k \in K$ such that $y(k) > s-\varepsilon$. Then $U = Y \cap S(B_s(K), \delta_k, \varepsilon)$ is a relatively weak open, non–empty subset of Y with λ_K^1 –diameter $< \varepsilon$. //

Example – $C(\mathbb{R})$ with the compact–open topology is weakly fragmentable.

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