YET ANOTHER CONSTRUCTION OF THE CENTRAL EXTENSION OF THE LOOP GROUP

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Abstract. We give a characterisation of central extensions of a Lie group G by $\mathbb{C}^\times$ in terms of a differential two-form on G and a differential one-form on $G \times G$. This is applied to the case of the central extension of the loop group.

1. Introduction

Let G and A be groups. A *central extension* of G by A is another group $\hat{G}$ and a homomorphism $\pi\colon \hat{G} \to G$ whose kernel is isomorphic to A and in the center of $\hat{G}$. Note that because A is in the center of $\hat{G}$ it is necessarily abelian. We will be interested ultimately in the case that $G = \Omega(K)$ the *loop group* of all smooth maps from the circle S^1 to a Lie group K with pointwise multiplication but the theory developed applies to any Lie group G.

2. Central extension of groups

Consider first the case when G is just a group and ignore questions of continuity or differentiability. In this case we can choose a *section* of the map π. That is a map $s\colon G \to \hat{G}$ such that $\pi(s(g)) = g$ for every $g \in G$. From this section we can construct a bijection

$$\phi\colon A \times G \to \hat{G}$$

by $\phi(g, a) = \iota(a)s(g)$ where $\iota\colon A \to \hat{G}$ is the identification of A with the kernel of π. So we know that as a set $\hat{G}$ is just the product $A \times G$. However as a group $\hat{G}$ is not generally a product. To see what it is note that $\pi(s(g)s(h)) = \pi(s(g))\pi(s(h)) = gh = \pi(s(gh))$ so that $s(g)s(h) = c(g, h)s(gh)$ where $c\colon G \times G \to A$. The bijection $\phi\colon A \times G \to \hat{G}$ induces a product on $A \times G$ for which ϕ is a homomorphism. To calculate this

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product we note that

$$\begin{aligned}\phi(a,g)\phi(b,h) &= \iota(a)s(g)\iota(b)s(h)\\ &= \iota(ab)s(g)s(h)\\ &= \iota(ab)c(gh)s(gh).\end{aligned}$$

Hence the product on $A \times G$ is given by $(a,g) \star (b,h) = (abc(g,h)gh)$ and the map ϕ is a group isomorphism between $\hat{G}$ and $A \times G$ with this product.

Notice that if we choose a different section $\tilde{s}$ then $\tilde{s} = sh$ were $h\colon G \to A$.

It is straightforward to check that if we pick any $c\colon G \times G \to A$ and define a product on $A \times G$ by $(a,g) \star (b,h) = (abc(g,h)gh)$ then this is an associative product if and only if c satisfies the *cocycle condition*

$$c(g,h)c(gh,k) = c(g,hk)c(h,k)$$

for all g, h and k in G.

If we choose a different section $\tilde{s}$ then we must have $\tilde{s} = sd$ for some $d\colon G \to A$. If $\tilde{c}$ is the cocycle determined by $\tilde{s}$ then a calculation shows that

$$c(g,h) = \tilde{c}(g,h)d(gh)d(g)^{-1}d(h)^{-1}. \tag{1}$$

We have now essentially shown that all central extensions of G by A are determined by cocycles c modulo identifying two that satisfy the condition (1). Let us recast this result in a form that we will see again in this talk.

Define maps $d_i\colon G^{p+1} \to G^p$ by

(2)

$$d_i(g_1,\dots,g_{p+1}) = \begin{cases}(g_2,\dots,g_{p+1}), & i=0,\\ (g_1,\dots,g_{i-1}g_i,g_{i+1},\dots,g_{p+1}), & 1 \le i \le p-1,\\ (g_1,\dots,g_p), & i=p.\end{cases}$$

If $M^p(G;A) = \mathrm{Map}(G^p,A)$ then we define $\delta\colon M^p(G;A) \to M^{p-1}(G;A)$ by $\delta(c) = (c \circ d_1)(c \circ d_2)^{-1}(c \circ d_3)\dots$. This satisfies $\delta^2 = 1$ and defines a complex

$$M^0(G;A) \xrightarrow{\delta} M^1(G;A) \xrightarrow{\delta} M^2(G;A) \xrightarrow{\delta} \dots$$

The cocycle condition can be rewritten as $\delta(c) = 1$ and the condition that two cocycles give rise to the same central extension is that $c = \tilde{c}\delta(d)$. If we define

$$H^p(G;A) = \frac{\text{kernel } \delta\colon M^p(G;A) \to M^{p-1}(G;A)}{\text{image } \delta\colon M^{p+1}(G;A) \to M^p(G;A)}$$

then we have shown that central extensions of G by A are classified by $H^2(G; A)$.

3. Central extensions of Lie groups

In the case that G is a topological or Lie group it is well-known that there are interesting central extensions for which no continuous or differentiable section exists. For example consider the central extension

$$\mathbb{Z}_2 \to SU(2) = \mathrm{Spin}(3) \to SO(3)$$

of the three dimensional orthogonal group $SO(3)$ by its double cover $\mathrm{Spin}(3)$. Here $SU(2)$ is known to be the three sphere but if a section existed then we would have $SU(2)$ homeomorphic to $\mathbb{Z}_2 \times SO(3)$ and hence disconnected.

From now on we will concentrate on the case when $A = \mathbb{C}^\times$. Then $\hat{G} \to G$ can be thought of as a $\mathbb{C}^\times$ principal bundle and a section will only exist if this bundle is trivial. The structure of the central extension as a $\mathbb{C}^\times$ bundle is important in what follows so we digress to discuss them in more detail.

3.1. $\mathbb{C}^\times$ **bundles.** Let $P \to X$ be a $\mathbb{C}^\times$ bundle over a manifold X. We denote the fibre of P over $x \in X$ by P_x. Recall [1] that if P is a $\mathbb{C}^\times$ bundle over a manifold X we can define the dual bundle P^* as the same space P but with the action $p^*g = (pg^{-1})^*$ and, that if Q is another such bundle, we can define the product bundle $P \otimes Q$ by $(P \otimes Q)_x = (P_x \times Q_x)/\mathbb{C}^\times$ where $\mathbb{C}^\times$ acts by $(p, q)w = (pw, qw^{-1})$. We denote an element of $P \otimes Q$ by $p \otimes q$ with the understanding that $(pw) \otimes q = p \otimes (qw) = (p \otimes q)w$ for $w \in \mathbb{C}^\times$. It is straightforward to check that $P \otimes P^*$ is canonically trivialised by the section $x \mapsto p \otimes p^*$ where p is any point in P_x.

If P and Q are $\mathbb{C}^\times$ bundles on X with connections μ_P and μ_Q then $P \otimes Q$ has an induced connection we denote by $\mu_P \otimes \mu_Q$. The curvature of this connection is $R_P + R_Q$ where R_P and R_Q are the curvatures of μ_P and μ_Q respectively. The bundle P^* has an induced connection whose curvature is $-R_P$.

Recall the maps $d_i \colon G^p \to G^{p-1}$ defined by (2). If $P \to G^p$ is a $\mathbb{C}^\times$ bundle then we can define a $\mathbb{C}^\times$ bundle over G^{p+1} denoted $\delta(P)$ by

$$\delta(P) = \pi_1^{-1}(P) \otimes \pi_2^{-1}(P)^* \otimes \pi_3^{-1}(P) \otimes \dots.$$

If s is a section of P then it defines $\delta(s)$ a section of $\delta(P)$ and if μ is a connection on P with curvature R it defines a connection on $\delta(P)$ which we denote by $\delta(\mu)$. To define the curvature of $\delta(\mu)$ let us denote by $\Omega^q(G^p)$ the space of all differentiable q forms on G^p. Then we define a map

$$\delta \colon \Omega^q(G^p) \to \Omega^q(G^{p+1}) \tag{3}$$

by $\delta = \sum_{i=0}^{p} d_i^*$, the alternating sum of pull-backs by the various maps $d_i \colon G^{p+1} \to G^p$. Then the curvature of $\delta(\mu)$ is $\delta(R)$. If we consider $\delta(\delta(P))$ it is a product of factors and every factor occurs with its dual so $\delta(\delta(P))$ is canonically trivial. If s is a section of P then under this identification $\delta\delta(s) = 1$ and moreover if μ is a connection on P then $\delta\delta(\mu)$ is the flat connection on $\delta\delta(P)$ with respect to $\delta(\delta(s))$.

4. Central extensions

Let G be a Lie group and consider a central extension

$$\mathbb{C}^\times \to \hat{G} \xrightarrow{\pi} G.$$

Following Brylinski and McLaughlin [2] we think of this as a $\mathbb{C}^\times$ bundle $\hat{G} \to G$ with a product $M \colon \hat{G} \times \hat{G} \to \hat{G}$ covering the product $m = d_1 \colon G \times G \to G$.

Because this is a central extension we must have that $M(pz, qw) = M(p,q)zw$ for any $p, q \in P$ and $z, w \in \mathbb{C}^\times$. This means we have a section s of $\delta(P)$ given by

$$s(g,h) = p \otimes M(p,q) \otimes q$$

for any $p \in P_g$ and $q \in P_h$. This is well-defined as $pw \otimes M(pw, qz) \otimes qz = pw \otimes M(p,q)(wz)^{-1} \otimes qz = p \otimes M(p,q) \otimes q$. Conversely any such section gives rise to an M.

Of course we need an associative product and it can be shown that M being associative is equivalent to $\delta(s) = 1$. To actually make $\hat{G}$ into a group we need more than multiplication we need an identity $\hat{e} \in \hat{G}$ and an inverse map. It is straightforward to check that if $e \in G$ is the identity then, because $M \colon \hat{G}_e \times \hat{G}_e \to \hat{G}_e$, there is a unique $\hat{e} \in \hat{G}_e$ such that $M(\hat{e}, \hat{e}) = \hat{e}$. It is also straightforward to deduce the existence of a unique inverse.

Hence we have the result from [2] that a central extension of G is a $\mathbb{C}^\times$ bundle $P \to G$ together with a section s of $\delta(P) \to G \times G$ such that $\delta(s) = 1$. In [2] this is phrased in terms of simplicial line bundles. Note that this is a kind of cohomology result analogous to that in the first section. We have an object (in this case a $\mathbb{C}^\times$ bundle) and δ of the object is 'zero' i.e. trivial as a $\mathbb{C}^\times$ bundle.

For our purposes we need to phrase this result in terms of differential forms. We call a connection for $\hat{G} \to G$, thought of as a $\mathbb{C}^\times$ bundle, a connection for the central extension. An isomorphism of central extensions with connection is an isomorphism of bundles with connection which is a group isomorphism on the total space $\hat{G}$. Denote by $C(G)$ the set of all isomorphism classes of central extensions of G with connection.

Let $\mu \in \Omega^1(\hat{G})$ be a connection on the bundle $\hat{G} \to G$ and consider the tensor product connection $\delta(\mu)$. Let $\alpha = s^*(\delta(\mu))$. We then have that

$$\begin{aligned} \delta(\alpha) &= (\delta(s)^*)(\delta(\mu)) \\ &= (1)^*(\delta^2(\mu)) \\ &= 0 \end{aligned}$$

as $\delta^2(\mu)$ is the flat connection on $\delta^2(P)$. Also $d\alpha = s^*(d\delta(\mu)) = \delta(R)$.

Let , (G) denote the set of all pairs (α, R) where R is a closed, $2\pi i$ integral, two form on G and α is a one-form on $G \times G$ with $\delta(R) = d\alpha$ and $\delta(\alpha) = 0$.

We have constructed a map $C(G) \to , (G)$. In the next section we construct an inverse to this map by showing how to define a central extension from a pair (α, R). For now notice that isomorphic central extensions with connection clearly give rise to the same (α, R) and that if we vary the connection, which is only possible by adding on the pull-back of a one-form η from G, then we change (α, R) to $(\alpha + \delta(\eta), R + d\eta)$.

4.1. **Constructing the central extension.** Recall that given R we can find a principal $\mathbb{C}^\times$ bundle $P \to G$ with connection μ and curvature R which is unique up to isomorphism. It is a standard result in the theory of bundles that if $P \to X$ is a bundle with connection μ which is flat and $\pi_1(X) = 0$ then P has a section $s\colon X \to P$ such that $s^*(\mu) = 0$. Such a section is not unique of course it can be multiplied by a (constant) element of $\mathbb{C}^\times$. As our interest is in the loop group G which satisfies $\pi_1(G) = 0$ we shall assume, from now on, that $\pi_1(G) = 0$. Consider now our pair (R, α) and the bundle P. As $\delta(R) = d\alpha$ we have that the connection $\delta(w) - \pi^*(\alpha)$ on $\delta(P) \to G \times G$ is flat and hence (as $\pi_1(G \times G) = 0$) we can find a section s such that $s^*(\delta(w)) = \alpha$.

The section s defines a multiplication by

$$s(p, q) = p \otimes M(p, q)^* \otimes q.$$

Consider now $\delta(s)$ this satisfies $\delta(s)^*(\delta(\delta(w))) = \delta(s^*(\delta(w)) = \delta(\alpha) = 0$. On the other hand the canonical section 1 of $\delta(\delta(P))$ also satisfies this so they differ by a constant element of the group. This means that there is a $w \in \mathbb{C}^\times$ such that for any p, q and r we must have

$$M(M(p, q), r) = wM(p, M(q, r)).$$

Choose $p \in \hat{G}_e$ where e is the identity in G. Then $M(p,p) \in \hat{G}_e$ and hence $M(p,p) = pz$ for some $z \in \mathbb{C}^\times$. Now let $p = q = r$ and it is clear that we must have $w = 1$.

So from (α, R) we have constructed P and a section s of $\delta(P)$ with $\delta(s) = 1$. However s is not unique but this is not a problem. If we change s to $s' = sz$ for some constant $z \in \mathbb{C}^\times$ then we have changed M to $M' = Mz$. As $\mathbb{C}^\times$ is central multiplying by z is an isomorphism of central extensions with connection. So the ambiguity in s does not change the isomorphism class of the central extension with connection. Hence we have constructed a map

$$, (G) \to C(G)$$

as required. That it is the inverse of the earlier map follows from the definition of α as $s^*(\delta(\mu))$ and the fact that the connection on P is chosen so its curvature is R.

5. Conclusion: Loop groups

In the case where $G = L(K)$ there is a well known expression for the curvature R of a left invariant connection on $L(\hat{K})$ — see [5]. We can also write down a 1-form α on $L(K) \times L(K)$ such that $\delta(R) = d\alpha$ and $\delta(\alpha) = 0$. We have:

$$R(g)(gX, gY) = \frac{1}{4\pi^2} \int_{S^1} \langle X, \partial_\theta Y \rangle d\theta$$
$$\alpha(g_1, g_2)(g_1 X_1, g_2 X_2) = \frac{1}{4\pi^2} \int_{S^1} \langle X_1, (\partial_\theta g_2) \partial_\theta g_2^{-1} \rangle d\theta.$$

Here $\langle\ ,\ \rangle$ is the Killing form on $\mathfrak{k}$ normalised so the longest root has length squared equal to 2 and ∂_θ denotes differentiation with respect to $\theta \in S^1$. Note that R is left invariant and that α is left invariant in the first factor of $G \times G$. It can be shown that these are the R and α arising in [3].

In [4] we apply the methods of this talk to give an explicit construction of the 'string class' of a loop group bundle and relate it to earlier work of Murray on calorons.

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