NON ISOMORPHISM OF THE DISC ALGEBRA WITH SPACES OF DIFFERENTIABLE FUNCTIONS

A. Pełczyński (Warszawa)

Abstract. It is proved that the Disc Algebra does not contain a complemented subspace isomorphic to the space $C_{(k)}(\mathbb{T}^d)$ of k times continuously differentiable functions on the d-dimensional torus ($k = 1, 2, \ldots; d = 2, 3, \ldots$).

Introduction.

Recall two interesting problems concerning the space $C_{(1)}(\mathbb{T}^2)$ of continuously differentiable functions on the 2-dimensional torus $\mathbb{T}^2$.

(I) Is $C_{(1)}(\mathbb{T}^2)$ isomorphic to a subspace of $C(K)$ (K-compact metric) with a separable annihilator?

(II) Does there exist a 1 - absolutely summing surjection from $C_{(1)}(\mathbb{T}^2)$ onto an infinite dimensional Hilbert space?

The negative answer on each of these questions implies the non-isomorphism of the Disc Algebra A with $C_{(1)}(\mathbb{T}^2)$. In the present paper we prove the latter fact. Precisely our main result (Theorem 2.1) says that the space $C_{(1)}(\mathbb{T}^2)$ is not isomorphic to any complemented subspace of A. The result seems to be interesting because of the method of its proof. We show that the natural embedding of $C_{(1)}(\mathbb{T}^2)$ into the Sobolev space $L^1_{(1)}(\mathbb{T}^2)$ does not factor through the natural embedding of A into H^1_μ for any finite Borel measure μ on the circle.

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1. Preliminaries.

1.1. In this paper we consider only finite non-negative Borel measures on compact spaces.

If μ is a measure then $I_\mu: L^\infty(\mu) \to L^1(\mu)$ denotes the natural embedding. If X is a subspace of $L^\infty(\mu)$ then I_μ^X denotes the restriction of I_μ to X regarded as an operator into the closure of $I_\mu(X)$ in $L^1(\mu)$.

1.2. A stands for the Disc Algebra which we identify with the subspace of $C(\mathbb{T})$ (= the space of complex valued continuous function on the circle $\mathbb{T} = \{z \in \mathbb{C}: |z| = 1\}$) consisting of all the boundary values of uniformly continuous analytic functions in the unit disc of the complex plane $\mathbb{C}$.

If μ is a measure on $\mathbb{T}$ then H_μ^p dene of A in $L^p(\mu)$ for $1 \leq p < \infty$.

By λ we denote the normalized Lebesgue measure on $\mathbb{T}$.

1.3. $C_{(1)}(\mathbb{T}^2)$ denotes the space of continuously differentiable complex-valued functions on the 2-dimensional torus $\mathbb{T}^2$ with the norm

$$\|f\|_{(1),\infty} = \sup_{t \in \mathbb{T}^2} \left(|f(t)|^2 + |D_{t(1)}f(t)|^2 + |D_{t(2)}f(t)|^2\right)^{1/2}.$$

We also consider for $1 \leq p < \infty$ the Sobolev space $L_{(1)}^p(\mathbb{T}^2)$ defined as the completion of $C_{(1)}(\mathbb{T}^2)$ in the norm

$$\|f\|_{(1),p} = \left(\int_{\mathbb{T}^2} (|f(t)|^2 + |D_{t(1)}f(t)|^2 + |D_{t(2)}f(t)|^2)^{p/2}\, dt\right)^{1/p}$$

where the integration is taken against the normalized Haar measure of $\mathbb{T}^2$.

It is convenient to identify the torus $\mathbb{T}^2$ with the square $[\pi, \pi)^2$ and the dual group of the torus group with the integer-valued lattice $\mathbb{Z}^2$ of $\mathbb{R}^2$. To each $n = (n(1), n(2)) \in \mathbb{Z}^2$ we assign the character $t \to \exp\, \mathrm{i}\, (n(1)t(1) + n(2)t(2)) = \exp i\langle t, n\rangle$. We put

$$e_n = Q_{(1)}(n)^{-\frac{1}{2}} \exp i\langle \cdot, n\rangle \quad \text{for } n \in \mathbb{Z}^2$$

where $Q_{(1)}(n) = 1 + \langle n, n\rangle = 1 + [n(1)]^2 + [n(2)]^2$.

Note that $\|e_n\|_{(1),p} = 1$ for $n \in \mathbb{Z}^2$ and for $1 \leq p \leq \infty$. Moreover the system $(e_n)_{n \in \mathbb{Z}^2}$ is an orthonormal basis for the space $L_{(1)}^2(\mathbb{T}^2)$. For $f \in L^1(\mathbb{T}^2)$ and for $n \in \mathbb{Z}^2$ we define the n-th Fourier coefficient by

$$\hat{f}(n) = \int_{\mathbb{T}^2} f(t) \exp(-i\langle t, n\rangle)\, dt.$$

1.4. For $a \in \mathbb{T}^2$ we denote by τ_a the translation operator defined by $\tau_a(f)(t) = f(t+a)$ for every measurable f and for almost every t with respect to the Haar measure of $\mathbb{T}^2$.

Let E and F denote one of the spaces $C_{(1)}(\mathbb{T}^2)$ and $L^p_{(1)}(\mathbb{T}^2)$ and let $W\colon E \to F$ be a linear operator. Recall that W is translation invariant provided $\tau_a W = W\tau_a$ for every $a \in \mathbb{T}^2$; It is well known and easy to check that W is translation invariant iff W is bounded and for each $n \in \mathbb{Z}^2$ there exists a complex number $w(n)$ such that

$$W(e_n) = w(n)e_n. \tag{1.1}$$

For arbitrary linear operator $W\colon E \to F$ we define the translation invariant linear operator W^{av} by

$$W^{av}(f) = \int_{\mathbb{T}^2} \tau_a W \tau_{-a}(f)\, da$$

where the integral is defined in the weak sense.

The following formula will be used in the proof of Step 1 of Theorem 2.1.

LEMMA 1.1. *For every linear operator $W\colon E \to F$ and for every $n \in \mathbb{Z}^2$ one has*

$$[W(e_n)]^\wedge(n) = w^{av}(n) Q^{\frac{1}{2}}_{(1)}(n) \tag{1.2}$$

where $w^{av}(n)$ is defined as in (1.1) by $W^{av}(e_n) = w^{av}(n)e_n$.

Proof. Clearly

$$w^{av}(n) Q^{\frac{1}{2}}_{(1)}(n) = [W^{av}(e_n)]^\wedge(n) \tag{1.3}$$

Put $\int_{\mathbb{T}^2} f\bar{g}\, dt = [f; g]$. Taking into account the identity

$$\tau_{-a}(e_n) = \exp(-i\langle a, n\rangle)e_n$$

we get

$$\begin{aligned}
Q^{-\frac{1}{2}}_{(1)}(n)[W^{av}(e_n)]^\wedge(n) &= [W^{av}(e_n);\, e_n] \\
&= \left[\int_{\mathbb{T}^2} \tau_a W \tau_{-a}(e_n)\, da;\, e_n\right] \\
&= \int_{\mathbb{T}^2} [\tau_a W \tau_{-a}(e_n);\, e_n]\, da \\
&= \int_{\mathbb{T}^2} [W \tau_{-a}(e_n);\, \tau_{-a}(e_n)]\, da \\
&= \int_{\mathbb{T}^2} [W(e_n);\, e_n]\, da \\
&= [W(e_n);\, e_n] \\
&= Q^{-\frac{1}{2}}_{(1)}(n)[W(e_n)]^\wedge(n)
\end{aligned}$$

which in view of $Q^{-\frac{1}{2}}_{(1)}(n) \neq 0$ and (1.3) gives (1.2). $\square$

1.5. We recall a variant of Grothendieck's theory of integral operators [6].

Definition 1.1. Given a linear operator $T: X \to Y$ (X,Y — Banach spaces) we write $T \in TCG(X,Y)$ provided for all Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and all linear operators $\alpha: \mathcal{H}_1 \to X$ and $\beta: Y \to \mathcal{H}_2$ the composition $\beta T \alpha$ is a nuclear operator (cf. [Pi], § 6.3 for definition). For $T \in TCG(X,Y)$ we put

$$tcg(T) = \sup n(\beta T \alpha)$$

where $n(\cdot)$ denotes the nuclear norm and supremum extends over all Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and all linear operators $\alpha: \mathcal{H}_1 \to X$ and $\beta: Y \to \mathcal{H}_2$ with $\|\alpha\| \, \|\beta\| \leq 1$. Here TCG abbreviates "traceclassgenic".

The next proposition collects the facts about traceclassgenic operators which are used in the proof of Theorem 2.1.

PROPOSITION 1.1. (a) (TCG, tcg) *is a Banach ideal in the sense of Pietsch* [Pi].

(b) *Let μ be a measure on a compact space K and let X be a closed linear subspace of $L^\infty(\mu)$ such that the closure of $I_\mu(X)$ in $L^1(\mu)$ coincides with the whole space $L^1(\mu)$. Then $I_\mu^X \in TCG(X, L^1(\mu))$ and $tcg(I_\mu^X) \leq \mu(K)$.*

(c) *Let E and F denote one of the spaces $C_{(1)}(\mathbb{T}^2)$ and $L^p_{(1)}(\mathbb{T}^2)$ for $1 \leq p < \infty$. Let $W \in TCG(E,F)$. Then $W^{av} \in TCG(E,F)$ and $tcg(W^{av}) \leq tcg(W)$.*

(d) *Let $W: C_{(1)}(\mathbb{T}^2) \to L^2_{(1)}(\mathbb{T}^2)$ be translation invariant operator. Then $W \in TCG(C_{(1)}(\mathbb{T}^2), L^2_{(1)}(\mathbb{T}^2))$ iff $\sum |w(n)|^2 < \infty$ where $(w(n))_{n \in \mathbb{Z}^2}$ satisfies with W* (1.1).

Moreover if $W \in TCG(C_{(1)}(\mathbb{T}^2), L^1_{(1)}(\mathbb{T}^2))$ then $tcg(W) = \left(\sum_{n \in \mathbb{Z}^2} |w(n)|^2\right)^{1/2}$.

We omit the routine proof of Proposition 1.1.

Remarks. Ad 1.3. Similarly one defines on the d-dimensional torus $\mathbb{T}^d$ the anisotropic Sobolev spaces $C_S(\mathbb{T}^d)$ and $L^p_S(\mathbb{T}^d)$ where S is an arbitrary smoothness and $1 \leq p \leq \infty$, cf. e.g. [P–W2].

Ad 1.4. The concepts discussed here can be extended to arbitrary vector valued translation invariant function spaces on compact (abelian) groups. Most of the folklore material can be found in the books [R] and [G–McG]. Also Proposition 1.1 (c) and (d) generalizes to this framework (cf. e.g. [K–P2] section 1.5; for our purpose one can adopt the proof of [K–P2], Proposition 1.1).

Ad 1.5. S. Kwapień has observed that the ideal (TCG, tcg) is in the trace duality (adjoint in the terminology of [Pi], chapt. 7) with the ideal of operators factorable through a Hilbert space.

2. Proof of the main result.

THEOREM 2.1. *The space $C_{(1)}(\mathbb{T}^2)$ is not isomorphic to a complemented subspace of the disc algebra.*

The proof of this result bases upon the following fact due to M. Wojciechowski (cf. [W], [P–W1]).

(W) *No infinite set of characters of $\mathbb{T}^2$ spans a complemented subspace of $L^1_{(1)}(\mathbb{T}^2)$ isomorphic to a Hilbert space.*

We also need the following

LEMMA 2.1. *Let $\mu = h\lambda + \nu$ be the Lebesgue decomposition of μ and let $\int_{\mathbb{T}} \log h\, d\lambda > -\infty$. Let a sequence $(f_k) \subset A$ satisfy*

$$\sup_k \sup_{t\in\mathbb{T}} |f_k(t)| = M < \infty; \tag{2.1}$$

$$\inf_{k\neq l} \int_{\mathbb{T}} |f_k - f_l| h\, d\lambda = c > 0. \tag{2.2}$$

Let f belong to the (obviously non-empty) set of limit points in the weak topology of $L^2(\mu)$ of the set $\{f_1, f_2, \dots\}$. Then $f \in H^2_\mu \cap L^\infty(\mu)$, there exist a projection $K: H^1_\mu \to H^1_\mu$ with $\ker K = \{zf : z \in \mathbb{C}\}$ and strictly increasing sequence of indices, say $(k(m))$, such that if $g_m = K(f_{k(m)})$ for $m = 1, 2, \dots$ then the sequence (g_m) considered in $H^1_{h\lambda}$ is equivalent to the standard unit vector basis of l^2 and there exists a projection $P: H^1_{h\lambda} \to H^1_{h\lambda}$ whose range is the closed linear subspace of $H^1_{h\lambda}$ generated by the set $\{g_1, g_2, \dots\}$.

Proof. By (2.1) the set $\{f_1, f_2, \dots\}$ is contained in the ball $\{g \in L^2(\mu): \|g\|_{L^2(\mu)} \le M\}$ which is weakly compact. Thus the set $\{f_1, f_2, \dots\}$ has limit points in the weak topology of $L^2(\mu)$. If f is a limit point of $\{f_1, f_2, \dots\}$ in the weak topology of $L^2(\mu)$ then by Mazur's theorem there is a sequence, say (φ_s), of finite convex linear combinations of the f_k's which tends to f in the norm topology of $L^2(\mu)$. Therefore a subsequence of the sequence (φ_s) tends to f μ-almost everywhere. By (2.1), $\sup_{t\in\mathbb{T}} |\varphi_s(t)| \le M$ for $s = 1, 2, \dots$ Thus $f \in L^\infty(\mu)$. Since $\varphi_s \in A$ for $s = 1, 2, \dots$ we infer that $f \in H^2_\mu$. Thus $f \in H^2_\mu \cap L^\infty(\mu)$. We define $K: H^1_\mu \to H^1_\mu$ as follows:

$$\text{if } f = 0 \text{ then } K(g) = g \text{ for } g \in H^1_\mu;$$

$$\text{if } f \neq 0 \text{ then } K(g) = g - \int_{\mathbb{T}} g\bar{f}\, d\mu \cdot f \left(\int_{\mathbb{T}} |f|^2\, d\mu \right)^{-1} \text{ for } g \in H^1_\mu.$$

Note that the relation $f \in H^2_\mu \cap L^\infty(\mu)$ implies that K is a bounded operator whose restriction to H^2_μ is also a bounded operator from H^2_μ into H^2_μ. Since f is a weak limit point of the set $\{f_1, f_2, \dots\}$, there exists a strictly increasing sequence of the indices, say $(k'''(m))$ such that the sequence $(f_{k'''(m)} - f)$ tends weakly to zero in $L^2(\mu)$, hence in $L^1(\mu)$ too as $m \to \infty$. Thus the sequence $(K(f_{k'''(m)}))$ tends weakly to zero in $L^2(\mu)$

and in $L^1(\mu)$ because $f \in \ker K$. Therefore using the "gliding hump" procedure one can define a subsequence $(k''(m))$ of the sequence $(k'''(m))$ so that the sequence $(K(f_{k''(m)}))$ consists of mutually "almost orthogonal" elements; in particular for arbitrary eventually zero sequence of scalars (c_m) we have

$$\Big\|\sum c_m K(f_{k''(m)})\Big\|_{L^2(\mu)} \leq 2\sup_m \|K(f_{k''(m)})\|_{L^2(\mu)}\Big(\sum_m |c_m|^2\Big)^{\frac{1}{2}}$$
$$\leq 2M\mu(\mathbb{T})^{\frac{1}{2}}\Big(\sum_m |c_m|^2\Big)^{\frac{1}{2}}.$$

Thus

$$\Big(\int_{\mathbb{T}}\Big|\sum_m c_m K(f_{k''(m)})\Big|^2 h\,d\lambda\Big)^{\frac{1}{2}} \leq 2M[\mu(\mathbb{T})]^{\frac{1}{2}}\Big(\sum_m |c_m|^2\Big)^{\frac{1}{2}}. \tag{2.3}$$

On the other hand note that $\lim_m \int_{\mathbb{T}} f_{k''(m)}\bar{f}h\,d\lambda = \int_{\mathbb{T}} |f|^2 h\,d\lambda$. Thus using (2.2) we can choose the subsequence $(k''(m))$ so that additionally $\int_{\mathbb{T}} |K(f_{k''(m)}) - K(f_{k''(r)})|h\,d\lambda > c/2$ whenever $m \neq r$. Hence without loss of generality we can also assume that for $1 \leq p \leq 2$

$$\Big(\int_{\mathbb{T}} |K(f_{k''(m)})|^p h\,d\lambda\Big)^{\frac{1}{p}} \geq c_* \quad \text{for } m = 1, 2, \ldots$$

where $c_* = \frac{c}{4}\big(\int_{\mathbb{T}} h\,d\lambda\big)^{\frac{1}{2}}$. Now taking into account that for $1 < p \leq 2$ the space $L^p(h\lambda)$ has an unconditional basis and cotype 2 the block basis technique of [BP] enables us to pick an infinite subsequence $k'(m)$ of $k''(m)$ so that for each p with $1 < p \leq 2$ there exists a constant $C(p) > 0$ such that for every eventually zero sequence of scalars (c_m).

$$\Big(\int_{\mathbb{T}}\Big|\sum c_m K(f_{k'(m)})\Big|^p h\,d\lambda\Big)^{\frac{1}{p}} \geq C(p)\Big(\sum |c_m|^2\Big)^{\frac{1}{2}} \tag{2.4}$$

Using (2.4) for $p = \frac{3}{2}$ and applying (2.3) together with the Cauchy-Schwarz inequality we get

$$\Big[C\Big(\frac{3}{2}\Big)\Big(\sum |c_m|^2\Big)^{\frac{1}{2}}\Big]^{\frac{3}{2}} \leq \int_{\mathbb{T}}\Big|\sum_m c_m K(f_{k'(m)})\Big|^{\frac{3}{2}} h\,d\lambda$$
$$\leq \Big[\int_{\mathbb{T}}\Big|\sum_m c_m K(f_{k'(m)})\Big|\,d\lambda\Big]^{\frac{1}{2}}\Big[\int_{\mathbb{T}}\Big|\sum_m c_m K(f_{k'(m)})\Big|^2 h\,d\lambda\Big]^{\frac{1}{2}}$$
$$\leq \Big[\int_{\mathbb{T}}\Big|\sum_m c_m K(f_{k'(m)})\Big|h\,d\lambda\Big]^{\frac{1}{2}} \cdot 2M[\mu(T)]^{\frac{1}{2}}\Big(\sum_m |c_m|^2\Big)^{\frac{1}{2}}$$

Thus there exist positive constants C_1 and C_2 such that for arbitrary eventually zero sequence of scalars (c_m) one has

$$C_1\Big(\sum |c_m|^2\Big)^{\frac{1}{2}} \le \int_{\mathbb{T}} \Big|\sum_m c_m K(f_{k'(m)})\Big| h\, d\lambda \le C_2\Big(\sum_m |c_m|^2\Big)^{\frac{1}{2}}. \tag{2.5}$$

The condition $\int_{\mathbb{T}} \log h\, d\lambda > -\infty$ implies that the existence of an outer function $F \in H^1 = H^1_\lambda$ with $|F| = |h|$ λ - a.e. ([H], p. 53 and p. 62). Thus $H^1_{h\lambda}$ is isometrically isomorphic to H^1 (The map $f \to f$. F defines the desired isometric isomorphism). The latter fact combined with (2.5) allows to apply [K–P1], Theorem 3.8 to extract a subsequence $(k(m))$ from the sequence $(k'(m))$ which satisfies the assertion of the Lemma. □

Remark. Alternatively one can prove the second part of the Lemma using the fact that H^1 has an unconditional basis ([Ma], [C]) and that in a Banach space of cotype 2 with an unconditional basis every infinite sequence separated from zero and satisfying the upper l^2-estimates contains an infinite subsequence equivalent to the unit vector basis of l^2 and generating a subspace complemented in the whole space.

Proof of Theorem 2.1. Introductory part. Assume to the contrary that there exist linear operators $V\colon C_{(1)}(\mathbb{T}^2) \to A$ and $U\colon A \to C_{(1)}(\mathbb{T}^2)$ with $UV =$ the identity on $C_{(1)}(\mathbb{T}^2)$. It is well known (cf. e.g. [Ki] and [P–S]) that the natural embedding $J\colon C_{(1)}(\mathbb{T}^2) \to L^1_{(1)}(\mathbb{T}^2)$ is 1-absolutely summing. Thus so is $JU\colon A \to L^1_{(1)}(\mathbb{T}^2)$. Therefore by the Pietsch Factorization Theorem (cf. e.g. [Wt], III.F.8.) there exists a measure μ on $\mathbb{T}$ and a linear operator $B\colon H^1_\mu \to L^1_\mu(\mathbb{T}^2)$ with $\|B\| \le 1$ such that $JU = I^A_\mu B$. Let $\mu = h\lambda + \nu$ be the Lebesgue decomposition of μ. Increasing if necessary the Pietsch measure μ one can assume without loss of generality that $h \ge 1$ λ - a.e. on $\mathbb{T}$, hence $\int_{\mathbb{T}} \log h\, d\lambda > -\infty$. It follows from the Rudin–Carleson theorem that H^1_μ decomposes into the direct sum, $H^1_\mu = H^1_{h\lambda} \oplus L^1(\nu)$ (cf. [M-P] and [P], § 2 for details). Let $P_1\colon H^1_\mu \to H^1_{h\lambda}$ and $P_2\colon H^1_\mu \to L'(\nu)$ denote the natural projections. We identify here $H^1_{h\lambda}$ with $H^1_{h\lambda} \oplus \{0\}$ and $L^1(\nu) = \{0\} \oplus L^1(\nu)$. In fact we are considering the following commutative diagram

$$\begin{array}{ccccccccc}
C_{(1)}(\mathbb{T}^2) & \xrightarrow{V} & A & \xrightarrow{U} & C_{(1)}(\mathbb{T}^2) & \xrightarrow{J} & L^1_{(1)}(\mathbb{T}^2) & \xrightarrow{\Lambda} & L^2_{(1)}(\mathbb{T}^2) \\
 & & \downarrow{\scriptstyle I^A_\mu} & & & & \uparrow{\scriptstyle B} & & \\
 & & H^1_\mu & & \xrightarrow{\mathrm{Id}=P_1\oplus P_2} & & H^1_{h\lambda} \oplus L'(\nu) & &
\end{array}$$

where the Sobolev embedding Λ will be defined in Step 1.

In the sequel, $f_n = V(e_n)$, for $n \in \mathbb{Z}^2$.

Step 1. *There exists an infinite sequence $(n_k) \subset \mathbb{Z}^2$ such that*

$$\int_{\mathbb{T}} |f_{n_k} - f_{n_l}| h\, d\lambda > 4^{-1} \quad \textit{whenever} \quad k \ne l.$$

Proof. For $\varrho = 1,2$ we put

$$W_\varrho = BP_\varrho I_\mu^A V, \quad W_\varrho^{av} = \int_{\mathbb{T}^2} \tau_a W_\varrho \tau_{-a}\, da$$

and we define for $n \in \mathbb{Z}^2$ the complex numbers $w_\varrho(n)$ by

$$W_\varrho^{av}(e_n) = w_\varrho(n) e_n.$$

Clearly $J = W_1^{av} + W_2^{av}$ because $\tau_a J \tau_{-a} = J$ for every $a \in \mathbb{T}^2$. Thus

$$w_1(n) + w_2(n) = 1 \quad \text{for} \quad n \in \mathbb{Z}^2.$$

Define $\Lambda: L^1_{(1)}(\mathbb{T}^2) \to L^2_{(1)}(\mathbb{T}^2)$ by

$$\Lambda(\varphi) = \sum_{n \in \mathbb{Z}^2} \hat{\varphi}(n) e_n;$$

in particular $\Lambda(e_n) = Q_{(n)}^{-\frac{1}{2}}(n) e_n$ for $n \in \mathbb{Z}^2$.

By the Sobolev embedding theorem (cf. [S], chapt. V, § 2.5) there exists $c > 0$ such that for every $\varphi \in L^1_{(1)}(\mathbb{T}^2)$ one has

$$\begin{aligned} \|\Lambda(\varphi)\|_{(1),2} &= \Big(\sum_{n \in \mathbb{Z}^2} |\Lambda(\varphi)^\wedge(n)|^2 Q_{(1)}(n) \Big)^{\frac{1}{2}} \\ &= \Big(\sum_{n \in \mathbb{Z}^2} |\hat{\varphi}(n)|^2 \Big)^{\frac{1}{2}} \\ &= \|\varphi\|_2 \\ &\le c \|\varphi\|_{(1),1}. \end{aligned}$$

Thus Λ is a well defined bounded translation invariant linear operator with $\|\Lambda\| \le c$.

By Proposition 1.1 (a) and (b) we get

$$tcg(W_2) \le \|V\| \, \|B\| tcg(I_\nu^A) \le \|V\| \nu(\mathbb{T}).$$

Now, by Proposition 1.1 (c) and (d), we obtain

$$\begin{aligned} \sum_{n \in \mathbb{Z}^2} |1 - w_1(n)|^2 Q_{(1)}^{-1}(n) &= \sum_{n \in \mathbb{Z}^2} |w_2(n)|^2 Q_{(1)}^{-1}(n) \\ &\le [tcg(\Lambda W_2^{av})]^2 \\ &\le [c\|V\| \nu(T)]^2 \\ &< \infty. \end{aligned}$$

Thus the set $\left\{n \in \mathbb{Z}^2 \colon |1 - w_1(n)| < 1 - \frac{\sqrt{3}}{2}\right\}$ is infinite because $\sum Q_{(1)}^{-1}(n) = \infty$. Hence the set

$$\Omega = \left\{n \in \mathbb{Z}^2 \colon |w_1(n)| > \frac{\sqrt{3}}{2}\right\}$$

is also infinite. By (1.2) for $n \in \Omega$ one has

$$\frac{\sqrt{3}}{2} < |w_1(n)| = \left|W_1(e_n)^\wedge(n) Q_{(1)}^{\frac{1}{2}}(n)\right|. \tag{2.6}$$

Next recall that if $\varphi \in L^1_{(1)}(\mathbb{T}^2)$ then

$$\left\{n \in \mathbb{Z}^2 \colon |\hat{\varphi}(n) Q_{(1)}^{\frac{1}{2}}(n)| > \varepsilon\right\} \quad \text{is finite for every } \varepsilon > 0. \tag{2.7}$$

$$\left|\hat{\varphi}(n) Q_{(1)}^{\frac{1}{2}}(n)\right| \leq \sqrt{3}\|\varphi\|_{(1),1}. \tag{2.8}$$

Using (2.7) we define inductively a sequence (n_k) in Ω so that if $k < l$ then

$$\left|W_1(e_{n_k})^\wedge(n_l) Q_{(1)}^{\frac{1}{2}}(n_l)\right| < \frac{\sqrt{3}}{4}.$$

Thus combining (2.6) with (2.8) we infer that $k < l$ implies

$$\frac{\sqrt{3}}{4} \leq \left|W_1(e_{n_k} - e_{n_l})^\wedge(n_l) Q_1^{\frac{1}{2}}(n_l)\right| \leq \sqrt{3}\|W_1(e_{n_k} - e_{n_l})\|_{(1),1}$$

Taking into account that $W_1(e_{n_k} - e_{n_l}) = BP_1 I_\mu^A(f_{n_k} - f_{n_l})$ and that $\|B\| \leq 1$ and $\|P_1 I_\mu^A(f_{n_k} - f_{n_l})\| = \int_{\mathbb{T}} |f_{n_k} - f_{n_l}| h\, d\lambda$ we get that $k < l$ implies $\int_{\mathbb{T}} |f_{n_k} - f_{n_l}| h\, d\lambda \geq \frac{1}{4}$. □

Step 2. Step 1 leads to a contradiction with (W).

Proof. Let $(k(m))$, K and P satisfy the assertion of Lemma 2.1 with (f_k) replaced by (f_{n_k}). Let G denote the subspace of $H^1_{h\lambda}$ generated by the set $\{g_m \colon m = 1, 2, \ldots\}$ where $g_m = K(f_{n_{k(m)}})$. Define the operator $E \colon G \to L^2_{(1)}(\mathbb{T}^2)$ by

$$E\left(\sum_m c_m g_m\right) = \sum_m c_m e_{n_{k(m)}} \qquad \text{for } (c_m) \in \ell^2.$$

Since the sequence (g_m) is equivalent to the unit vector basis of ℓ^2, there exists $C_1 > 0$ such that

$$\left\|E\left(\sum_m c_m g_m\right)\right\|_{(1),2} = \left(\sum_m |c_m|^2\right)^{\frac{1}{2}} \leq C_1^{-1}\left\|\sum_m c_m g_m\right\|_{L^1(h\lambda)}.$$

Hence E is a well defined bounded linear operator. Now let us consider the operator $\mathcal{J}_G$: $C_{(1)}(\mathbb{T}^2) \to L^2_{(1)}(\mathbb{T}^2)$ defined by $\mathcal{J}_G = EPP_1KI^A_\mu V$. Clearly $\mathcal{J}_G$ is 1-absolutely summing because I^A_μ is so. A direct computation shows that $\mathcal{J}_G(e_n) = a(n)e_n$ for $n \in \mathbb{Z}^2$ where $a(n) = 1$ for $n = n_{k(m)}$ $(m = 1, 2, \dots)$ and $a(n) = 0$ otherwise. Thus $\mathcal{J}_G$ is a translation invariant operator. Combining the Pietsch Factorization Theorem with the averaging technique (cf. [P–W 2], Corollary 3.1 for details) we infer that the Haar measure of $\mathbb{T}^2$ is the Pietsch measure for $\mathcal{J}_G$, precisely there is $C > 0$ such that

$$\|\mathcal{J}_G(f)\|_{(1),2} \leq C\,\|f\|_{(1),1} \qquad \text{for } f \in C_{(1)}(\mathbb{T}^2).$$

Since $C_{(1)}(\mathbb{T}^2)$ regarded as a subset of $L^1_{(1)}(\mathbb{T}^2)$ is norm dense in $L^1_{(1)}(\mathbb{T}^2)$, the latter inequality implies that $\mathcal{J}_G$ uniquely extends to a translation invariant projection on $L^1_{(1)}(\mathbb{T}^2)$ denoted also by $\mathcal{J}_G$. Obviously the range of $\mathcal{J}_G$ is infinite. Thus $\mathcal{J}_G$ is a Paley projection, i.e. a translation invariant projection on $L^1_{(1)}(\mathbb{T}^2)$ whose range is isomorphic to an infinite dimensional Hilbert space (cf. [P–W1], Proposition 0.1). This contradicts (W). □

3. Generalizing to $C_S(\mathbb{T}^d)$.

Recall that a smoothness is a finite non-empty subset of partial derivatives in d-variables identified with a subset S of points with non-negative coordinates of the integer-valued lattice $\mathbb{Z}^d$ provided $a \in S$ and $b \in \mathbb{Z}^d$ with $0 \leq b(j) \leq a(j)$ for $j = 1, 2, \dots, d$ implies $b \in S$.

An $a \in S$ is called maximal if the condition $c \in S$ and $c(j) \geq a(j)$ for $j = 1, 2, \dots, d$ implies $c = a$.

By $C_S(\mathbb{T}^d)$ we denote the space of functions f: $\mathbb{T}^d \to \mathbb{C}$ having continuous derivatives $D^a f$ for $a \in S$.

First observe that if S has one maximal element then, by a result of [Si] and [P–S], $C_S(\mathbb{T}^d)$ is isomorphic to $C(\mathbb{T}^d)$ hence by Milutin's Theorem (cf. [Wt], III. D.19) and a linear extension version of the Rudin–Carleson Theorem (cf. [Wt], III.E.3) $C_S(\mathbb{T}^d)$ is isomorphic to a complemented subspace of A.

In the sequel we shall always assume that S satisfies:

(i) there is more than one maximal element in S.

Clearly (i) implies that $S \subset \mathbb{Z}^d$ with $d \geq 2$. We begin with the case $d = 2$. Then we have:

THEOREM 3.1. *Assume that a smoothness $S \subset \mathbb{Z}^2$ satisfies* (i) *and*

(ii) *if a and b are maximal elements in S then $a(1) + a(2) - b(1) - b(2)$ is an even number. Then $C_S(\mathbb{T}^2)$ is not isomorphic to a complemented subspace of A.*

The proof of Theorem 3.1 is a slight modification of the proof of Theorem 2.1. The condition (ii) implies that the Sobolev space $L^1_S(\mathbb{T}^2)$ satisfies the assertion of (W) (cf. [P–W1], Corollary 3.1). Next we put $e_n = Q_S^{-\frac{1}{2}}(n)\exp(i\langle\cdot, n\rangle)$ where

$$Q_S(n) = \sum_{a \in S} n(1)^{2a(1)} n(2)^{2a(2)} \qquad \text{for } n \in \mathbb{Z}^2.$$

We modify the argument of Step I replacing the classical Sobolev Embedding by a result of [So] and [P–S] (cf. [P–S], Theorem 4.2, Lemma 0.1 and the proof of Lemma 5.1) which for our purpose can be restated as follows:

If a smoothness $S \subset \mathbb{Z}^2$ satisfy (i) then

(iii) there are distinct a and b in S such that: the line passing through a and b supports the convex hull of S in $\mathbb{R}^2$ and is not parallel to any axis of $\mathbb{R}^2$;

$$\sum_{n \in \mathbb{Z}^2} |n(1)|^{a(1)+b(1)-1} \, |n(2)|^{a(2)+b(2)-1} \, Q_S^{-1}(n) = \infty; \tag{3.1}$$

the operator Λ_S: $L^1_S(\mathbb{T}^2) \to L^2_S(\mathbb{T}^2)$ defined by

$$\Lambda_S(e_n) = \left(|n(1)|^{a(1)+b(1)-1} \, |n(2)|^{a(2)+b(2)-1} \, Q_S^{-1}(n) \right)^{\frac{1}{2}} e_n \qquad \text{for } \; n \in \mathbb{Z}^2$$

is bounded. □

In fact analyzing the proof of [P–W1], Proposition 2.1 and using the fact that a part of the series (3.1) consisting of terms indexed by the lattice point belonging to an arbitrary small angle around the direction perpendicular to the line passing through a and b diverges to infinity one can prove

THEOREM 3.1a. *If there are a and b in a smoothness $S \subset \mathbb{Z}^2$ which satisfy* (iii) *and such that $a(1) + a(2) - b(1) - b(2)$ is an even number then $C_S(\mathbb{T}^2)$ is not isomorphic to a complemented subspace of A.*

It is plausible that already the condition (i) itself implies the assertion of Theorem 3.1. The simplest case for which we are unable to verify this conjecture is the smoothness in $\mathbb{Z}^2$ generated by the "pure" derivatives D_{xx} and D_y.

Finally we consider smoothnesses in $\mathbb{Z}^d$ for $d \geq 3$.

By an observation due to Kislyakov and Sidorenko cf. [Ki–Si], § 3) for every smoothness $S \subset \mathbb{Z}^d$ ($d \geq 3$) the space $C_S(\mathbb{T}^d)$ contains a complemented subspace isomorphic to $C_{S'}(\mathbb{T}^2)$ for every smoothness $S' \subset \mathbb{Z}^2$ which is one of the forms: either

$$S' = \left\{ (a(p), a(q)) \in \mathbb{Z}^2 \colon \; a \in S; \; p, q \text{ fixed with } 1 \leq p < q \leq d \right\},$$

or

$$S' = \left\{ \Big(\sum_{j \in C} a(j), \sum_{j \neq C} a(j) \Big) \in \mathbb{Z}^2 \colon \; a \in S; \; C \text{ fixed proper subset of } \{1, 2, \dots, d\} \right\}.$$

Thus if one of the smoothnesses S' satisfies the assumption of Theorem 3.1 (or Theorem 3.1a) then $C_S(\mathbb{T}^d)$ is not isomorphic to a complemented subspace of A. In particular using the terminology of [P–W1] we have

COROLLARY 3.1. *Let $S \subset \mathbb{Z}^d$ $(d \geq 2)$ be a smoothness which satisfies* (i). *Assume either* (a) *there is no Paley projection on $L^1_S(\mathbb{T}^d)$, or* (b) *the fundamental polynomial Q_S is elliptic where*

$$Q_S(\xi) = \sum_{a \in S} \prod_{j=1}^{d} |\xi(j)|^{2a(j)} \qquad (\xi = (\xi(j)) \in \mathbb{R}^d),$$

or (c)

$$S = (k) = \left\{ a = (a(j)) \in \mathbb{Z}^d \colon \sum_{j=1}^{d} a(j) \leq k; a(j) \geq 0 \right\}$$

is the classical smoothness of all derivatives of order $\leq k$ in d variables.

Then $C_S(\mathbb{T}^d)$ is not isomorphic to a complemented subspace of A.

Note that (c)⇒(b)⇒(a) (cf. [P–W1], section 3) and that the case (a) of the Corollary easily reduces by the procedure described above to Theorem 3.1.

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INSTITUTE OF MATHEMATICS,
POLISH ACADEMY OF SCIENCES
ŚNIADECKICH 8, IP.
00–950 WARSZAWA,
POLAND