and

$$\dim(A/\mathfrak{p}) \leqq \dim(A).$$

The latter is geometrically interpreted as follows: If $x \in \overline{y}$, then $\dim(V(j_x)) \leqq \dim(V(j_y))$.

Dimension is a very coarse invariant, i.e. were we to consider the equivalence classes of affine varieties of a given dimension, we would obtain huge classes of highly non isomorphic varieties.

§3. DEPTH

The next numerical invariant we shall study in the notion of depth. We assume throughout this section that A is a noetherian local ring with maximal ideal $\mathfrak{m}$, and that M is a finitely generated A-module.

Definition 3.1. a) an element $x \in A$ is called M-regular if the homomorphism $\varphi : M \to M$ given by $\varphi(m) = xm$ is injective.

b) a sequence $\{x_1, \ldots, x_n\}$ of elements of A is called M-regular if x_i is $M/x_1 M + \ldots + x_{i-1} M$ regular, $1 \leqq i \leqq n$.

Remark. Clearly every $x \notin \mathfrak{m}$ being invertible is M-regular for every module M. Hence we shall confine our attention to those M-regular elements which belong to $\mathfrak{m}$. With regard to b) we state, without proof, the fact that the sequence $\{x_1, \ldots, x_n\}$ is M-regular if, and only if all sequences $\{x_{\sigma(1)}, \ldots, x_{\sigma(n)}\}$ $\sigma \in S_n$ are M-regular, where S_n denotes the group of permutations on n symbols. (Grothendieck, E.G.A., Ch. 0, §15.1, I.H.E.S. no 20) The above statement is false if A is not noetherian.

Clearly any sequence $\{x_1, \ldots, x_n\}$ with $x_1 \notin \mathfrak{m}$ is M-regular for every M (since $M/x_1 M = 0$), hence, keeping in mind the above remark, we shall confine our attention to M-regular sequences $\{x_1, \ldots, x_n\}$ with $x_i \in \mathfrak{m}$.

Definition 3.2. Depth (M) = maximal number of elements in all possible M-regular sequences (of elements of $\mathfrak{m}$!).

We investigate first some of the properties of the notion of M-regularity.

Proposition 3.1.

1) x is M-regular if, and only if, $x \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$.
2) if x is M-regular, $\dim(M/xM) = \dim(M) - 1$.
3) any M-regular sequence is contained in a system of parameters of M.
4) the sequence $\{x_1, \ldots, x_r\}$ is a maximal M-regular sequence if, and only if, one of the following two equivalent conditions hold
   i) $\mathrm{Hom}_A(k, M/x_1 M + \ldots + x_r M) \neq 0$, where $k = A/\mathfrak{m}$.
   ii) $M/x_1 M + \ldots + x_r M$ contains a submodule isomorphic to k.
5) let $\{x_1, \ldots, x_r\}$ be an M-regular sequence. Then $\mathrm{Hom}_A(k, M/x_1 M + \ldots + x_r M) \cong \mathrm{Ext}_A{}^r(k, M) \cong \mathrm{Ext}_A{}^{r-1}(k, M/x_1 M)$.

Proof: 1) Assume x is M-regular, and $x \in \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$.
Then $x \in \mathfrak{p} \in \mathrm{Ass}(M)$, for some $\mathfrak{p}$. Now $\mathfrak{p}$ is the annihilator of some $m \neq 0$, $m \in M$. Therefore the homomorphism

$\varphi : M \to M$, $\varphi(m') = xm'$ is not injective ($\varphi(m) = 0$, $m \neq 0$).

Conversely, assume $x \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$, and let $m \neq 0$, $m \in M$ such that $xm = 0$. Since $m \neq 0$, $0 \neq Am \subset M$, hence $\mathrm{Ass}(Am) \neq \emptyset$ (in fact $M = 0 \iff \mathrm{Ass}(M) = \emptyset$). Now $\mathrm{Ass}(Am) \subset \mathrm{Ass}(M)$ trivially, and $x \in \mathrm{Ann}(Am)$, whence $x \in \bigcap_{\mathfrak{p} \in \mathrm{Ass}(Am)} \mathfrak{p}$, a contradiction.

2) This is an immediate consequence of 1) and proposition 2.6.

3) We prove this by induction on k, where $\{x_1, \ldots, x_k\}$ is an M-regular sequence. If $k = 1$, then x_1 is M-regular and, by 2) above and Proposition 2.7, $\{x_1\}$ can be imbedded in a system of parameters of M. Let $k > 1$. By induction assumption and Proposition 2.7,

$$\dim(M/x_1 M + \ldots + x_{k-1} M) = \dim(M) - k + 1,$$

and from $M/x_1 M + \ldots + x_k M = (M/x_1 M + \ldots + x_{k-1} M)/ x_k(M/x_1 M + \ldots + x_{k-1} M)$ and 2) above we get (since x_k is $M/x_1 M + \ldots + x_{k-1} M$-regular):

$$\dim(M/x_1 M + \ldots + x_k M) = \dim(M) - k$$

whence, again from Proposition 2.7, $\{x_1, \ldots, x_k\}$ can be imbedded in a system of parameters of M.

4) We observe that a sequence $\{x_1, \ldots, x_r\}$ is M-regular and maximal if, and only if, the sequence $\{x_2, \ldots, x_r\}$ is $M/x_1 M$-regular and maximal, hence we are reduced by induction to the case $r = 0$. We observe furthermore that conditions i) and ii) are obviously equivalent, since a non zero A-homomorphism

of $k = A/\mathfrak{m}$ is injective.

Now $r = 0$ (and maximality), implies that there are <u>no</u> M-regular elements in $\mathfrak{m}$, and by 1) above $\mathfrak{m} = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$. Therefore $\mathfrak{m} \in \mathrm{Ass}(M)$ and, $\mathfrak{m}$ being the annihilator of some non zero $x \in M$, $k = A/\mathfrak{m} \simeq Ax \subset M$ and ii) follows. Conversely, if $M \supset N \simeq A/\mathfrak{m} = k$, let $x \in M$ be a generator of N. Then $\mathfrak{m}$ is <u>the</u> annihilator of x, whence $\mathfrak{m} \in \mathrm{Ass}(M)$ and there are <u>no</u> M-regular elements in $\mathfrak{m}$, i.e. $\emptyset$ is a maximal sequence of M-regular elements, Q.E.D.

5) Let $N = M/x_1 M$. We have an exact sequence

$$0 \to M \xrightarrow{\varphi} M \xrightarrow{c} N \to 0$$

where $\varphi(m) = x_1 m$. Hence we get

$$\ldots \to \mathrm{Ext}_A^{r-1}(k, M) \xrightarrow{\tilde{\varphi}} \mathrm{Ext}_A^{r-1}(k, M) \xrightarrow{\tilde{c}}$$

$$\mathrm{Ext}_A^{r-1}(k, N) \xrightarrow{\partial} \mathrm{Ext}_A^{r}(k, M) \xrightarrow{\tilde{\varphi}} \mathrm{Ext}_A^{r}(k, M) \to \ldots$$

Now, since $x_1 \in \mathfrak{m}$, $\tilde{\varphi} = 0$ (multiplication by x_1 annihilates all elements of k). On the other hand, by induction

$$\mathrm{Ext}_A^{r-1}(k, M) \cong \mathrm{Hom}(k, M/x_1 M + \ldots + x_{r-1} M) = 0$$

since $\{x_1, \ldots, x_{r-1}\}$ is <u>not</u> a maximal M-regular sequence. Therefore $\mathrm{Ext}_A^{r-1}(k, N) \cong \mathrm{Ext}_A^{r}(k, M)$. As was pointed out in the proof of 4), $\{x_2, \ldots, x_r\}$ is a maximal N-regular sequence, whence we can proceed by induction and obtain

$$\mathrm{Ext}_A^{r-1}(k, N) \cong \mathrm{Ext}_A^{r-2}(k, M/x_1 M + x_2 M) \cong \ldots$$

$$\cong \mathrm{Hom}(k, M/x_1 M + \ldots + x_r M),$$

and 5) is proved.

Corollary 3.1. Maximal M-regular sequences have the same cardinality.

Proof: Obvious from 5).

Corollary 3.2. Let $M^n = \bigoplus_{i=1}^{n} M_i$, $M_i \simeq M$. Then Depth (M^k) = Depth (M).

Proof: The isomorphism

$$M^n/x_1M^n+\ldots+x_rM^n = M/x_1M+\ldots+x_rM \oplus\ldots\oplus M/x_1M+\ldots+x_rM$$

shows that any maximal M^n regular sequence is a maximal M-regular sequence. The corollary follows from Corollary 3.1.

We now come to the main theorem concerning the notion of depth, namely:

Theorem 3.1. Let A be a noetherian local ring, M a finitely generated A-module. Then

i) depth (M) = 0 is equivalent to $\mathfrak{m} \in \mathrm{Ass}(M)$.

ii) if $x \in \mathfrak{m}$ is M-regular then depth (M/xM) = depth (M) - 1.

iii) depth (M) $\leqq \inf_{\mathfrak{p} \in \mathrm{Ass}(M)} \dim(A/\mathfrak{p}) \leqq \sup_{\mathfrak{p} \in \mathrm{Ass}(M)} \dim(A/\mathfrak{p})$ = dim(M).

Proof: i) This is a restatement of 1), Proposition 3.1.

ii) Let $\{x_2,\ldots,x_r\}$ be a maximal M/xM-regular sequence. If x is M-regular, then $\{x, x_2,\ldots,x_r\}$ is a maximal M-regular sequence, whence depth (M) = depth (M/xM) + 1.

iii) We prove this by induction on n = depth (M). If n = 0, then $\mathfrak{m} \in \mathrm{Ass}(M)$, whence, trivially

$$0 = \inf_{\mathfrak{p} \in \mathrm{Ass}(M)} \dim(A/\mathfrak{p}) \leqq \sup_{\mathfrak{p} \in \mathrm{Ass}(M)} \dim(A/\mathfrak{p}) = \dim(M).$$

In the induction step we shall make use of the following:

Lemma 3.1. Let $t \in \mathfrak{m}$ be M-regular, $\mathfrak{p} \in \mathrm{Ass}(M)$. Then any minimal prime containing $\mathfrak{p} + At$ belongs to $\mathrm{Ass}(M/tM)$.

Proof: By Proposition 4 of B.C.A., IV, §1, there exists a submodule $M' \subset M$ and an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

such that $\mathrm{Ass}(M') = \{\mathfrak{p}\}$; $\mathrm{Ass}(M'') = \mathrm{Ass}(M) - \{\mathfrak{p}\}$. By 1 of Proposition 3.1, t is both M'-regular and M"-regular and the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M' & \to & M & \to & M'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M' & \to & M & \to & M'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M'/tM' & \to & M/tM & \to & M''/tM'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

is obviously commutative and exact, whence $\mathrm{Ass}(M'/tM') \subset \mathrm{Ass}(M/tM)$. We have

$$\mathrm{Supp}(M'/tM') = \mathrm{Supp}(M') \cap V(t).$$

If $\mathfrak{q}$ is a minimal prime containing $\mathfrak{p} + At$, then from the above $\mathfrak{q}$ is a minimal prime of $\mathrm{Supp}(M'/tM')$, whence $\mathfrak{q} \in \mathrm{Ass}(M'/tM)$ and we are done.

We return to the proof of iii) of theorem 3.1. Assume depth $(M) = n$. Let $x \in \mathfrak{m}$ be M-regular, $N = M/xM$. By ii) of theorem 3.1, depth $(N) = n - 1$. Let $\mathfrak{p}$ be any point in

Ass(M), and let $\mathfrak{q}$ be a minimal prime containing $\mathfrak{p}$ + Ax. Clearly $\mathfrak{q} \supsetneq \mathfrak{p}$ (since $x \notin \mathfrak{p}$) and by the lemma $\mathfrak{q} \in$ Ass(N). By the induction assumption we have

$$n - 1 \leqq \dim(A/\mathfrak{q})$$

and clearly $\dim(A/\mathfrak{q}) \leqq \dim(A/\mathfrak{p}) - 1$. Hence $n \leqq \dim(A/\mathfrak{p})$, for all $\mathfrak{p} \in$ Ass(M), iii) follows.

$\longleftrightarrow$

Appendix

Not only is the function d:Spec(A) $\rightarrow$ N given by $d(\mathfrak{p})$ = depth $(A_{\mathfrak{p}})$ not continuous, but the concept of depth is a considerably more sensitive invariant than dimension. In particular depth $(A_{\mathfrak{p}})$ bears no relation to depth (A), contrary to the behavior of dimension. To see this, let A be any local ring, which is an integral domain, $\mathfrak{p} \in$ Spec(A), say $\mathfrak{p}$ = (0). Then $A_{\mathfrak{p}}$ is a field, and has hence depth 0, while depth (A) is arbitrary. On the other hand let A_o be any local integral domain, $\mathfrak{m}_o$ its unique maximal ideal, $k_o = A_o/\mathfrak{m}_o$. Consider the A_o-module $A = A_o \oplus k_o$, and define on A a ring structure by defining $(a, x)\cdot(a', x') = (aa', ax' + a'x)$. One easily checks that A is a local ring, with $\mathfrak{m}_o \oplus k_o$ as unique maximal ideal, and that every non-unit in A is a zero divisor, whence depth (A) = 0. However, if $\dim(A_o) \geqq 2$, and $\mathfrak{p}_o$ is a non zero, non maximal prime ideal of A_o then $\mathfrak{p} = \mathfrak{p}_o \oplus k_o$ is a prime ideal in A and $A_{\mathfrak{p}} \simeq A_{o\,\mathfrak{p}_o}$. Now depth $(A_{o\,\mathfrak{p}_o}) \geqq 1$ since $A_{o\,\mathfrak{p}_o}$ is an integral domain.

The following result is due to Hartshorne and gives a geometrical significance to the notion of depth.

(Hartshorne) Let A be a local ring with depth (A) $\geqq 2$. Then Spec(A) - $\{\mathfrak{m}\}$ is a connected topological space.

In particular, the local ring of the unique point of intersection of two sufficiently general planes in four dimensional affine space is a 2-dimensional ring whose depth (by Hartshorne's result) is $\leqq 1$. This shows that, in the inequalities iii) of Theorem 3.1, strict inequality is possible. This justifies the following:

Definition 3.3. Let A be a noetherian local ring, M a finitely generated A-module. M is said to be a Cohen-Macaulay module (C-M module) if depth (M) = dim(M). If A is an arbitrary noetherian ring (not necessarily local), A is said to be a Cohen-Macaulay ring if, for every maximal ideal $\mathfrak{m}$ of A, the local ring $A_{\mathfrak{m}}$ is Cohen-Macaulay.

We illustrate the notion of C-M modules with a few examples.

1) dim(M) = 0, $M \neq 0$. Then, from iii) of theorem 3.1, M is C-M. Here the notion of C-M modules is redundant.
2) dim(A) = 1, A a noetherian local ring. Then, if A is C-M, depth (A) = 1, which is equivalent to saying, since dim(A) = 1, that $\mathfrak{m} \notin$ Ass(A). Hence a non C-M ring of dimension 1 is a local ring in which all non-units are zero divisors. For example if A = k[x, y], where k is any field and $x^2 y = xy^2 = 0$, and $\mathfrak{m} = xA + yA$, one easily checks that $A_{\mathfrak{m}}$ is a non C-M ring of dimension 1.

3) dim(A) = 2. Here we limit ourselves to showing that every 2-dimensional, integrally closed local integral domain is C-M. To see this, let $x \in \mathfrak{m}$, $x \neq 0$. Since A is an integral domain, x is A-regular and, since A is integrally closed, none of the prime ideals associated to xA is imbedded (see B.C.A., VII, §1). Then, if $\mathfrak{p} \in \mathrm{Ass}(A/xA)$, it follows by the Hauptidealsatz that $\mathfrak{p} \neq \mathfrak{m}$.

Therefore $\mathfrak{m} = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(A/xA)} \mathfrak{p}$ is impossible, and a $y \in \mathfrak{m}$, $y \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}(A/xA)} \mathfrak{p}$ can be found. Therefore depth (A) $\geqq$ 2, and hence depth (A) = 2 = dim(A) which proves our assertion.

We now investigate some of the consequences of knowing that a ring A, or a module M, are C-M.

Proposition 3.2. Let M be a C-M A-module. Then

1) For every $\mathfrak{p} \in \mathrm{Ass}(M)$,

$$\dim(A/\mathfrak{p}) = \dim(M) = \operatorname{depth}(M)$$

2) The following three conditions are equivalent:

(i) x is M-regular

(ii) dim(M/xM) = dim(M) - 1

(iii) x belongs to no prime of Ass(M)

3) If x is M-regular, M/xM is a C-M module

Proof: 1) is a trivial consequence of the definition of C-M modules and of (iii) of Theorem 3.1.

2) (i) implies (ii) by (ii) of Theorem 3.1, and (i) is equivalent to (iii) by 1) of Proposition 3.1. It remains to prove that (ii) implies (i). This follows immediately from 1)

above (all primes in Ass(M) are equidimensional) and proposition 2.6.

3) By (ii) of Theorem 3.1 we have

$$\text{depth }(M/xM) = \text{depth }(M) - 1 = \dim(M) - 1 = \dim(M/xM)$$

hence M/xM is a C-M module.

We state without proof (an easy application of proposition 2.7 and 3.1) the generalization of 2) and 3) above to M-regular sequences.

<u>Proposition</u> 3.3. Let M be a C-M module. Then the following three conditions are equivalent:

(i) $\{x_1, \ldots, x_r\}$ is an M-regular sequence

(ii) $\dim(M/x_1 M + \ldots + x_r M) = \dim(M) - r$

(iii) $\{x_1, \ldots, x_r\}$ is embeddable in a system of parameters.

Furthermore, if $\{x_1, \ldots, x_r\}$ is an M-regular sequence, then $M/x_1 M + \ldots + x_r M$ is a C-M module.

<u>Proposition 3.4</u>. A module M, for which conditions (i), (ii), (iii) of the previous proposition are equivalent, and such that $M/x_1 M + \ldots + x_r M$ is C-M whenever $\{x_1, \ldots, x_r\}$ is an M regular sequence, is a C-M module.

<u>Proof</u>: Let $n = \dim(M)$. If $n = 0$ there is nothing to prove. Assume $n \geqq 1$, let $\{x_1, \ldots, x_n\}$ be a system of parameters of M. Since (iii) $\Longrightarrow$ (i), x_1 is M-regular and $M/x_1 M$ is a C-M module. Now since x_1 is M-regular $x_1 \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$, whence $\dim(M/x_1 M) = \dim(M) - 1$. Therefore

$$\dim(M) = \dim(M/x_1M) + 1 = \operatorname{depth}(M/x_1M) + 1 = \operatorname{depth}(M)$$

and M is C-M, Q.E.D.

Corollary 3.3. If M is a C-M module, every maximal M-regular sequence is a system of parameters and conversely.

Proof: Obvious.

Remark. If A is a (not necessarily local) C-M integral domain, and $x \in A$, $x \neq 0$, clearly x is A-regular, whence A/xA is again C-M. Since $k[X_1,\ldots,X_n]$ is a C-M ring (we shall prove this later), it follows from the above remark that, if $f(X_1,\ldots,X_n)$, $g(X_1,\ldots,X_n)$ are relatively prime irreducible elements of $k[X_1,\ldots,X_n]$, then $k[X_1,\ldots,X_n]/(f, g)$ is again C-M. This throws a better light on example 2) given after definition 3.3.

We now examine the behavior of the notion of C-M under localization. We have

Proposition 3.5. Let M be a C-M module, $\mathfrak{p} \in \operatorname{Supp}(M)$. Then

1) $M_{\mathfrak{p}} = M \otimes A_{\mathfrak{p}}$ is a C-M module
2) $\dim(M) = \dim(M_{\mathfrak{p}}) + \dim(M/\mathfrak{p}M)$

Proof: We shall obtain proposition 3.5 as a consequence of the following:

Proposition 3.6. Let M be a C-M module, $\mathfrak{p} \in \operatorname{Supp}(M)$, $r = \dim(M) - \dim(M/\mathfrak{p}M)$. Then

1) There exists an M-regular sequence $\{x_1,\ldots,x_r\}$ with $x_i \in \mathfrak{p}$ and

2) any such sequence gives

$$\dim(M/x_1 M + \ldots + x_r M) = \dim(M/\mathfrak{p} M) = \dim(A/\mathfrak{p}).$$

Proof: To prove 1) we proceed by induction on r. When $r = 0$ the statement is trivial. Let $r \geqq 1$. Then $\dim(M/\mathfrak{p}M) < \dim(M)$, hence $\mathfrak{p} \notin \mathrm{Ass}(M)$ (since the primes in Ass(M) are equidimensional), and therefore $\mathfrak{p} \not\subset \bigcup_{\mathfrak{q} \in \mathrm{Ass}(M)} \mathfrak{q}$. Let $x_1 \in \mathfrak{p}$, $x_1 \notin \bigcup_{\mathfrak{q} \in \mathrm{Ass}(M)} \mathfrak{q}$. Then x_1 is M-regular and the module $N = M/x_1 M$ is C-M. Furthermore $\dim(N) = \dim(M) - 1$ and $N/\mathfrak{p}N \simeq M/\mathfrak{p}M$. We can hence apply the induction assumption to N and find an N-regular sequence $\{x_1, \ldots, x_r\}$ with $x_i \in \mathfrak{p}$. Now trivially $\{x_1, \ldots, x_r\}$ is an M-regular sequence with $x_i \in \mathfrak{p}$, and 1) is proved. 2) Let $\{x_1, \ldots, x_r\}$ be an M-regular sequence with $x_i \in \mathfrak{p}$. Let $P = M/x_1 M + \ldots + x_r M$ ($P = M$ if $r = 0$). Now $P/\mathfrak{p}P = M/\mathfrak{p}M$ and from proposition 3.3 we get that

$$\dim(P/\mathfrak{p}P) = \dim(M) - r = \dim(P)$$

and that P is a C-M module. Now clearly $\mathfrak{p} \in \mathrm{Supp}(P)$, hence $\mathfrak{p} \supset \mathfrak{p}_0'$ for some $\mathfrak{p}_0' \in \mathrm{Ass}(P)$. Furthermore we have $\dim(P/\mathfrak{p}P) = \dim(P) = \dim(A/\mathfrak{p}')$ for <u>all</u> $\mathfrak{p}' \in \mathrm{Ass}(P)$ (since P is C-M). Since clearly $\mathfrak{p} \subset \mathrm{Ann}(P/\mathfrak{p}P)$

$$\dim(P) = \dim(P/\mathfrak{p}P) \leqq \dim(A/\mathfrak{p})$$

and $\dim(A/\mathfrak{p}) \leqq \dim(A/\mathfrak{p}_0') = \dim(P)$. Hence $\mathfrak{p} = \mathfrak{p}_0'$ i.e. $\mathfrak{p} \in \mathrm{Ass}(P)$ and 2) follows.

We now prove Proposition 3.5. Let $x_1, \ldots, x_r$ be an M-regular sequence in $\mathfrak{p}$, where $r = \dim(M) - \dim(M/\mathfrak{p}M)$. Since localization is a flat operation we have that the images of $x_1, \ldots, x_r$ in $\mathfrak{p}M_{\mathfrak{p}}$ are still an $M_{\mathfrak{p}}$-regular sequence. Hence by proposition 1.1

$$\dim(M_{\mathfrak{p}}) \leqq \dim(M) - \dim(M/\mathfrak{p}M) = r \leqq \operatorname{depth}(M_{\mathfrak{p}}) \leqq \dim(M_{\mathfrak{p}})$$

whence 1) and 2) of proposition 3.5 follow.

Corollary 3.4. If A is a local C-M ring, A is catenary, and for every local epimorphism $A \to B$, B is catenary.

Proof: The quotient of a catenary local ring by a prime ideal being catenary, it is enough to prove A is catenary. Let $\mathfrak{w}$ be a minimal prime ideal of A, $\mathfrak{p}$, $\mathfrak{q}$ two prime ideals of A such that $\mathfrak{w} \subset \mathfrak{p} \subset \mathfrak{q}$. Then $A_{\mathfrak{q}}$ is a C-M ring and

$$\dim(A_{\mathfrak{q}}) = \dim(A_{\mathfrak{p}}) + \dim(A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}})$$

by proposition 3.5. If $A' = A/\mathfrak{w}$, and $\mathfrak{p}'$, $\mathfrak{q}'$ are the images of $\mathfrak{p}$, $\mathfrak{q}$ in A', this relation is equivalent to

$$\dim(A'_{\mathfrak{q}'}) = \dim(A'_{\mathfrak{p}'}) + \dim(A'_{\mathfrak{q}'}/\mathfrak{p}'A'_{\mathfrak{q}'})$$

hence A' is catenary by proposition 1.2; this shows that A itself is catenary.

Remark. The notion of C-M rings still is insufficient to distinguish the three local rings considered in the introduction, i.e. $\mathbb{C}[X, Y]/(Y^2 - X^3 - X^2)$; $\mathbb{C}[X, Y]/(Y^2 - X^3)$; $\mathbb{C}[X, Y](X - Y)$, localized at the origin. One easily checks that all three are C-M rings, following the procedure used in the remark after Corollary 3.3.

We shall obtain one notion which distinguishes the three local rings in the next section.