§4. REGULAR RINGS

We let A be a noetherian local ring, $\mathfrak{m}$ its maximal ideal, $k = A/\mathfrak{m}$. We denote by $S_k(\mathfrak{m}/\mathfrak{m}^2)$ the symmetric algebra of the k-vector space $\mathfrak{m}/\mathfrak{m}^2$. If $\operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2) = r$ one trivially has $S_k(\mathfrak{m}/\mathfrak{m}^2) \simeq k[T_1,\ldots,T_r]$ = where $T_1,\ldots,T_r$ are indeterminates over k.

We proceed to define a homomorphism

$$\theta: S_k(\mathfrak{m}/\mathfrak{m}^2) \to gr_{\mathfrak{m}}(A) = \bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

as follows:

Let $\overline{x}_1,\ldots,\overline{x}_r$ be a k-basis of $\mathfrak{m}/\mathfrak{m}^2$, and let $x_1,\ldots,x_r \in \mathfrak{m}$ be their representatives. By Nakayama's Lemma (see the remark on page 35) $x_1,\ldots,x_r$ forms a set of generators of $\mathfrak{m}$. Hence $\mathfrak{m}^i$ is generated by elements of the form $x_1^{\alpha_1} \ldots x_r^{\alpha_r}$ with $\alpha_1 +\ldots+ \alpha_r = i$. θ is defined by $\theta(\overline{x}_1^{\alpha_1} \ldots \overline{x}_r^{\alpha_r})$ = the class of $x_1^{\alpha_1} \ldots x_r^{\alpha_r} \bmod \mathfrak{m}^{i+1}$. Trivially θ is a homogeneous homomorphism of degree 0, and an epimorphism.

Theorem 4.1. Let A be a noetherian local ring of dimension n, $\mathfrak{m}$ its maximal ideal $k = A/\mathfrak{m}$. The following four conditions are equivalent.

a) $\theta: S_k(\mathfrak{m}/\mathfrak{m}^2) \to gr_{\mathfrak{m}}(A)$ is bijective

b) $\operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2) = n$

c) $\mathfrak{m}$ is generated by n elements

d) There exists an A-regular system which generates $\mathfrak{m}$.

<u>Proof</u>: b) $\Longrightarrow$ c) follows from the remark above that every k-basis of $\mathfrak{m}/\mathfrak{m}^2$ lifts back (in $\mathfrak{m}$) to a set of generators of $\mathfrak{m}$ (by Nakayama's Lemma). Conversely, any set of generators of $\mathfrak{m}$ gives rise (mod $\mathfrak{m}^2$) to a set of generators of $\mathfrak{m}/\mathfrak{m}^2$ over k, whence $\operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2) \leqq n$. But, by proposition 2.5, $\operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2) \geqq n$, whence c) $\Longrightarrow$ b). We have proved <u>b) $\Longleftrightarrow$ c)</u>.

a) $\Longrightarrow$ d). Let $\overline{z}_1,\ldots,\overline{z}_r \in \mathfrak{m}/\mathfrak{m}^2$ be a basis of $\mathfrak{m}/\mathfrak{m}^2$ over k. We use the symbol $\overline{z}^\alpha$ for $\overline{z}_1^{\alpha_1} \ldots \overline{z}_r^{\alpha_r}$, and $|\alpha| = \alpha_1 +\ldots+ \alpha_r$. Let $z_1,\ldots,z_r \in \mathfrak{m}$ be representatives of $\overline{z}_1,\ldots,\overline{z}_r$. We already know that $z_1,\ldots,z_r$ generate $\mathfrak{m}$ (Nakayama's Lemma), and shall show that they form an A-regular sequence. We begin by asserting that, obviously,

$$\theta\Big(\sum_{|\alpha|=j} \overline{c}_\alpha \overline{z}^\alpha\Big) = \sum_{|\alpha|=j} c_\alpha z^\alpha \pmod{\mathfrak{m}^{j+1}}$$

where $\overline{c}_\alpha \in k = A/\mathfrak{m}$, $c_\alpha \in A$, their representatives. Hence, since θ is injective, the relation $\sum_{|\alpha|=j} c_\alpha z^\alpha \in \mathfrak{m}^{j+1}$, $c_\alpha \in A$ implies $\theta\big(\sum_{|\alpha|=j} \overline{c}_\alpha \overline{z}^\alpha\big) = 0$, whence $c_\alpha \in \mathfrak{m}$.

Assume now that $z_1,\ldots,z_r$ do not form an A-regular sequence. Then, for some j, $1 \leqq j \leqq r$, there exists an $x \in A$, $x \notin A z_1 +\ldots+ A z_{j-1}$ and $xz_j \in A z_1 +\ldots+ A z_{j-1}$. That, is we have an equation of the form

$$xz_j = y_1 z_1 +\ldots+ y_{j-1} z_{j-1}.$$

Since θ is surjective, we have, for some t,

$$xz_j = \sum_{|\alpha|=t} c_\alpha z_j z^\alpha \qquad (\text{mod. } \mathfrak{m}^{t+2})$$

where at least one c_α for an α with $\alpha_1 = \alpha_2 = \ldots = \alpha_{j-1} = 0$ is such that $c_\alpha \notin \mathfrak{m}$. However, in the expression of $y_1 z_1 + \ldots + y_{j-1} z_{j-1}$ as $\sum_{|\alpha| \leqq t+1} d_\alpha z^\alpha$ (mod. $\mathfrak{m}^{t+2}$), all the coefficients d_α such that $d_\alpha \notin \mathfrak{m}$ correspond to multiindices α for which $\alpha_1, \alpha_2, \ldots, \alpha_{j-1}$ are not all 0. We thus reach a contradiction.

d) $\Longrightarrow$ c). Let $z_1, \ldots, z_r$ be an A-regular sequence which forms a set of generators of $\mathfrak{m}$. Then, by proposition 2.5,

$$r \geqq \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2) \geqq n$$

and by the definition of depth (A) and theorem 3.1

$$n \geqq \text{depth (A)} \geqq r.$$

Hence $r = \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2) = n$, and c) follows:

c) $\Longrightarrow$ a). We proceed by contradition, i.e. we assume $\ker \theta \neq 0$. For brevity's sake we write $S = S_k(\mathfrak{m}/\mathfrak{m}^2)$; $G = gr_{\mathfrak{m}}(A)$. We have the exact sequence

$$0 \to \mathfrak{J} \to S \xrightarrow{\theta} G \to 0$$

with $\mathfrak{J} \neq 0$. Since θ is homogeneous, $\mathfrak{J}$ is a homogeneous ideal in S, and $\mathfrak{J}_0 = \mathfrak{J}_1 = 0$, since $S_0 = G_0 = k$, $S_1 = G_1 = \mathfrak{m}/\mathfrak{m}^2$. Let h be the smallest positive integer such that $\mathfrak{J}_h \neq 0$. Let $u \in \mathfrak{J}_h$, $u \neq 0$. Then clearly, S being an integral domain, $S_{s-h} \xrightarrow{\sim} uS_{s-h}$, $s \geqq h$ $(a \rightarrow ua)$ and $uS_{s-h} \subset \mathfrak{J}_s$. Hence, (since $\mathrm{rank}_k(\mathfrak{m}/\mathfrak{m}^2) = n$, by c) $\Longrightarrow$ b)),

$$\mathrm{length}_k(\mathfrak{J}_s) \geqq \mathrm{length}_k(S_{s-h}) = \binom{s-h+n-1}{n-1}$$

The exact sequence

$$0 \rightarrow \mathfrak{J}_s \rightarrow S_s \rightarrow G_s \rightarrow 0$$

shows $\mathrm{length}_k(G_s) = \mathrm{length}_k(S_s) - \mathrm{length}_k(\mathfrak{J}_s) =$

$$= \binom{s+n-1}{n-1} - \mathrm{length}_k(\mathfrak{J}_s) \leqq \binom{s+n-1}{n-1} - \binom{s-h+n-1}{n-1}$$

and $\binom{s+n-1}{n-1} - \binom{s-h+n-1}{n-1}$ is a polynomial in s of degree at most $(n-2)$.

From the exact sequence

$$0 \rightarrow G_s \rightarrow A/\mathfrak{m}^{s+1} \rightarrow A/\mathfrak{m}^s \rightarrow 0$$

we have, with the notations of section 2,

$$\mathrm{length}(G_s) = P_{\mathfrak{m}}(A, s+1) - P_{\mathfrak{m}}(A, s).$$

By theorem 2.3 and a well-known result of polynomial theory we have

$$P_{\mathfrak{m}}(A, s) = c_n\binom{s+n}{n} + c_{n+1}\binom{s+n-1}{n-1} + \dots c_0,$$

with $c_i \in \mathbb{Q}$ (actually, since $P_{\mathfrak{m}}(A, s) \in \mathbb{Z}$ one easily sees that

$c_i \in Z$), and $c_n \neq 0$. Hence $P_{\mathfrak{m}}(A, s+1) - P_{\mathfrak{m}}(A, s) = c_n\binom{s+n}{n-1}$ + terms of lower degree. Hence length(G_s) is a polynomial of degree $n - 1$ for $s \gg 0$. We have reached a contradiction and a) is proved. If $\dim(A) = 0$, $\mathfrak{m} = (0)$ and the theorem is trivial. The theorem is proved.

Definition 4.1. A local ring A is said to be regular if it satisfies either a), b), c), or d) of theorem 4.1.

Corollary 4.1. Let A be a regular local ring. Then

i) A is an integral domain

ii) A is C-M

iii) A is integrally closed.

Proof: i) $S_k(\mathfrak{m}/\mathfrak{m}^2)$ is trivially an integral domain; by a) of theorem 4.1 so is $gr_{\mathfrak{m}}(A)$. Hence A cannot have zero divisors. (B.C.A., III, 2,3).

ii) In the proof of d) $\Longrightarrow$ c) in theorem 4.1 we showed

$$r \leqq \operatorname{depth}(A) \leqq \dim(A) \leqq \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2) \leqq r$$

where r is the number of elements in an A-regular sequence which generates $\mathfrak{m}$. Hence depth(A) = dim(A) and A is C-M.

iii) $S_k(\mathfrak{m}/\mathfrak{m}^2)$ is trivially integrally closed B.C.A., V., §1 Corollary 3. Hence so is $gr_{\mathfrak{m}}(A)$, and by proposition 15 of B.C.A., V, §1, A is integrally closed.

We give some examples of regular local rings. It is clear from c) of theorem 4.1 that if $\dim(A) = 0$, then the regularity of A implies that A is a field, and conversely.

If A is a regular local ring and $\dim(A) = 1$, then A is a discrete valuation ring. In fact, by theorem 4.1, $\mathfrak{m}$ is principal, and we can apply proposition 9 of B.C.A., VI, §3.

Finally, any ring A of power series in n variables $T_1,\ldots,T_n$ over a field is a regular local ring. This follows from the fact that $T_1,\ldots,T_n$ generate $\mathfrak{m}$ and form an A-regular sequence.

We globalize the notion of regular rings as follows:

Definition 4.2. A ring A is said to be regular if, for every maximal ideal $\mathfrak{m}$ of A, the local ring $A_{\mathfrak{m}}$ is regular.

We shall show later on that the polynomial ring in n variables over a field k is a regular ring.

Definition 4.3. Let A be a regular local ring. A set of generators of $\mathfrak{m}$ which forms an A-regular sequence is said to be a regular system of parameters of A.

Remark. Theorem 4.1 guarantees the existence of regular systems of parameters in any regular local ring A.

We also observe that, due to linguistical shortcomings, not every system of parameters of A which forms an A-regular sequence is necessarily a regular system of parameters, (see Definition 2.5) while every regular system of parameters is a system of parameters and an A-regular sequence.

We investigate the properties of regularity under quotient operations. We have

Proposition 4.1. Let A be a noetherian local ring, $x_i \in \mathfrak{m}$, $i = 1,\ldots,r$, $\mathfrak{J} = x_1 A +\ldots+ x_r A$. The following three conditions are equivalent:

a) A is regular and $\{x_1,\ldots,x_r\}$ is contained in a regular system of parameters.

b) A is regular and the equivalence classes of $x_1,\ldots,x_r$

in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent

c) $\{x_1,\ldots,x_r\}$ is contained in a system of parameters, and $A/\mathfrak{I}$ is regular.

Furthermore the above three conditions imply that $\mathfrak{I}$ is prime.

Proof: a) $\Longleftrightarrow$ b). By Nakayama's lemma and the proof of theorem 4.1, any regular system of parameters gives rise to a k-basis of $\mathfrak{m}/\mathfrak{m}^2$ and conversely.

a) $\Longrightarrow$ c). Let $\mathfrak{n} = \mathfrak{m}\cdot A/\mathfrak{I}$, the maximal ideal of $A/\mathfrak{I}$. Consider the exact sequence

$$0 \to (\mathfrak{m}^2 + \mathfrak{I})/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0$$

(since we have the exact sequence $0 \to \mathfrak{m}^2 + \mathfrak{I} \to \mathfrak{m} \to \mathfrak{n}/\mathfrak{n}^2 \to 0$, we have $\mathfrak{m}/(\mathfrak{m}^2 + \mathfrak{I}) \simeq \mathfrak{n}/\mathfrak{n}^2$).

Let $n = \dim(A)$. Now, by a) and proposition 2.7 we have $\dim(A/\mathfrak{I}) = n - r$, and by b) (which has been shown to follow from a)) $\operatorname{rank}_k((\mathfrak{m}^2 + \mathfrak{I})/\mathfrak{m}^2) = r$ (since the equivalence classes of $x_1,\ldots,x_r$ in $(\mathfrak{m}^2 + \mathfrak{I})/\mathfrak{m}^2$ clearly generate it). Hence $\operatorname{rank}_k(\mathfrak{n}/\mathfrak{n}^2) = n - r = \dim(A/\mathfrak{I})$, and $A/\mathfrak{I}$ is regular. Hence c) is proved, since it is already assumed in a) that $\{x_1,\ldots,x_r\}$ is contained in a system of parameters.

c) $\Longrightarrow$ a). Since $A/\mathfrak{I}$ is regular, by proposition 2.7 and theorem 4.1 applied to $A/\mathfrak{I}$ we have

$$n - r = \dim(A/\mathfrak{I}) = \operatorname{rank}(\mathfrak{n}/\mathfrak{n}^2)$$

Since $x_1,\ldots,x_r$ generate $((\mathfrak{m}^2 + \mathfrak{I})/\mathfrak{m}^2)$ we have $\operatorname{rank}((\mathfrak{m}^2 + \mathfrak{I})/\mathfrak{m}^2) \leq r$. Hence $\operatorname{rank}(\mathfrak{m}/\mathfrak{m}^2) \leq n$. But rank$(\mathfrak{m}/\mathfrak{m}^2) \geq n$ always, whence rank$(\mathfrak{m}/\mathfrak{m}^2) = n$ and A is regular.

Trivially, if $A/\mathfrak{J}$ is regular, $\mathfrak{J}$ is a prime ideal, since $A/\mathfrak{J}$ is an integral domain. The proposition is proved.

Corollary 4.2. Let A be a noetherian local ring, $t \in \mathfrak{m}$. Then the following conditions are equivalent:

a) A is regular, $t \notin \mathfrak{m}^2$

b) A/tA is regular and t does not belong to any minimal prime of A.

Proof: Apply propositions 4.1 and 3.1.

By proposition 4.1, we have that, if A is regular, and $\mathfrak{J}$ is generated by a subset of a regular system of parameters, then $A/\mathfrak{J}$ is regular. We sharpen this result in the following

Proposition 4.2. Let A be a noetherian regular local ring, $\mathfrak{J}$ an ideal of A. Then $A/\mathfrak{J}$ is regular if, and only if, $\mathfrak{J}$ is generated by a subset of a regular system of parameters.

Proof: The "if" part has been proved in proposition 4.1. Assume now that $A/\mathfrak{J}$ is regular, and let $n = \dim(A)$, $n - r = \dim(A/\mathfrak{J})$. Again we consider the exact sequence

$$0 \to ((\mathfrak{m}^2 + \mathfrak{J})/\mathfrak{m}^2) \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0$$

where $\mathfrak{n}$ is as in the proof of proposition 4.1. We know that rank$(\mathfrak{m}/\mathfrak{m}^2) = n$, and rank$(\mathfrak{n}/\mathfrak{n}^2) = n - r$. Hence rank$((\mathfrak{m}^2 + \mathfrak{J})/\mathfrak{m}^2) = r$. Let $x_1,\ldots,x_r$ be elements of $\mathfrak{J}$ which are linearly independent mod $\mathfrak{m}^2$ and whose equivalence classes mod $\mathfrak{m}^2$ form a k-basis of $((\mathfrak{m}^2 + \mathfrak{J})/\mathfrak{m}^2)$. By extending the set of such equivalence classes to a k-basis of

$\mathfrak{m}/\mathfrak{m}^2$, and using theorem 4.1 we see that $\{x_1,\ldots,x_r\}$ is contained in a regular system of parameters. Let $\mathfrak{J}' = x_1 A + \ldots + x_r A$. Clearly $\mathfrak{J}' \subset \mathfrak{J}$. By proposition 4.1 $\mathfrak{J}'$ is a prime ideal and $\dim(A/\mathfrak{J}') = n - r$. But $\mathfrak{J}$ is also a prime ideal (since $A/\mathfrak{J}$ is regular) and we have $\dim(A/\mathfrak{J}) = \dim(A/\mathfrak{J}')$. The exact sequence

$$0 \to \mathfrak{J}/\mathfrak{J}' \to A/\mathfrak{J}' \to A/\mathfrak{J} \to 0$$

shows that $\mathfrak{J} = \mathfrak{J}'$ (otherwise $\mathfrak{J}\cdot A/\mathfrak{J}'$ is a non zero prime ideal of $A/\mathfrak{J}'$ and $\dim(A/\mathfrak{J}') > \dim(A/\mathfrak{J})$).

$\longleftrightarrow$

We now wish to show that, in the classical case, the notion of regularity we have given is equivalent to the classical one given in terms of the rank of a certain Jacobian.

We let $B = \mathbb{C}[X_1,\ldots,Y_n]$, $\mathfrak{a} \subset B$ an ideal, $\mathfrak{m} \supset \mathfrak{a}$ a maximal ideal, $A = B/\mathfrak{a}$. Then $\mathfrak{m}$ is generated by n linear polynomials of the form $X_i - \alpha_i$, $i = 1,\ldots,n$. Let $\mathfrak{a}$ be generated by the polynomials

$$P_\lambda,\ \lambda = 1,\ldots,t.$$

Let $\dim A_{\mathfrak{m}/\mathfrak{a}} = n - r$. We assert:

<u>Proposition 4.3</u>. $A_{\mathfrak{m}/\mathfrak{a}}$ is regular if, and only if, the rank of the matrix $(\frac{\partial P_\lambda}{\partial X_i}(\alpha_1,\ldots,\alpha_n))$ is r.

<u>Proof</u>: We have $A_{\mathfrak{m}/\mathfrak{a}} \cong B_{\mathfrak{m}}/\mathfrak{a}B_{\mathfrak{m}}$. By proposition 4.2 it follows that $A_{\mathfrak{m}/\mathfrak{a}}$ is regular, if, and only if, $\mathfrak{a}B_{\mathfrak{m}}$ is

generated by r elements, which can be imbedded in a $B_{\mathfrak{m}}$-regular system of parameters (since $B_{\mathfrak{m}}$ can be seen to be regular, $\mathfrak{m}B_{\mathfrak{m}}$ being generated by $\{X_1 - \alpha_1, \ldots, X_n - \alpha_n\}$). Furthermore we may assume that such r elements are actually in B, say $Q_1, \ldots, Q_r$. Since both sets $\{Q_1, \ldots, Q_r\}$ and $\{P_\lambda\}$ $\lambda = 1, \ldots, t$ generate $\mathfrak{a}B_{\mathfrak{m}}$ one easily sees that the ranks of the two matrices $((\frac{\partial Q_i}{\partial X_j}(\alpha_1, \ldots, \alpha_n)))$, $((\frac{\partial P_\lambda}{\partial X_j}(\alpha_1, \ldots, \alpha_n)))$ are equal.

Now, if $D: B_{\mathfrak{m}} \to B_{\mathfrak{m}}$ is any derivation, then clearly $D(\mathfrak{m}^2) \subset \mathfrak{m}$. Hence if φ denotes the composition

$$B_{\mathfrak{m}} \xrightarrow[D]{} B_{\mathfrak{m}} \to B_{\mathfrak{m}}/\mathfrak{m}B_{\mathfrak{m}} = \mathbb{C}$$

we have $\varphi(\mathfrak{m}^2) = 0$, and hence φ defines a $\mathbb{C}$-linear form

$$\tilde{\varphi}: \mathfrak{m}/\mathfrak{m}^2 \to \mathbb{C}$$

If $\varphi_j = \frac{\partial}{\partial X_j}$, $Q(X_1, \ldots, X_n) \in \mathfrak{m}$, then one immediately sees that $\tilde{\varphi}_j(Q) = \frac{\partial Q}{\partial X_j}(\alpha_1, \ldots, \alpha_n)$. Also it is clear that $\{\tilde{\varphi}_j\}$ $j = 1, \ldots, n$ is a set of n linearly independent forms over $\mathfrak{m}/\mathfrak{m}^2$. Since the equivalence classes of $Q_1, \ldots, Q_r$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent, it follows that rank $((\tilde{\varphi}_j(Q_i))) = r$, whence rank $((\frac{\partial P_\lambda}{\partial X_j}(\alpha_1, \ldots, \alpha_n))) = r$.

Conversely, if rank $((\frac{\partial P_\lambda}{\partial X_j}(\alpha_1, \ldots, \alpha_n))) = r$, then r of the P_λ's are linearly independent mod $\mathfrak{m}^2$, and by theorem 4.1 (since $B_{\mathfrak{m}}$ is regular of dimension n), they are a subset of a regular system of generators of $\mathfrak{m}$. Furthermore they generate $\mathfrak{a} + \mathfrak{m}^2/\mathfrak{m}^2$. Hence, by Nakayama's lemma, they generate $\mathfrak{a}B_{\mathfrak{m}}$ and we are done.

Classically, a point $(\alpha_1,\ldots,\alpha_n) \in \mathbb{C}^n$, belonging to the algebraic set defined by the ideal $\mathfrak{a}$ is called <u>simple</u> if the matrix $((\frac{\partial P_\lambda}{\partial X_j}(\alpha_1,\ldots,\alpha_n)))$ has rank equal to $n - \dim(A_{\mathfrak{m}/\mathfrak{a}})$.

Thus we have that <u>a point is simple if, and only if, its local ring is regular</u>.

We recall briefly the definition of a parametric representation of a variety, again in the classical case.

Let $\mathfrak{a} \subset \mathbb{C}[X_1,\ldots,X_n]$ be an ideal, and let V be the subset of $\mathbb{C}^n$ consisting of the common zeros of $\mathfrak{a}$. We say that V admits the parametric representation by polynomials

$$(*)\quad \begin{cases} X_1 = P_1(T_1,\ldots,T_m) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ X_n = P_n(T_1,\ldots,T_m) \end{cases}$$

if the homomorphism $\varphi: \mathbb{C}[X_1,\ldots,X_n] \to \mathbb{C}[T_1,\ldots,T_m]$ defined by $\varphi(X_i) = P_i(T_1,\ldots,T_m)$ has kernel $\mathfrak{a}$. Using the Hilbert Nullstellensatz one easily sees that this means that exactly all points of V are obtained by substituting some appropriate values for $T_1,\ldots,T_m$ in (*). Let now $\mathfrak{m} \subset \mathbb{C}[X_1,\ldots,X_n]$ be a maximal ideal with $\mathfrak{m} \supset \mathfrak{a}$, and let $\dim(A_{\mathfrak{m}/\mathfrak{a}}) = n - r$, where $A = \mathbb{C}[X_1,\ldots,X_n]/\mathfrak{a}$. Let $(\alpha_1,\ldots,\alpha_n)$ be the point of V corresponding to $\mathfrak{m}$, and let $\mathfrak{a}$ be generated by $\{Q_\lambda\}$ $1 \leqq \lambda \leqq t$. Let $(t_1,\ldots,t_m) \in \mathbb{C}^m$ such that $P_i(t_1,\ldots,t_m) = \alpha_i$. If the matrix $((\frac{\partial P_i}{\partial T_j}(t_1,\ldots,t_m)))$ has rank $n - r$, then the homomorphism

$$\theta : \mathbb{C}dX_1 \oplus \ldots \oplus \mathbb{C}dX_n \to \mathbb{C}dT_1 \oplus \ldots \oplus \mathbb{C}dT_m$$

given by $\theta\left(\sum_{i=1}^{n} c_i \, dX_i\right) = \sum_{i=1}^{n} c_i \sum_{j=1}^{m} \frac{\partial P_i}{\partial T_j}(t_1,\ldots,t_m)\, dT_j$ has image of dimension $n - r$ and kernel generated by $\sum_{i=1}^{n} \frac{\partial Q_\lambda}{\partial X_i}(\alpha_1,\ldots,\alpha_n)\, dX_i$. Hence rank $\left(\frac{\partial Q_\lambda}{\partial X_i}(\alpha_1,\ldots,\alpha_n)\right) = r$, and $\{\alpha_1,\ldots,\alpha_n\}$ is a regular point of V. The example

$$\begin{cases} X = T^2 \\ Y = T^2 \\ Z = T^2 \end{cases}$$

where $n = 3$, $r = 2$, easily show (take $X = Y = Z = T = 0$) that the converse of the above statement is false. (In fact here V is the line $X = Y = Z$, and proposition 4.1 shows that the origin is a simple point on such line, while rank $((0,0,0)) = 0$).

Remark. The concept of regularity enables us to solve the problem of distinguishing the local ring of the three examples given in the introduction. In fact, while the third local ring is regular, the first two are not (apply Proposition 4.3).

We introduce one last numerical notion to be attached to a local ring.

Definition 4.4. Let A be a ring, M an A-module. A projective resolution of M of length n is an exact sequence

$$0 \to L_n \to L_{n-1} \to \ldots \to L_1 \to L_0 \to M \to 0$$

where L_i is a projective A-module, $i = 0, \ldots, n$.

Definition 4.5. Let M be an A-module. Then the projective dimension of M, dim. proj. (M) is defined as the infimum of the lengths of all projective resolutions of M. The cohomological dimension of A, coh. dim(A), is defined as the supremum of the projective dimensions of all A-modules.

We state, without proof, two of the fundamental theorems concerning the notion of coh. dim(A). The proofs involve tools whose introduction would take us far afield, and of which we shall have no need in the remaining part of this work.

Theorem 4.2. (Hilbert-Serre) Let A be a noetherian local ring. Then one (and only one) of the following two alternatives hold

1) coh. $\dim(A) = \infty$
2) A is regular and coh. $\dim(A) = \dim(A)$

Corollary 4.3. If A is a noetherian regular local ring, and $\mathfrak{p} \in \operatorname{Spec}(A)$, then $A_{\mathfrak{p}}$ is regular.

Proof: The homomorphism $A \to A_{\mathfrak{p}}$ shows that every $A_{\mathfrak{p}}$-module is an A-module. Now, for noetherian local rings the notions of projective and flat modules are equivalent. Since $A_{\mathfrak{p}}$ is A-flat, if L is $A_{\mathfrak{p}}$-flat and

$$0 \to M \to N$$

is an exact sequence of A-modules, we have

$$0 \to A_{\mathfrak{p}} \otimes_A M \to A_{\mathfrak{p}} \otimes_A N \text{ is exact}$$

and

$$0 \to L \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \otimes_A M) \to L \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \otimes_A N) \quad \text{is exact}$$

or

$$0 \to L \otimes_A M \to L \otimes_A N \quad \text{is exact,}$$

and L is A-flat. Hence every projective resolution of an $A_{\mathfrak{p}}$-module M is a projective resolution of the <u>A-module</u> M, and we obtain the following inequality

$$\text{coh dim}(A_{\mathfrak{p}}) \leq \text{coh dim}(A)$$

from which the corollary follows immediately via Theorem 4.2.

<u>Theorem 4.3</u>. (Auslander-Buchsbaum) Every noetherian regular local ring is a unique factorization domain.

For the proofs of Theorems 4.2 and 4.3 we refer the reader to A. Grothendieck's "Elements de Geometrie Algebrique", Chapter 0_{IV} (The portion of Chapter 0 preceding Chapter IV), section 17.3, and Chapter IV, section 21.11.

The problem of classifying all regular local rings is at the moment unsolved, and probably unsolvable as stated. In fact, if X, Y, are two irreducible schemes and $\varphi: X \to Y$ a morphism such that, for some $x \in X$, $\mathcal{O}_{x,X} \simeq \mathcal{O}_{\varphi(x),Y}$ and both are regular, then, under certain appropriate finiteness conditions, φ is birational. Hence to classify regular local rings requires first a classification of birationally equivalent schemes, a very tall order at the moment.

We complete this section with some results concerning the two notions of depth and regularity.

We call a noetherian ring A normal if A is the direct sum of integrally closed integral domains, and reduced if its nilradical is 0.

Definition 4.6. Let A be a noetherian ring, k a non-negative integer.

1) We say that A satisfies condition (S_k) if, for every $\mathfrak{p} \in \operatorname{Spec}(A)$

$$\operatorname{depth}(A_{\mathfrak{p}}) \geqq \min[k, \dim(A_{\mathfrak{p}})]$$

2) We say that A satisfies condition (R_k) if, for every $\mathfrak{p} \in \operatorname{Spec}(A)$

$$\dim A_{\mathfrak{p}} \leqq k \text{ implies } A_{\mathfrak{p}} \text{ is regular.}$$

Corollary 4.4. a) S_o always holds:

b) A satisfies (S_k) if, and only if, for every $\mathfrak{p} \in \operatorname{Spec}(A)$, depth $A_{\mathfrak{p}} \geqq k$ and, if $\dim(A_{\mathfrak{p}}) \leqq k$, then $A_{\mathfrak{p}}$ is C-M.

Proof: a) is obvious. To prove b) we recall that $\operatorname{depth}(A_{\mathfrak{p}}) \leq \dim(A_{\mathfrak{p}})$. Therefore, if $k < \dim(A_{\mathfrak{p}})$, $\operatorname{depth}(A_{\mathfrak{p}}) \geqq k$ is equivalent to the requirement of (S_k), and if $k \geqq \dim(A_{\mathfrak{p}})$, then $\operatorname{depth}(A_{\mathfrak{p}}) = \dim(A_{\mathfrak{p}})$ (i.e. $A_{\mathfrak{p}}$ is C-M) is again equivalent to the requirement of (S_k).

Proposition 4.4. (S_k) is equivalent to the following condition: For every $t \in A$ and every A_t-regular sequence $\{x_1,\ldots,x_r\}$, $r < k$, the A_t-module $A_t/x_1A_t + \ldots + x_rA_t$ has no immersed primes.

Proof: $k = 1$, whence $r = 0$. We will show that S_1 is equivalent to saying that A has no immersed primes. Let $\mathfrak{p}$ be a prime of A which is not minimal. Then $\dim(A_{\mathfrak{p}}) \geqq 1$, whence by (S_1) $\operatorname{depth}(A_{\mathfrak{p}}) \geqq 1$.

Hence $\mathfrak{p} \notin \operatorname{Ass}(A)$ (if $\mathfrak{p}$ is the annihilator of $a \in A$, then $\frac{a}{1} \neq 0$ in $A_{\mathfrak{p}}$ and $\mathfrak{p}A_{\mathfrak{p}}$ is the annihilator of it).

Conversely, if A has no immersed primes, let $\mathfrak{p} \in \mathrm{Spec}(A)$.

If $\mathfrak{p} \in \mathrm{Ass}(A)$, then $\mathfrak{p}$ is minimal, hence $\min[1, \dim A_{\mathfrak{p}}] = 0$ and $\mathrm{depth}(A_{\mathfrak{p}}) \geqq 0$. If $\mathfrak{p} \notin \mathrm{Ass}\, A$, then $\mathfrak{p}$ is not minimal and $\min[1, \dim A_{\mathfrak{p}}] = 1$. If $\mathrm{depth}(A_{\mathfrak{p}}) = 0$, then by theorem 3.1, $\mathfrak{p}A_{\mathfrak{p}} \in \mathrm{Ass}(A_{\mathfrak{p}})$ whence $\mathfrak{p} \in \mathrm{Ass}(A)$, a contradiction. Hence A satisfies (S_1).

We proceed by induction on k. Let $k > 1$.

Let A satisfy (S_k), and let $\{x_1, \ldots, x_r\}$, $r < k$ be an A_t-regular sequence. Let $B = A_t/x_1A_t$. From proposition 3.1 and theorem 3.1 we see that B satisfies (S_{k-1}) (since, for every $\mathfrak{p} \in \mathrm{Spec}(A_t)$ with $x_1 \in \mathfrak{p}$, x_1 is $A_{\mathfrak{p}}$-regular) hence $B/x_2 B + \ldots + x_r B = A_t/x_1A_t + \ldots + x_rA_t$ has no imbedded primes. Conversely, assume that for $t \in A$, the A_t-module $A_t/x_1A_t + \ldots + x_rA_t$ has no immersed primes, for every A_t-regular sequence $\{x_1, \ldots, x_r\}$ with $r < k$.

By the induction assumption, A satisfies (S_{k-1}). Let $\mathfrak{p} \in \mathrm{Spec}(A)$. We proceed in steps.

<u>Case 1</u>. $\dim(A_{\mathfrak{p}}) = r < k$. Since A satisfies (S_{k-1}) we have

$$\mathrm{depth}(A_{\mathfrak{p}}) \geqq \min(k-1, r) = r$$

whence $\mathrm{depth}(A_{\mathfrak{p}}) \geqq \min(k, \dim(A_{\mathfrak{p}}))$.

<u>Case 2</u>. $\dim(A_{\mathfrak{p}}) = r \geqq k$. Again, since A satisfies (S_{k-1}) we have $\mathrm{depth}(A_{\mathfrak{p}}) \geqq \min(k-1, r) = k - 1$. Hence there exists a sequence $x_1, \ldots, x_{k-1} \in \mathfrak{p}A_{\mathfrak{p}}$ which is $A_{\mathfrak{p}}$-regular, and we may assume $x_i \in \mathfrak{p}$. Then $x_1, \ldots, x_{k-1}$ is an A_t-regular sequence for some $t \notin \mathfrak{p}$. Therefore, by assumption $B_t = A_t/x_1A_t + \ldots + x_{k-1} A_t$ has no immersed primes. Since

$\dim(B_{\mathfrak{p}B}) = \dim(A_{\mathfrak{p}}/x_1 A_{\mathfrak{p}} + \ldots + x_{k-1} A_{\mathfrak{p}}) = \dim(A_{\mathfrak{p}}) - (k-1) \geqq 1$, and B_t has no immersed primes, it follows that $\mathfrak{p} \notin \mathrm{Ass}(B_t)$. Hence $\mathrm{depth}(B_{\mathfrak{p}}) \geqq 1$. We then obtain

$$1 \leqq \mathrm{depth}(A_{\mathfrak{p}}/x_1 A_{\mathfrak{p}} + \ldots + x_{k-1} A_{\mathfrak{p}}) = \mathrm{depth}(A_{\mathfrak{p}}) - (k-1)$$

whence $\mathrm{depth}(A_{\mathfrak{p}}) \geqq k$, and (S_k) is proved.

We are now in the position of obtaining two criterions for A to be normal, and reduced respectively.

Proposition 4.5. A is reduced if, and only if, A satisfies both (S_1) and (R_0).

Proof: We observe that clearly (R_0) is equivalent to saying that, for all minimal primes $\mathfrak{p}$ of A, (whence $\dim(A_{\mathfrak{p}}) = 0$) $A_{\mathfrak{p}}$ is a field.

Now assume that A is reduced. Then, if $\mathfrak{p}$ is a minimal prime of A, $\mathfrak{p} A_{\mathfrak{p}} = (0)$ (since $0 = \bigcap_{\substack{\mathfrak{q} \subset A \\ \mathfrak{q} \text{ minimal}}} \mathfrak{q}$, and $\mathfrak{q} A_{\mathfrak{p}} = 0$ for $\mathfrak{q} \neq \mathfrak{p}$ and minimal), whence $A_{\mathfrak{p}}$ is a field and (R_0) follows. To prove that A satisifes (S_1) we proceed by contradiction. If A does not satisfy (S_1) then, by proposition 4.4, there exists a prime $\mathfrak{q} \in \mathrm{Ass}(A)$ which is not minimal. Let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k$ be the minimal primes of A. Then $\mathfrak{q} \not\subset \bigcup_{i=1}^{k} \mathfrak{p}_i$, (since $\mathfrak{q}$ is not minimal) whence there exists $x \in \mathfrak{q}$, $x \notin \bigcup_{i=1} \mathfrak{p}_i$. Since $x \in \mathfrak{q} \in \mathrm{Ass}(A)$, x is a zero divisor in A. Let x_i be the image of x under $A \xrightarrow{\varphi_i} A_{\mathfrak{p}_i}$ $i = 1, \ldots, k$. We have $xt = 0$ for some non zero t. Then $x_i \varphi_i(t) = 0$. Since

$x \notin \mathfrak{p}_i$, x_i is a unit in $A_{\mathfrak{p}_i}$, whence $\varphi_i(t) = 0$, $i = 1,\ldots,k$. Then (by the definition of $A_{\mathfrak{p}_i}$) $t \in \mathfrak{p}_i$, $i = 1,\ldots,k$. Since A is reduced, $\bigcap_{i=1}^{k} \mathfrak{p}_i = 0$, whence $t = 0$ a contradiction.

Assume, conversely, that A satisfies both (S_1) and (R_o). Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_k$ be again the minimal prime ideals of A. We wish to show that A is reduced, i.e. that $\bigcap_{i=1}^{k} \mathfrak{p}_i = 0$. Assume that there exists a non zero $z \in \bigcap_{i=1}^{k} \mathfrak{p}_i$. By (R_o), $A_{\mathfrak{p}_i}$ is a field, whence $\mathfrak{p}_i A_{\mathfrak{p}_i} = 0$, $i = 1,\ldots,k$, whence $\varphi_i(z) = 0$, $i = 1,\ldots,k$. Therefore, for every i, there exists $s_i \notin \mathfrak{p}_i$ such that $s_i \cdot z = 0$, i.e. $\operatorname{ann}(z) \not\subset \mathfrak{p}_i$, $i = 1,\ldots,k$, whence $\operatorname{ann}(z) \not\subset \bigcup_{i=1}^{k} \mathfrak{p}_i$. By (S_1), since A has no imbedded primes, $\bigcup_{i=1}^{k} \mathfrak{p}_i = \bigcup_{\mathfrak{p} \in \operatorname{Ass}(A)} \mathfrak{p}$ = the set of zero divisors of A. We have that, for a $z \neq 0$, there exists a non zero divisor of A which annihilates z, clearly a contradiction, Q.E.D.

<u>Proposition 4.6</u>. (Serre) Let A be noetherian. Then A is normal if, and only if, A satisfies both (S_2) and (R_1).

<u>Proof</u>: We remark first of all that A satisfies both (S_2) and (R_1) if, and only if, the following holds:

(*) Let $\mathfrak{p} \in \operatorname{Spec}(A)$. If $\dim(A_{\mathfrak{p}}) \leq 1$, then $A_{\mathfrak{p}}$ is regular. If $\dim A_{\mathfrak{p}} \geq 2$, then $\operatorname{depth}(A_{\mathfrak{p}}) \geq 2$.

We leave the verification of our remark to the reader.

Now, if A is normal, so is $A_{\mathfrak{p}}$. Hence, if $\dim(A_{\mathfrak{p}}) \leq 1$, then $A_{\mathfrak{p}}$ is either a field (which is regular) or, by the discussion on page 38, a valuation ring, hence by proposition 9 in B.C.A., VI, §3, no. 6, A is a discrete valuation ring. Hence $A_{\mathfrak{p}}$ is regular, and (R_1) is satisfied.

To prove that (S_2) is satisfied we have to prove, in addition to the above, that depth$(A_{\mathfrak{p}}) \geqq 2$ when dim$(A_{\mathfrak{p}}) \geqq 2$. This was proved during the proof of remark 3) after definition 3.3.

Assume now that (*) above is satisfied. We remark first of all that, trivially (R_k) implies (R_{k-j}), $j = 0,\ldots,k$, and also that (S_k) implies (S_{k-j}), $j = 0,\ldots,k$. Hence, since (S_2) and (R_1) hold, so do (S_1) and (R_o), and A is reduced by proposition 4.5.

Let $\{\mathfrak{p}_i\}_{i \in I}$ be the minimal primes of A. Note that I is finite and that, since A is reduced $\bigcap\limits_{i \in I} \mathfrak{p}_i = (0)$. Let K_i be the field of fractions of $A/\mathfrak{p}_i$, and let $R = \prod\limits_{i \in I} K_i$. Then the canonical homomorphism $A \to R$ is an injection. Identifying A with its image, we see that we have to prove that A is integrally closed in R. Let $h \in R$ be integral over A. Since R is the total ring of fractions of A, $h = f/g$ for some $f, g \in A$, g is not a zero divisor of A.

From an equation of integral dependence of h over A we get, by multiplication by an appropriate power of g

$$(*) \qquad f^n + \sum_j a_j f^{n-j} g^j = 0 \qquad a_j \in A$$

Let $\mathfrak{p} \in \text{Spec}(A)$ be such that dim$(A_{\mathfrak{p}}) = 1$

By (R_1) $A_{\mathfrak{p}}$ is regular, whence, by corollary 4.1, it is integrally closed. Let $f_{\mathfrak{p}}$, $g_{\mathfrak{p}}$ denote the images of f, g under $A \to A_{\mathfrak{p}}$. Note that $g_{\mathfrak{p}}$ is not a zero divisor in $A_{\mathfrak{p}}$, hence $f_{\mathfrak{p}}/g_{\mathfrak{p}}$ belongs to the field of fractions of $A_{\mathfrak{p}}$. From (*) above, first localizing at $\mathfrak{p}$ and then dividing by $g_{\mathfrak{p}}^n$ we see that $f_{\mathfrak{p}}/g_{\mathfrak{p}}$ is integral over $A_{\mathfrak{p}}$, hence $f_{\mathfrak{p}}/g_{\mathfrak{p}} \in A_{\mathfrak{p}}$ and $f_{\mathfrak{p}}A_{\mathfrak{p}} \subset g_{\mathfrak{p}}A_{\mathfrak{p}}$, whence $(fA)_{\mathfrak{p}} \subset (gA)_{\mathfrak{p}}$. Now, since g is not a zero divisor of A, g is A-regular and, by proposition 4.4, A/gA has no immersed primes containing gA. If $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ denote the minimal primes of A/gA, by the Hauptidealsatz we have $\dim A_{\mathfrak{q}_j} = 1$, and by the previous discussion $(fA)_{\mathfrak{q}_j} \subset (gA)_{\mathfrak{q}_j}$. Let $\mu_j : A \to A_{\mathfrak{q}_j}$ be the canonical homomorphisms. Let $gA = \bigcap_j \mathfrak{q}'_j$ be a primary irredundant decomposition of gA in A. Then $\{\mathfrak{q}_j\} = \mathrm{Ass}(A/\mathfrak{q}'_j)$ and the $\mathfrak{q}_j$ are minimal in Ass(A/gA), $j = 1, \ldots, r$. Then, by proposition 5 of B.C.A., 4, §2, no. 3, we have $\mathfrak{q}'_j = \mu_j^{-1}[(gA)_{\mathfrak{q}_j}]$, i.e. $gA = \bigcap_j \mu_j^{-1}[(gA)_{\mathfrak{q}_j}]$. Clearly $fA \subset \bigcap_j \mu_j^{-1}[(fA)_{\mathfrak{q}_j}]$, whence, by $(fA)_{\mathfrak{q}_j} \subset (gA)_{\mathfrak{q}_j}$, $fA \subset gA$, i.e. $h = f/g \in A$, Q.E.D.

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We end this section with a few examples from classical Algebraic Geometry. Let $A = \mathbb{C}[X_1, \ldots, X_n]/\mathfrak{a}$ be reduced (whence (R_0) and (S_1) hold). In this case the geometrical interpretation of the fact that R_1 holds for A is that the local ring of the generic point of any irreducible subvariety of codimension 1 of Spec(A) is regular, hence a valuation ring. If R_1 does not hold, then there exists a prime $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $\dim(A_{\mathfrak{p}}) = 1$ and $A_{\mathfrak{p}}$ is not regular. In this case $V(\mathfrak{p})$ consists entirely of singular points, i.e. points whose local rings are not regular. To see this let $\mathfrak{q} \in V(\mathfrak{p})$ and assume $A_{\mathfrak{q}}$ is regular. We have $\mathfrak{q} \supset \mathfrak{p}$, whence $A_{\mathfrak{p}} \simeq (A_{\mathfrak{q}})_{\mathfrak{p}A_{\mathfrak{q}}}$. If $A_{\mathfrak{q}}$ is regular, it follows from corollary 4.3 that $A_{\mathfrak{p}}$ is regular, contrary to assumption. In particular, all closed points $\mathfrak{m}$ of $V(\mathfrak{p})$ must be singular, and the problem of determining whether A satisfies (R_1) or not is reduced, via proposition 4.3, to the examination of the rank of the Jacobian of a set of generators of $\mathfrak{a}$.

We illustrate the above by studying the following example: Let

$$\begin{cases} T_0 = X^4 \\ T_1 = X^3Y \\ T_2 = X^2Y^2 \\ T_3 = XY^3 \\ T_4 = Y^4 \end{cases}$$

be the parametric representation of a <u>cone</u> in five dimensional affine space, i.e. we consider the inclusion

$$\mathbb{C}[X^4,\ X^3Y,\ X^2Y^2,\ XY^3,\ Y^4] \to \mathbb{C}[X,\ Y].$$

Let V denote such a cone. The ideal of V is the kernel $\mathfrak{a}$ of the homomorphism $\varphi : \mathbb{C}[T_0, T_1, \ldots, T_4] \to \mathbb{C}[X, Y]$ given by $\varphi(T_i) = X^{4-i}\, Y^i$.

It is a rewarding exercise for the reader to check that $\mathfrak{a}$ is generated by $(T_0 T_2 - T_1^2)$, $(T_1 T_3 - T_2^2)$, $(T_2 T_4 - T_3^2)$, and that V is a two-dimensional cone. The discussion after proposition 4.3 tells us that the origin is the only possible singular point of V. whence (R_1) holds for $\mathbb{C}[T_0,T_1,T_2,T_3,T_4]/\mathfrak{a} \simeq \mathbb{C}[X^4,X^3Y,X^2Y^2,XY^3,Y^4]$.

To see that (S_2) also holds, we need only check that the depth of the local ring of every closed point of V is 2. This is clear for non singular points, since the local ring is then regular, and it is also true at the origin, since X^4, Y^4 $\in$ $\mathbb{C}[X^4,X^3Y,X^2Y^2,XY^3,Y^4]$ is a $\mathbb{C}[X^4,X^3Y,X^2Y^2,XY^3,Y^4]_{\mathfrak{m}}$ - regular sequence, where $\mathfrak{m}$ denotes the maximal ideal generated by X^4,X^3Y,X^2Y^2,XY^3,Y^4.

Consider now $A = \mathbb{C}[X^4,X^3Y,XY^3,Y^4] \subset \mathbb{C}[X,Y]$. Here Spec A is a two dimensional cone in 4-dimensional space, and the discussion after proposition 4.3 tells us that the origin is the only possible singular point of Spec(A). Hence (R_1) holds for A.

Now $(X^2Y^2)^2 = X^4Y^4$ shows that X^2Y^2 is integral over A. However one easily checks $X^2Y^2 \notin A$, whence A is not integrally closed, and (S_2) does not hold for A. Note that this implies $\text{depth}(A_{\mathfrak{m}}) \leq 1$, where $\mathfrak{m}$ denotes the maximal ideal of the origin in Spec(A).

Finally consider $A = \mathbb{C}[X^4,X^3Y,X^3Y,XY^3,Y^4,Z] \subset \mathbb{C}[X,Y,Z]$. Here Spec(A) is a three dimensional variety in five dimensional space, and, again by the discussion after proposition 4.3, (R_1) holds for A.

If $\mathfrak{p} \in \text{Spec}(A)$ and $\dim(A_{\mathfrak{p}}) = 2$, then $\text{Spec}(A/\mathfrak{p}) \nmid \{\mathfrak{m}_a\}$ where $\mathfrak{m}_a$ denotes the maximal ideal of the point (0, 0, a). Hence $A_{\mathfrak{p}}$ is regular and $\text{depth}(A_{\mathfrak{p}}) = 2$.

If $\dim(A_{\mathfrak{p}}) = 3$, and $\mathfrak{p} \neq \mathfrak{m}_a$, then $A_{\mathfrak{p}}$ is again regular and $\mathrm{depth}(A_{\mathfrak{p}}) = 3$. At $\mathfrak{m}_a$ we have $\dim(A_{\mathfrak{m}_a}) = 3$, and $\mathrm{depth}(A_{\mathfrak{m}_a}) \geq 2$, since clearly Y^4, $Z - a$ form an $A_{\mathfrak{m}_a}$-regular sequence. Hence (S_2) holds for A.

Actually $\mathrm{depth}(A_{\mathfrak{m}_a}) = 2$, which gives us an example of a local integral domain which is not a C-M ring, whence A itself is not a C-M ring.

That $\mathrm{depth}(A_{\mathfrak{m}_a}) = 2$ is proved as follows. One can take n=0. Let $A' = \mathbb{C}[X^4, X^3Y, XY^3, Y^4]$. Then $A/ZA \simeq A'$. Let $\mathfrak{m}'$ be the maximal ideal of A' corresponding to the origin of Spec(A'). We know from above that $\mathrm{depth}(A'_{\mathfrak{m}'}) \leq 1$, and $\mathrm{depth}(A_{\mathfrak{m}_o}) \geq 2$. Furthermore we have

$$A'_{\mathfrak{m}'} = A_{\mathfrak{m}_o}/ZA_{\mathfrak{m}_o}$$

and since Z is $A_{\mathfrak{m}_o}$-regular, $1 \geq \mathrm{depth}(A'_{\mathfrak{m}'}) = \mathrm{depth}(A_{\mathfrak{m}_o}) - 1$, whence $\mathrm{depth}(A_{\mathfrak{m}_o}) \leq 2$. We are done.

It is a rewarding exercise for the reader to check that the kernel $\mathfrak{a}$ of the homomorphism $\varphi:\mathbb{C}[T_1,T_2,T_3,T_4] \to \mathbb{C}[X^4,X^3Y,XY^3,Y^4]$ - defined by $\varphi(T_1) = X^4$, $\varphi(T_2) = X^3Y$, $\varphi(T_3) = XY^3$, $\varphi(T_4) = Y^4$ is generated by $T_1{}^2\, T_3 - T_2{}^3$, $T_2\, T_4{}^2 - T_3{}^3$, T_1, $T_4{}^3 - T_3{}^4$, and that no two of the above three polynomials generate $\mathfrak{a}$.

§5. BEHAVIOR UNDER LOCAL HOMOMORPHISM

In this section we let A, B be local rings, unless otherwise specified, with unique maximal ideals $\mathfrak{m}$, $\mathfrak{n}$ respectively.

We recall that a homomorphism $\varphi: A \to B$ is called <u>local</u> if