SEPARATION OF GRADIENT YOUNG MEASURES AND THE BMO

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Dedicated to Professor Alan McIntosh for his 60th birthday

Abstract. Let $K = \{A, B\} \subset M^{N\times n}$ with $\mathrm{rank}(A - B) > 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded arcwise connected Lipschitz domain. We show that there is a direct estimate of the size of the ϵ-neighborhood K_ϵ of K such that $K_\epsilon = \bar{B}_\epsilon(A) \cup \bar{B}_\epsilon(B)$ separates gradient Young measures, that is, if $(u_j) \subset W^{1,1}(\Omega, \mathbb{R}^N)$ is bounded and $\int_\Omega \mathrm{dist}(Du_j, K_\epsilon)dx \to 0$ as $j \to \infty$, then up to a subsequence, either $\int_\Omega \mathrm{dist}(Du_j, \bar{B}_\epsilon(A))dx \to 0$ or $\int_\Omega \mathrm{dist}(Du_j, \bar{B}_\epsilon(B))dx \to 0$.

In this note I present some direct estimate on neighborhoods K_ϵ of a two matrix set $K = \{A, B\}$ that separate gradient Young measures. I also show how one can use the BMO seminorm of approximate solutions of linear elliptic systems to control the oscillation of sequences gradients approaching K_ϵ. The note is based on a recent work [19] with a slightly different approach. In the case of the two matrix set, we can establish our main result (Theorem 1 below) without quoting a recent deep approximation theorem obtained by Müller [14], instead, most of the tools we use will be standard in the calculus of variations. The problem we consider is motivated from the variational approach of material microstructure [4, 5], in particular, the study of metastablility, hysteresis and numerical analysis related to them.

Let $M^{N\times n}$ be the space of $N \times n$ real matrices with $N, n \geq 2$. Let Ω be a bounded arcwise connected Lipschitz domain throughout this note. We denote by $\rightharpoonup$ and $\stackrel{*}{\rightharpoonup}$ weak convergence and weak-$*$ convergence respectively. The characteristic function of a set $V \subset \mathbb{R}^n$ is denoted by χ_V. The following result is now well-known [4, 17].

Theorem A. *Let $A, B \in M^{N\times n}$ with $\mathrm{rank}(A - B) > 1$. Let $(u_j) \subset W^{1,p}(\Omega, \mathbb{R}^N)$ $(1 \leq p < \infty)$ be a bounded sequence such that $\lim_{j\to\infty} \int_\Omega \mathrm{dist}^p(Du_j, \{A, B\}) \to 0$. Then, up to a subsequence either $Du_j \to A$ a.e. or $Du_j \to B$ a.e..*

In the study of metastablility and hysteresis of material microstructure, Ball and James [6] established the following:

Theorem B. *If a compact set $K = K_1 \cup K_2 \subset M^{N\times n}$ $(K_1 \cap K_2 = \emptyset)$ separates gradient Young measures in the sense that* $\operatorname{supp}\nu_x \subset K$ *a.e.* $\Rightarrow \operatorname{supp}\nu_x \subset K_1$ *or* $\operatorname{supp}\nu_x \subset K_2$ *a.e. Then there exists $\epsilon > 0$, such that K_ϵ still separates gradient Young measures.*

In other words, K separates gradient Young measures if $\operatorname{dist}(Du_j, K) \to 0$ in $L^1(\Omega)$ implies, up to a subsequence either $\operatorname{dist}(Du_j, K_1) \to 0$ in $L^1(\Omega)$ or $\operatorname{dist}(Du_j, K_2) \to 0$ in $L^1(\Omega)$. Theorem B then claims that there exists $\epsilon > 0$ such that K_ϵ still separates gradient Young measures.

Theorem B was established by using a contradiction argument in [6]. In this note I give an estimate of $\epsilon > 0$ for the special case $K = \{A, B\}$ with $\operatorname{rank}(A-B) > 1$ such that $K_\epsilon = \bar{B}_\epsilon(A) \cup \bar{B}_\epsilon(B)$ separates gradient Young measures, i.e. we give an estimate of the size of the balls such that a sequence of gradients approaching two balls can only approach one. We have,

Theorem 1. *Let $K = \{A, B\} \subset M^{N\times n}$ with $\operatorname{rank}(A - B) > 1$. Let $K_\epsilon = \bar{B}_\epsilon(A) \cup \bar{B}_\epsilon(B)$, and $\lambda_{\max}$ be the largest eigenvalue of $(A - B)^t(A - B)/|A - B|^2$. Then there exists an estimate of $\epsilon > 0$ depending on n, N, $|A - B|$, $\lambda_{\max}$ such that $u_j \rightharpoonup u$ in $W^{1,1}(\Omega, \mathbb{R}^N)$, $\int_\Omega \operatorname{dist}(Du_j, K_\epsilon)dx \to 0$ as $j \to \infty$ implies that up to a subsequence, either*

$$\lim_{j\to\infty} \int_\Omega \operatorname{dist}(Du_j, \bar{B}_\epsilon(A))dx = 0, \quad \text{or} \quad \lim_{j\to\infty} \int_\Omega \operatorname{dist}(Du_j, \bar{B}_\epsilon(B))dx = 0.$$

In the language of gradient Young measures [11, 15], this means $\operatorname{supp}\nu_x \subset K_\epsilon$, a.e. $x \in \Omega$ implies either $\operatorname{supp}\nu_x \subset \bar{B}_\epsilon(A)$ a.e. or $\operatorname{supp}\nu_x \subset \bar{B}_\epsilon(B)$ a.e.

We use the following standard tools to establish Theorem 1.

(i) A non-negative quasiconvex function vanishing exactly on K_ϵ [10, 17], (ii) homogeneous Young measures [15], (iii) BMO estimates for elliptic systems [9], (iv) Besicovitch's covering lemma [8].

More precisely, we have an explicit non-negative quasiconvex function f satisfying

$$0 \le f(X) \le C(1 + |X|^2), \qquad f^{-1}(0) = K_\epsilon.$$

So if $\int_\Omega \operatorname{dist}^2(Du_j, K_\epsilon)dx \to 0$ and $u_j \rightharpoonup u$ in $W^{1,2}$, then by the well-known weak lower semicontinuity theorem of Acerbi and Fusco [1],

$$0 = \liminf_{j\to\infty} \int_\Omega f(Du_j)dx \ge \int_\Omega f(Du)dx.$$

In the language of gradient Young measures, this gives

$$\int_\Omega \int_{M^{N\times n}} f(\lambda) d\nu_x(\lambda) dx = \int_\Omega f(\bar\nu_x) dx, \quad \text{where } \bar\nu_x = \int_{M^{N\times n}} \lambda d\nu_x = Du(x),$$

hence $\operatorname{supp} \nu_x \subset K_\epsilon$, and more importantly, we can **locate** the weak limit

$$\bar\nu_x = Du(x) \in K_\epsilon \text{ a.e.} \tag{1}$$

Recall that a continuous function $f : M^{N\times n} \to \mathbb{R}$ is quasiconvex [13, 3] if

$$\int_\Omega f(A + D\phi) \geq f(A)|\Omega|, \quad \text{for } A \in M^{N\times n},\ \phi \in C_0^1(\Omega, \mathbb{R}^N).$$

It is well known that if $0 \leq f(X) \leq C(1+|X|^p)$, the variational integral $u \to \int_\Omega f(Du)dx$ is sequentially weakly lower semicontinuous in $W^{1,p}$ if and only if f is quasiconvex [1].

We may construct quasiconvex functions by calculating the quasiconvex envelope QF for a given function $F : M^{N\times n} \to \mathbb{R}$: $QF = \sup\{g \leq F,\ g \text{ quasiconvex}\}$. In our case, we let $F(X) = \operatorname{dist}^2(X, \{A, B\})$, then QF can be explicitly calculated [10] and

$$QF(X) = F(X) = \operatorname{dist}^2(X, \{A, B\}), \qquad \text{if} \quad \operatorname{dist}^2(X, \{A, B\}) \leq |A - B|^2(1 - \lambda_{\max})/4.$$

Let

$$f(X) = \left[Q \operatorname{dist}^2(X, \{A, B\}) - \epsilon\right]_+, \qquad \epsilon \leq \frac{|A-B|^2(1-\lambda_{\max})}{4},$$

then $f \geq 0$ is quasiconvex and $f^{-1}(0) = K_\epsilon$.

Next need the $W^{1,\infty}$-Gradient Young measure and the homogeneous Young measure [11, 15] to **localize** our problem: We only need a special case of the general theorem of gradient Young measures. Let $K \subset M^{N\times n}$ be compact, $u_j \rightharpoonup u$ in $W^{1,1}(\Omega, \mathbb{R}^N)$ such that $\operatorname{dist}(Du_j, K) \to 0$ in L^1. Then

(I) up to a subsequence, there exists a family of probability measures (gradient Young measures) $\operatorname{supp} \nu_x \subset K$, a.e. $x \in \Omega$, $f(Du_j) \rightharpoonup \int_K f(\lambda) d\nu_x$ weakly in L^1, for all continuous functions f satisfying $|f(X)| \leq C(|X|+1)$ and the weak limit can be identified as $Du(x) = \int_K \lambda d\nu_x := \bar\nu_x$ a.e. $(\nu_x)_{x\in\Omega}$ is called the family of $W^{1,\infty}$-gradient Young measures generated by (a subsequence) of gradients (Du_j).

(II) There is a bounded sequence $v_j \in W^{1,\infty}$ such that (Dv_j) generates the same family of Young measures (ν_x), $\|Dv_j - Du_j\|_{L^1} \to 0$, $f(Dv_j) \overset{*}{\rightharpoonup} \int_K f(\lambda) d\nu_x$ for all continuous functions f.

(III) For a.e. $x_0 \in \Omega$, there exists a bounded sequence (ϕ_k) in $W_0^{1,\infty}(D, \mathbb{R}^N)$ where D is the unit disk in $\mathbb{R}^n$, such that the corresponding gradient Young measures $\{\hat{\nu}_y\}$ of the sequence $(Du(x_0) + D\phi_k)$ satisfy $\hat{\nu}_y = \nu_{x_0}$ a.e. $y \in D$. We call $\hat{\nu}_y$ the homogeneous Young measure and simply denote it by $\nu := \hat{\nu}_y$, $y \in D$. The homogeneous Young measure enables us to **localize** our problem. We may decompose Theorem 1 into

Lemma 1. *('Existence') Suppose ν is a homogeneous Young measure satisfying* $\operatorname{supp} \nu \subset K_\epsilon$ *and $\bar{\nu} = X \in \bar{B}_\epsilon(A)$. Then* $\operatorname{supp} \nu \subset \bar{B}_\epsilon(A)$.

Lemma 1 implies that for a fixed x, $\operatorname{supp} \nu_x \subset K_\epsilon$, $\bar{\nu}_x \in \bar{B}_\epsilon(A)$ implies $\operatorname{supp} \nu_x \subset \bar{B}_\epsilon(A))$.

Lemma 2. *('Regularity') There is an estimate of $\epsilon > 0$ depending on $|A - B|$, $\lambda_{\max}$, n and N such that if $Du(x) \in K_\epsilon$ a.e. in Ω. Then either $Du(x) \in \bar{B}_\epsilon(A)$ a.e. or $Du(x) \in \bar{B}_\epsilon(B)$ a.e.*

Lemma 2 shows that if $\bar{\nu}_x \in K_\epsilon$ then $\bar{\nu}_x \in \bar{B}_\epsilon(A)$ a.e. or $\bar{\nu}_x \in \bar{B}_\epsilon(B)$. Combining (1), Lemma 1 and Lemma 2, we reach the conclusion of Theorem 1.

To prove Lemma 1 we need to control the large scale oscillation of the gradients of a sequence of mappings with a fixed affine boundary condition. For Lemma 2, we need to rule out large scale oscillation of the gradient of a fixed mapping without prescribed boundary condition. Note that in either cases the best possible we can reach is some partial rigidity of the gradients.

The tool we use is the Schauder estimates in BMO and Campanato spaces for linear elliptic systems with constant coefficients. Notice the ellipticity property of linear subspaces of $M^{N\times n}$ without rank-one matrices [2]. Let $E = \operatorname{span}[A - B]$. Then E is a subspace without rank-one matrices. Let $E^\perp$ be the orthogonal complement of E and $P_{E^\perp}$ be the orthogonal projection to $E^\perp$, then there is some constant $c_0 > 0$ such that for any rank-one matrix $a \otimes b$, $|P_{E^\perp}(a \otimes b)|^2 \geq c_0 |a|^2 |b|^2$, where $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$.

Let us recall some basic facts about elliptic system with constant coefficients [9]:

$$\operatorname{div} A^{ij}_{\alpha\beta} D_\beta u^j = \operatorname{div} F^i_\alpha \text{ in } \Omega \text{ with } F^i_\alpha \in L^\infty. \tag{2}$$

We say that the system satisfy the Legendre-Hadamard strong ellipticity condition if

$$A^{ij}_{\alpha\beta} \xi_\alpha \xi_\beta \eta^i \eta^j \geq \lambda_0 |\xi|^2 |\eta|^2, \; \xi \in \mathbb{R}^n, \, \eta \in \mathbb{R}^N. \tag{3}$$

We denote by $\Lambda_0 > 0$ a constant such that $A^{ij}_{\alpha\beta} \xi_\alpha \xi_\beta \eta^i \eta^j \leq \Lambda_0 |\xi|^2 |\eta|^2$.

Example. Let $E \subset M^{N\times n}$ be a subspace without rank-one matrices. Then the second order elliptic system $\operatorname{div} P_{E^\perp} Du = \operatorname{div} F$ satisfies (3). In our case $\lambda_0 = 1 - \lambda_{\max}$, and $\Lambda_0 = 1$.

Let $u \in W^{1,2}$ be a weak solution of (2). The following are some estimates of Du.

(A) **Interior Estimate**

Let $x_0 \in \Omega$ and $0 < \rho < R$ such that $B_\rho(x_0) \subset B_R(x_0) \subset \bar{B}_R(x_0) \subset \Omega$,

$$\fint_{B_\rho(x_0)} |Du - [Du]_{x_0,\rho}|^2 dx \leq C\left[\left(\frac{\rho}{R}\right)^\tau \fint_{B_R(x_0)} |Du - [Du]_{x_0,R}|^2 dx + [F]^2_{\mathcal{L}^{2,n}(\Omega)}\right], \qquad (4)$$

where $C = C(n, N, \lambda_0, \Lambda_0) > 0$, $\tau = \tau(n, N, \lambda_0, \Lambda_0)$, $0 < \tau < 2$, and $[F]_{\mathcal{L}^{2,n}(\Omega)}$ is the Campanato seminorm.

(B) **Global Estimate**

Under Dirichlet condition $u|_{\partial\Omega} = 0$,

$$\|Du\|_{BMO(\Omega)} \leq C\|F\|_{L^\infty}. \qquad (5)$$

We prove Lemma 2 first which depends on the interior estimate (a) above.

Proof of Lemma 2. Without loss of generality, we may assume that $A = 0$, $E = \operatorname{span}[B]$. This can be done by a simple translation in $M^{N\times n}$. Note that $|P_{E^\perp}(Du)| \leq 2\epsilon$, hence $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ is a weak solution of

$$\operatorname{div} P_{E^\perp}(Du) = \operatorname{div} F, \quad \text{where} \quad F = P_{E^\perp}(Du), \quad \|F\|_{L^\infty} \leq 2\epsilon.$$

Let $V = \{x \in \Omega,\ Du(x) \in \bar{B}_\epsilon(B)\}$. $Du(x) \in K_\epsilon$ implies that

$$Du = B\chi_V + H, \qquad \|H\|_{L^\infty} \leq 4\epsilon. \qquad (6)$$

Then (a) implies, for a fixed $x_0 \in \Omega$ and $0 < \rho < R$ such that $B_{2R}(x_0) \subset \Omega$,

$$\begin{aligned}\fint_{B_\rho(x_0)} |Du - [Du]_{x_0,\rho}|^2 dx &\leq C\left[\left(\frac{\rho}{R}\right)^\tau \fint_{B_R(x_0)} |Du - [Du]_{x_0,R}|^2 dx + [F]^2_{\mathcal{L}^{2,n}(\Omega)}\right] \\ &\leq C\left(\frac{\rho}{R}\right)^\tau (|B| + 2\epsilon)^2 + C\epsilon^2 \leq C\epsilon^2\end{aligned}$$

if ρ/R is sufficiently small. Consequently, for every $x_0 \in \Omega$, there is a small cubes $Q(x_0, r) \subset \Omega$ centered at x_0 with side-length $r > 0$ depending on x_0 and Ω,

$$\|Du\|^2_{BMO(Q(x_0,r))} \leq C\epsilon^2. \tag{7}$$

By applying the definition of BMO and (3), we have, for each cube $Q \subset Q(x_0, r)$ and let

$$G(Q) = \frac{|Q \cap V|}{|Q|}, \qquad \text{then} \qquad G(Q)(1 - G(Q)) \leq \frac{C}{|B|^2}\epsilon^2 < \frac{3}{16},$$

as long as $\epsilon > 0$ is small. Hence for $Q \subset Q(x_0, r)$,

$$\text{either} \qquad G(Q) \leq \frac{1}{4} \qquad \text{or} \qquad G(Q) \geq \frac{3}{4}. \tag{8}$$

Now we use the Intermediate Value Theorem and a density argument to finish the proof.

Without loss of generality, we assume $x_0 \in \Omega$ is a point of density 1 for V. Then we show that there is no points of density 1 for $\Omega \setminus V$ in Ω. Since x_0 is a point of density 1 for V, there is a cube $Q_0 \subset Q(x_0, r)$ centered at x_0, such that $G(Q_0) > 3/4$. We first show that there is no point of density 1 for $Q_0 \setminus V$ in Q_0. Otherwise, let $x_1 \in Q_0$ be an interior point such that there is some $Q_1 \subset Q_0$ centered at x_1 and satisfies $G(Q_1) < 1/4$. Then we may construct a continuous family of decreasing cubes $Q(t)$ in Q_0 such that $Q(0) = Q_0$ and $Q(1) = Q_1$. It is then easy to see that $t \to G(Q(t))$ is a continuous function. By the Intermediate Value Theorem, there is some cube $Q(t_0) \subset Q_0 \subset Q(x_0, r)$, such that $G(Q(t_0)) = 1/2$. This contradicts to (8).

Since Ω is arcwise connected, for each $x \in \Omega$, there is a piecewise affine curve $\gamma : [0, 1] \to \Omega$, such that $\mathrm{dist}(\gamma, \partial\Omega) = \delta_0 > 0$, $\gamma(0) = x_0$, $\gamma(1) = x$. If we choose $r > 0$ sufficiently small and let $Q(t)$ be the cube centered at $\gamma(t)$ with radius $0 < r < \delta_0$, we may claim that (8) holds for $Q \subset Q(t)$, $0 \leq t \leq 1$. Then it is easy to see that x is not a point of density 1 for $\Omega \setminus V$, hence $Du(x) \in B_\epsilon(B)$ a.e. □

Proof of Lemma 1. We may assume $A = 0$ as before and $\mathrm{supp}\,\nu \subset K_\epsilon$ and $\bar{\nu} = X \in \bar{B}_\epsilon(0)$. Let $X + Du_j$ generates ν with $u_j \in W^{1,\infty}_0$ bounded. Define

$$F_j = \begin{cases} P_{E^\perp}(Du_j), & \text{if } |P_{E^\perp}(Du_j)| \leq 4\epsilon, \\ 0, & \text{otherwise,} \end{cases}$$

where $E = \mathrm{span}[B]$. Solving $\mathrm{div}\, P_{E^\perp}(Dv_j) = \mathrm{div}\, F_j$ in D, $v_j|_{\partial D} = 0$,

$$\|Dv_j\|_{BMO(D)} \leq C\epsilon, \qquad \|Du_j - Dv_j\|_{L^2} \to 0 \tag{9}$$

Let $W_j = \{x \in D, \mathrm{dist}(Dv_j, K) \geq 4\epsilon\}$ so $|W_j| \to 0$ and let $V_j = \{x \in D, |Dv_j - B| < 4\epsilon\}$, then $Dv_j = B\chi_{V_j} + Dv_j\chi_{W_j} + O(\epsilon)$. Let $G_j(Q) = |V_j \cap Q|/|Q|$, then (9) implies

$$G_j(Q)(1 - G_j(Q)) \leq C\epsilon^2 + C \fint_Q (1 + |Dv_j|^2)\chi_{W_j} dx. \tag{10}$$

Also notice that $\int_D Dv_j dx = 0$ which implies $G_j(D) \leq C\epsilon < 1/4$ for large j.

Our aim is to show that $|V_j| \to 0$ so that $\mathrm{dist}^2(X + Dv_j, \bar{B}_\epsilon) \to 0$ in L^1, hence $\nu \operatorname{supp} \bar{B}_\epsilon(0)$. Now we use a slightly different argument as in the proof of Lemma 1 by using $\int_{W_j}(1 + |Dv_j|^2)dy$ to bound $|V_j|$.

For each point $x \in D$ of density 1 for V_j, there exists a cube $Q \subset D$ centered at x such that $G_j(Q) > 3/4$. Note that $G_j(D) < 1/4$, we can then prove that there is an open cube Q_x containing Q in D such that $G_j(Q_x) = 1/2$ which maximize the left hand side of (10). If we further require that $1/4 - C\epsilon^2 := \gamma > 0$, then from (10),

$$\gamma|Q_x| = |Q_x| \left(\frac{1}{4} - C\epsilon^2\right) \leq C \int_{Q_x} (1 + |Dv_j|^2)\chi_{W_j} dy. \tag{11}$$

Clearly $\{Q_x\}$ is a covering of the points of density 1 for V_j by open cubes. By Besicovitch's covering lemma (see e.g. [8]), it is then easy to prove that

$$\gamma|V_j| \leq C \int_{W_j} (1 + |Dv_j|^2)dy \to 0.$$

Therefore $|V_j| \to 0$ so that $\operatorname{supp} \nu \subset \bar{B}_\epsilon(0)$. □

Theorem 1 can be generalized to any finite sets contained in a subspace without rank-one matrices [19]. For a finite set $K = \{A_i\} \subset M^{N\times n}$, we define the diameter of K $d_K = \max\{|A_i - A_j|,\, i \neq j\}$, and the smallest distance $g_K = \min\{|A_i - A_j|,\, i \neq j\}$.

Theorem 2. *Suppose $E \subset M^{N\times n}$ be a linear subspace without rank-one matrices. Let $K = \{A_i\} \subset E$ be a finite subset. Let*

$$\lambda_E = \min\{|P_{E^\perp}(a \otimes b)|^2,\ a \in \mathbb{R}^N,\ b \in \mathbb{R}^n,\ |a| = |b| = 1\},$$
$$\frac{1}{\mu_E} = \inf_{|a|=|b|=1} \frac{|P_E(a \otimes b)|^2}{|P_{E^\perp}(a \otimes b)|^2} = \frac{1 - \lambda_E}{\lambda_E}.$$

Then there exists an estimate of $\epsilon > 0$ depending on d_K, g_K, λ_E, μ_E, n and N, such that $u_j \rightharpoonup u$ in $W^{1,1}$, $\int_\Omega \mathrm{dist}(Du_j, K_\epsilon) \to 0$ implies, up to a subsequence, for some fixed $A_{i_0} \in K$,

$$\lim_{j\to\infty} \int_\Omega \mathrm{dist}(Du_j,\, \bar{B}_\epsilon(A_{i_0})) \to 0.$$

Remark 1. Theorem 2 was proved in [19] by using S. Müller's improved approximation lemma for sequences of gradients approximating a compact set [14]. In the special case of Theorem 1, we may avoid using this result, instead, establishing Lemma 1 directly form the global estimate for the Dirichlet problem.

Remark 2. For the incompatible multi-elastic well structure $K = \bigcup_{i=1}^{m} SO(2)H_i$ with H_i positive definite, the theory for linear elliptic system still works provided that the wells are sufficiently 'flat' [20].

Remark 3. For the two well structure $K = SO(n) \cup SO(n)H$, it is known [12, 16] that under a technical assumption on H, the compactness result holds, that is, if $\mathrm{dist}^2(Du_j, K) \to 0$ in L^1, then there exists some $A \in K$, such that $Du_j \to A$ a.e. A nonlinear elliptic system is involved. However, I do not know any interior BMO estimates for the elliptic system $\mathrm{div}\, A(Du) = \mathrm{div}\, f$, $u|_{\partial\Omega} = 0$, where $c|Y|^2 \le DA(X)YY \le C|Y|^2$, $\|f\|_{L^\infty} \le \epsilon$?

As remarked in [12], if $H = \lambda I$ with $\lambda > 0$, $\lambda \neq 1$, and I being the identity matrix, one may simply use the n-Laplace operator to study convergent sequences of gradients to K, that is, $\mathrm{div}\, |Du|^{n-2}Du = \mathrm{div}\, F$. However, even for this explicit system, I do not know any BMO estimate of the weak solutions in $W^{1,n}$ given that $\|F\|_{L^\infty}$ is small.

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