DIVERGENT SUMS OF SPHERICAL HARMONICS

CHRISTOPHER MEANEY

ABSTRACT. We combine the Cantor-Lebesgue Theorem and Uniform Boundedness Principle to prove a divergence result for Cesàro and Bochner-Riesz means of spherical harmonic expansions.

1. BACKGROUND

Fix an integer $d > 1$ and consider the unit sphere S^d in $\mathbb{R}^{d+1}$, equipped with normalized rotation-invariant measure. For each $n \geq 0$ let $\mathcal{H}_n$ denote the space of spherical harmonics of degree n restricted to S^d, so that $L^2(S^d) = \oplus_{n=0}^{\infty}\mathcal{H}_n$. See [22, Section 4.2] for details. Every distribution ψ on S^d has a spherical harmonic expansion

$$(1)\qquad \sum_{n=0}^{\infty} Y_n(\psi)(x), \qquad \forall x \in S^d, \text{ where } Y_n(\psi) \in \mathcal{H}_n, \quad \forall n \geq 0.$$

This is the expansion of ψ in eigenfunctions of the Laplace-Beltrami operator on S^d. It is known [14] that if $1 \leq p < 2$ then there is an $\psi \in L^p(S^d)$ for which (1) diverges almost everywhere. That leaves open the general behaviour of spherical harmonic expansions for elements of $L^2(S^d)$. A partial step in this direction follows from the localization principle [18].

Theorem 1.1 (Localization)**.** *Suppose ψ is a distribution on S^d and $U \subset S^d$ is an open set disjoint from the support of ψ. For each $x \in U$, the expansion $\sum_{n=0}^{\infty} Y_n(\psi)(x)$ converges if and only if $Y_n(\psi)(x) \to 0$ as $n \to \infty$.*

Corollary 1.2. *If $\psi \in L^2(S^d)$ and $U \subset S^d$ is an open set on which ψ is zero almost everywhere, then the expansion $\sum_{n=0}^{\infty} Y_n(\psi)(x)$ converges to zero almost everywhere on U.*

There are special cases where a function $\psi \in L^2(S^d)$ can be guaranteed to have an almost everywhere convergent spherical harmonic expansion, if ψ is in an L^2- Sobolev space $W^{2,s}$ of positive index s [16] or if it is zonal [1]. (Recall that a function f on S^d is said to be *zonal* about a point $y \in S^d$ when $f(x)$ depends only on $x \cdot y$ for all $x \in S^d$.)

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Carleson's theorem [3] has been extended to zonal functions [11]. Let p_c be the critical index

$$p_c = \frac{2d}{d+1}.$$

Theorem 1.3. *If $p_c < p \leq 2$ and $f \in L^p(S^d)$ is zonal about a point $y \in S^d$, then its spherical harmonic expansion is convergent almost everywhere.*

Corollary 1.4. *Suppose $\psi \in L^2(S^d)$, $U \subset S^d$ is an open set, $f_1 \in \bigcup_{s>0} W^{2,s}(S^d)$, $f_2 \in L^2(S^d)$ is a finite sum of zonal functions, and $\psi = f_1 + f_2$ almost everywhere on U. Then $\sum_{n=0}^{\infty} Y_n(\psi)(x)$ converges almost everywhere on U.*

The two corollaries 1.2 and 1.4 would be rendered trivial if there where a higher dimensional version of Carleson's theorem.

They do suggest that when considering convergence of expansions, we should examine the term-wise behaviour away from the support of a distribution.

In the early 1980's we showed [17] that Theorem 1.3 is sharp and that localization fails at the critical index.

Theorem 1.5. *For each $y \in S^d$ and $1 \leq p \leq p_c$ there is a $\psi \in L^p(S^d)$, supported in the hemisphere $\{x : x \cdot y \geq 0\}$ whose spherical harmonic expansion diverges almost everywhere.*

This was proved by a combination of the Cantor-Lebesgue theorem, knowledge of the $L^{p'}$-norms of the zonal spherical functions, and the uniform boundedness principle. Kanjin [13] showed that these methods could be combined with a result of Hardy and Riesz [12] to deal with Riesz means for radial functions on Euclidean space. This approach was also used in [20] for Riesz means of radial functions on non-compact rank one symmetric spaces.

Here we prove a similar result for Cesàro and Riesz means of spherical harmonic expansions of zonal functions. This shows the sharpness of the results in [4]. See [2, 7] for earlier work on Cesàro means of spherical harmonic expansions. See [21, 5] for results in a more general setting.

2. Cesàro & Riesz means

2.1. **Cesàro means.** The Cesàro means [24, pages 76–77] of order δ of the expansion (1) are defined by

$$\sigma_N^\delta \psi(x) = \sum_{n=0}^{N} \frac{A_{N-n}^\delta}{A_N^\delta} Y_n(\psi)(x), \qquad \forall N \geq 0, x \in S^d, \tag{2}$$

where $A_n^\delta = \binom{n+\delta}{n}$. Theorem 3.1.22 in [24] says that if the Cesàro means converge, then the terms of the series have controlled growth.

Lemma 2.1. *Suppose that* $\lim_{N\to\infty} \sigma_N^\delta \psi(x)$ *exists for some* $x \in X$ *and* $\delta > -1$. *Then*

$$|Y_N(\psi)(x)| \leq C_\delta \, N^\delta \max_{0\leq n\leq N} \left|\sigma_n^\delta \psi(x)\right|, \qquad \forall n \geq 0.$$

2.2. **Riesz means.** Hardy and Riesz [12] had proved a similar result for Riesz means. Recall that the Riesz means of order $\delta \geq 0$ are defined for each $r > 0$ by

$$S_r^\delta \psi(x) = \sum_{0\leq n<r} \left(1 - \frac{k}{r}\right)^\delta Y_n(\psi)(x). \tag{3}$$

Theorem 21 of [12] tells us how the convergence of $S_r^\delta \psi(x)$ controls the size of the partial sums $S_r^0 \psi(x)$.

Lemma 2.2. *Suppose that* ψ *is a distribution on the sphere for which there is some* $\delta > 0$ *and* $x \in X$ *at which its Riesz means* $S_r^\delta \psi(x)$ *converges to* c *as* $r \to \infty$ *then*

$$\left|S_r^0 \psi(x) - c\right| \leq A_\delta \, r^\delta \sup_{0<t\leq r+1} \left|S_t^\delta \psi(x)\right|.$$

Note that this implies

$$Y_n(\psi)(x) = \mathbf{O}(n^\delta)$$

and we have the same growth estimates as in Lemma 2.1.

Gergen[9] wrote formulae relating the Riesz and Cesàro means of order $\delta \geq 0$, from which it follows that the two methods of summation are equivalent.

3. Zonal Functions and Jacobi Polynomials

3.1. **Notation.** Suppose that f is a function on S^d with $f(x)$ depending only on $x{\cdot}y$, for a fixed $y \in S^d$, so that $f(x) = f_0(x \cdot y)$. The spherical harmonic expansion of f is

$$\sum_{n=0}^{\infty} c_n(f_0) h_n^{-1} P_n^{(\alpha,\alpha)}(x \cdot y) \tag{4}$$

where $\alpha = (d-2)/2$, $P_n^{(\alpha,\alpha)}$ is the Jacobi polynomial of degree n and index (α, α),

$$h_n = \int_{-1}^{1} \left|P_n^{(\alpha,\alpha)}(t)\right|^2 \left(1-t^2\right)^\alpha \, dt,$$

and the coefficients are

$$c_n(f_0) = \int_{-1}^{1} f_0(t) P_n^{(\alpha,\alpha)}(t) \left(1-t^2\right)^\alpha \, dt, \quad \forall n \geq 0.$$

See section 4.7 of Szegő's book [23] for details about these special functions. Let m_α be the measure on $[-1, 1]$ given by

$$dm_\alpha(t) = (1-t^2)^\alpha \, dt,$$

so that $\{P_n^{(\alpha,\alpha)} : n \geq 0\}$ is a orthogonal basis of $L^2(m_\alpha)$. From (4.3.3) in [23] we know that the normalization constants h_n satisfy

$$h_n \sim A\, n^{-1} \text{ as } n \to \infty \tag{5}$$

3.2. **Uniform Boundedness.** Suppose there is a number $1 < q \leq \infty$ and some positive number A with

$$\|P_n^{(\alpha,\alpha)}\|_{L^q(m_\alpha)} \geq cn^A, \qquad \forall n \geq 1.$$

The formation of the coefficient

$$F \mapsto c_n(F) = \int_{-1}^{1} F(t) P_n^{(\alpha,\alpha)}(t) dm_\alpha(t)$$

is then a bounded linear functional on the dual of $L^q(m_\alpha)$ with norm bounded below by a constant multiple of n^A. The uniform boundedness principle implies that for p conjugate to q and each $0 \leq \varepsilon < A$ there is an $F \in L^p(m_\alpha)$ so that

$$c_n(F)/n^\varepsilon \to \infty \text{ as } n \to \infty. \tag{6}$$

3.3. **Cantor-Lebesgue Theorem.** This idea is explained in [19] and is based on [24, Section IX.1]. Suppose we have a sequence of functions F_n on an interval in the real line with the asymptotic property

$$F_n(\theta) = c_n \left(\cos(M_n\theta + \gamma_n) + \mathbf{o}(1)\right), \qquad \forall n \geq 0$$

uniformly on a set E of finite positive measure, and with $M_n \to \infty$ as $n \to \infty$. Integrating $|F_n|^2$ over E gives

$$\begin{aligned}\int_E |F_n(\theta)|^2\, d\theta &= |c_n|^2 \left(\int_E \cos^2(M_n\theta + \gamma_n)\, d\theta + \mathbf{o}(1) \right) \\ &= |c_n|^2 \left(\frac{|E|}{2} + \frac{e^{2i\gamma_n}}{4} \widehat{\chi_E}(2M_n) + \frac{e^{-2i\gamma_n}}{4} \widehat{\chi_E}(-2M_n) + \mathbf{o}(1) \right).\end{aligned}$$

The Riemann-Lebesgue Theorem [24, Thm. II.4.4] says that the Fourier transforms $\widehat{\chi_E}(\pm 2M_n) \to 0$ as $M_n \to \infty$. If we know that there is some function G for which $|F_n(\theta)| \leq G(n)$ uniformly on E for all n then there is an $n_0 > 0$ for which

$$\frac{|E|}{4}|c_n|^2 \leq \int_E |F_n(\theta)|^2\, d\theta \leq G(n)^2|E|, \quad \forall n \geq n_0.$$

This shows that $|c_n| \leq 2G(n)$ for all $n \geq n_0$.

3.4. **Asymptotics.** Theorem 8.21.8 in Szegő's book[23] gives the following asymptotic behaviour for the Jacobi polynomials $P_n^{(\alpha,\alpha)}$. For $\alpha \geq -1/2$ and $\varepsilon > 0$ the following estimate holds uniformly for all $\varepsilon \leq \theta \leq \pi - \varepsilon$ and $n \geq 1$.

$$P_n^{(\alpha,\alpha)}(\cos\theta) = n^{-1/2} k(\theta) \cos\left(M_n\theta + \gamma\right) + \mathbf{O}\left(n^{-3/2}\right). \tag{7}$$

Here $k(\theta) = \pi^{-1/2} \left(\sin(\theta)/2\right)^{-\alpha - 1/2}$, $M_n = n + (2\alpha+1)/2$, and $\gamma = -(\alpha + 1/2)\pi/2$.

From Egoroff's theorem and Lemma 2.1 we can say that if the series (4) is Cesàro summable of order δ on a set of positive measure in S^d then there is a set of positive measure $E \subset [0,\pi]$ on which

$$\left|c_n(f_0)h_n^{-1}P_n^{(\alpha,\alpha)}(\cos\theta)\right| \leq An^\delta$$

and hence

$$\left|c_n(f_0)n^{(1/2)-\delta}\left(\cos\left(M_n\theta+\gamma\right)+\mathbf{O}(n^{-1})\right)\right| \leq A \tag{8}$$

uniformly for $\theta \in E$. The argument of subsection 3.3 shows that

$$\left|c_n(f_0)n^{(1/2)-\delta}\right| \leq A, \qquad \forall n \geq 1. \tag{9}$$

Lemma 3.1. *If f is a zonal function on the unit sphere whose spherical harmonic expansion is Cesàro summable of order δ on a set of positive measure, then there is a constant $A > 0$ for which*

$$|c_n(f_0)| \leq An^{\delta-(1/2)}, \quad \forall n \geq 1.$$

3.5. **Norm Estimates.** Markett[15] has calculated estimates on the L^p norms of Jacobi polynomials. Let

$$q_c = \frac{4(\alpha+1)}{2\alpha+1} = \frac{2d}{d-1}.$$

Equation (2.2) in [15] gives the following lower bounds on these norms.

Lemma 3.2. *For real number $\alpha > -1/2$, $1 \leq q < \infty$, and $r > -1/q$,*

$$\left(\int_0^1 \left|P_n^{(\alpha,\alpha)}(x)\right|^q (1-x)^\alpha\,dx\right)^{1/q} \sim \begin{cases} n^{-1/2} & \text{if } q < q_c, \\ n^{-1/2}(\log n)^{1/q} & \text{if } q = q_c, \\ n^{\alpha-(2\alpha+2)/q} & \text{if } q > q_c. \end{cases}$$

Notice that these integrals are taken over $[0,1]$ rather than all of $[-1,1]$.

4. Main Result

Theorem 4.1. *For each $1 \leq p < p_c = 2d/(d+1)$,*

$$0 \leq \delta < \frac{d}{p} - \frac{d+1}{2},$$

and $y \in S^d$, there is a function in $L^p(S^d)$ which is zonal about y, supported in the hemisphere $\{x : x \cdot y \geq 0\}$, and whose spherical harmonic expansion has Cesàro and Riesz means which diverge almost everywhere.

Proof. Suppose that a series (4) has Cesàro means of order δ which converge on a set of positive measure. Then Lemma 3.1 implies that

$$c_n(f_0) = \mathbf{O}\left(n^{\delta-(1/2)}\right), \quad \text{as } n \to \infty. \tag{10}$$

Compare this inequality with the last line of Lemma 3.2 and section 3.2. If $q > q_c$, $(1/p) + (1/q) = 1$ and

$$\alpha - \frac{(2\alpha + 2)}{q} > \delta - \frac{1}{2}$$

then there must be a zonal function $f \in L^p(S^d)$ with f_0 supported on $[0, 1]$ for which the estimate (10) fails. Remembering the definition of α in terms of the dimension d, we are considering

$$\delta - \frac{1}{2} < \frac{d-1}{2} - d\left(1 - \frac{1}{p}\right)$$

which means

$$\delta < \frac{d}{p} - \frac{(d+1)}{2}.$$

□

Remark 4.1. In [19] we applied this technique to produce an analogous theorem for Laguerre expansions.

5. Central function on $SU(2)$

We conclude with a simple three dimensional example. Suppose that $G = SU(2)$ is equipped with the normalized translation invariant measure μ and that T is the maximal torus of diagonal elements of G. For each $\ell \in \widehat{G} = \{k/2 \; : \; k \in \mathbb{Z}, k \geq 0\}$ there is an irreducible unitary representation of G with dimension $2\ell + 1$ and character

$$\chi_\ell\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\right) = \frac{\sin\left((2\ell+1)\theta\right)}{\sin(\theta)}.$$

Every central function on G is determined by its restriction to T. The Fourier series of central functions are expansions in the characters. If $f \in L^1(G, \mu)$ is central then

$$f \sim \sum_{\ell=0}^{\infty} c_\ell \chi_\ell \tag{11}$$

with

$$c_\ell = \int_G f(x)\overline{\chi_\ell(x)}\, d\mu(x), \qquad \forall \ell \in \widehat{G}. \tag{12}$$

In [8] and [10], Dooley, Giulini, Soardi, and Travaglini estimated the Lebesgue norms of characters of compact Lie groups. The group $SU(2)$ provides the simplest case of these estimates. For each $q > 3$

$$\|\chi_\ell\|_q \geq c(2\ell+1)^{1-3/q}, \qquad \forall \ell \in \widehat{G}. \tag{13}$$

If $1/p + 1/q = 1$ then

$$1 - \frac{3}{q} = 1 - 3\left(1 - \frac{1}{p}\right) = \frac{3}{p} - 2.$$

Uniform boundedness then says that if $1 \leq p < 3/2$ and $a < (3/p) - 2$ then there is a central function $f \in L^p(G)$ for which the coefficients in (11) have

$$c_\ell/(2\ell+1)^a \text{ unbounded as } \ell \to \infty.$$

Suppose that (11) is Cesàro summable of order δ on a set of positive measure. Then Lemma 2.1 says that

$$c_\ell \sin\left((2\ell+1)\theta\right) = \mathbf{O}(\ell^\delta) \text{ as } \ell \to \infty,$$

on a set of positive measure. The Cantor-Lebesgue Theorem then says that

$$c_\ell = \mathbf{O}(\ell^\delta) \text{ as } \ell \to \infty.$$

Theorem 5.1. *For $1 \leq p < 3/2$ and $0 \leq \delta < (3/p) - 2$ there is a central function $f \in L^p(SU(2))$ for which the Cesàro and Riesz means of order δ are divergent almost everywhere.*

This shows the sharpness of results in Clerc's paper [6].

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Dept. Mathematics, Macquarie University, North Ryde, NSW 2109 Australia
E-mail address: chrism@maths.mq.edu.au