RECENT WORK ON NEVANLINNA THEORY AND DIOPHANTINE APPROXIMATIONS

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What I will describe here is a formal analogy between value distribution theory and various diophantine questions in number theory. In particular, there is a dictionary which can be used to translate, e. g., the First and Second Main Theorems of Nevanlinna theory into the number field case. For example, we shall see that the number theoretic counterpart to the Second Main Theorem combines Roth's theorem and Mordell's conjecture (proved by Faltings in 1983).

This analogy is only formal, though: it can only be used to translate the statements of main results, and the proofs of some of their corollaries. The proofs of the main results, though, cannot be translated due to a lack of a number theoretic analogue of the derivative of a meromorphic function, among other reasons. All that I can say at this point is that negative curvature plays a role in the proofs in both cases.

Thus, until recently the analogy was good only for producing conjectures, by translating statements of theorems in value distribution theory into number theory. But in 1989 it has played a role in finding a new proof of the Mordell conjecture, via the suggestion that the Mordell conjecture and Roth's theorem should have a common proof, as is the case with the Second Main Theorem.

We begin by briefly describing this analogy, but only briefly as it has been described elsewhere in [**V 1**] and [**V 2**], as well as in the book [**V 3**]. Likewise, more recent results will be described in [**V 6**]; therefore we refer the reader to [**V 3**] and [**V 6**] for details.

Let $f : \mathbb{C} \to C$ be a holomorphic curve in a compact Riemann surface (which we may assume is connected). Let D be an effective reduced divisor on C; i.e., a finite set of points, and let $\operatorname{dist}(D, P)$ be some function measuring the distance from P to a fixed divisor D. Then we have the usual definition

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Partially supported by the National Science Foundation Grants DMS-8610730 and DMS-9001372.

$$m(D,r) = \int_0^{2\pi} -\log \operatorname{dist}(D, f(re^{i\theta}))\frac{d\theta}{2\pi}.$$

Assuming that $f(0) \notin \operatorname{Supp} D$, the definition of the counting function can be rewritten as

$$N(D,r) = \sum_{w\in\mathbb{D}_r} \operatorname{ord}_w f^*D \cdot \log \frac{r}{|w|}.$$

Finally, let the characteristic function be given by the more classical definition:

$$T_D(r) = m(D,r) + N(D,r).$$

Note in particular that in the above definitions, we only needed the restriction of f to the closed disc $\bar{\mathbb{D}}_r$, of radius r. Thus we are actually regarding f as an infinite family of maps $f_r : \bar{\mathbb{D}}_r \to C$, obtained by restriction from f. In the analogy with number theory, let each f_r correspond to one of (countably many) rational points, so that a holomorphic function $f : \mathbb{C} \to C$ corresponds to an *infinite* set of rational points on C. For example there are no infinite sets of (distinct) rational points on a curve of genus > 1 (Mordell's conjecture), just as there are no nontrivial holomorphic maps from $\mathbb{C}$ to a Riemann surface of genus > 1. Both these facts follow from the appropriate version of the Second Main Theorem, as defined below.

To make the number theoretic counterparts to the standard definitions as above, let C be a smooth connected projective curve, and let D be a reduced effective divisor on C. Assume that both C and D are defined over a number field k. For each place v of k (i.e., for each complex embedding $\sigma : k \to \mathbb{C}$ and for each non-archimedean absolute value corresponding to a prime ideal in the ring of integers of k), let $\operatorname{dist}_v(D, P)$ again be the distance from P to a fixed divisor D in the v-adic topology. These distances should be chosen consistently, as in ([**L 2**], Ch. 10, Sect. 2). For example, if $C = \mathbb{P}^1$ and $D = [\alpha]$, then the various $\operatorname{dist}_v([\alpha], P)$ functions can be written as $\min(1, |x-\alpha|_v)$.

Then the proximity function is defined as

$$m(D,P) = \frac{1}{[k:\mathbb{Q}]} \sum_{v|\infty} -\log \operatorname{dist}_v(D,P)$$

where the notation $v|\infty$ means the sum is taken over the (finitely many) archimedean places of k. Thus, we are comparing the absolute values of f on the boundary of $\mathbb{D}_r$, with the absolute values "at infinity" of a number field.

The formula for the counting function is similar:

$$N(D,P) = \frac{1}{[k:\mathbb{Q}]} \sum_{v\nmid\infty} -\log \operatorname{dist}_v(D,P).$$

This is more clearly a counterpart to the definition in the Nevanlinna case if we write it as

$$N(D,P) = \frac{1}{[k:\mathbb{Q}]} \sum_{v\nmid\infty} \operatorname{ord}_{\mathfrak{p}} g(P) \cdot \log N\mathfrak{p},$$

where g is a function which locally defines the divisor D, and $\mathfrak{p}$ is the prime ideal corresponding to the valuation v. Thus the points inside $\mathbb{D}_r$, correspond to non-archimedean places, and the summands (for fixed $w \in \mathbb{D}_r$ or fixed v) take on discrete sets of values.

Finally, we again let

$$\begin{aligned} T_D(P) &= m(D,P) + N(D,P) \\ &= \frac{1}{[k:\mathbb{Q}]} \sum_{v} -\log \operatorname{dist}_v(D,P) \\ &= h_D(P), \end{aligned}$$

which is a well-known definition in number theory known as the Weil height.

As before, we can define the defect $\delta(D) = \liminf m(D,P)/h_D(P)$. The assumption that D is defined over k implies that $\delta(D) < 1$.

Then the following theorem holds with either set of definitions above, replacing "?" by r or P, as appropriate.

Theorem (Second Main Theorem). *Let D be a reduced effective divisor on a curve C. Let A be an ample divisor on C, let K be a canonical divisor on C, and let $\epsilon > 0$ be given. Then for almost all "?",*

$$m(D, ?) + T_K(?) \leq \epsilon T_A(?) + O(1).$$

Of course, in the Nevanlinna case, this is true with $(1 + \epsilon) \log T_A(r)$ in place of $\epsilon T_A(r)$, but this is only conjectured in the number field case.

In the number field case, when $g = 0$ this is Roth's theorem, which is the following.

Theorem (Roth, 1955). *Let k be a number field; for each archimedean place v of k let $\alpha_v \in \bar{\mathbb{Q}}$ be given. Also let $\epsilon > 0$. Then for all but finiteiy many $x \in k$,*

$$\prod_{v|\infty} \min(1, |x - \alpha_v|_v) > \frac{1}{H(x)^{2+\epsilon}}.$$

Here $H(x) = \prod_v \max(1, |x|_v)$, so that $h_{\mathcal{O}(1)}(x) = (1/[k : \mathbb{Q}]) \log H(x)$.

To see how this theorem follows from the Second Main Theorem, let A be a divisor corresponding to $\mathcal{O}(1)$, let D be the union of all conjugates over k of all α_v, and take $-\log$ of both sides. For details, see ([**V 3**], 3.2).

When $g(C) > 1$, the Second Main Theorem is equivalent to Mordell's conjecture. Indeed, take $D = 0$, so that $m(D, P) = 0$, and we can take $A = K$ since K is ample. Then let $\epsilon < 1$; this gives a bound for $h_K(P)$, which is unbounded for infinite sets of rational points. This gives a contradiction. Conversely, if there are only finitely many rational points, then any statement will hold up to $O(1)$.

If the genus of C equals 1, then the Second Main Theorem corresponds to an approximation statement on elliptic curves proved by Lang.

Note that in number theory the Second Main Theorem is viewed as an upper bound on $m(D, P)$ instead of a lower bound on $N(D, P)$ as is the case in value distribution theory.

The fact that the Second Main Theorem of Nevanlinna theory has just one proof valid for all values of $g(C)$ suggests that the same should hold for number fields. This led to a new proof of the Mordell conjecture ([**V 4**] and [**V 5**]), using methods closer to Roth's. Work on obtaining a truly combined proof is progressing. This new proof led Faltings [**F**] to generalize the methods to give two new theorems:

Theorem (Faltings). *Let X be an affine variety, defined over a number fieid k, whose projective closure is an abelian variety. Then the set of integral points on X (relative to the ring of integers in k) is finite.*

Theorem (Faltings). *Let X be a closed subvariety of an abelian variety A. Assume that both are defined over k, and that X does not contain any translates of any nontrivial abelian subvarieties of A. Then the set $X(k)$ of k-rational points on X is finite.*

This is still an incomplete answer, because if X does contain a nontrivial translated abelian subvariety of A, then this theorem provides no information. Instead, the following conjecture should hold:

Conjecture (Lang, [**L 1**]). *Let X be a closed subvariety of an abelian variety A. Then $X(k)$ is contained in the union of finitely many translated abelian subvarieties of A contained in X.*

Before discussing this further, let us recall some facts about the geometry of this situation. For all that follows, assume that X is a closed subvariety of an abelian variety A.

Theorem (Ueno, ([**Ii**], Ch. 10, Thm. 10.13)). *There exists an abelian subvariety B of A such that the map $\pi : A \longrightarrow A/B$ has the properties that $X = \pi^{-1}(\pi(X))$ and $\pi(X)$ is a variety of general type.*

The map $\pi|X$ is calied the **Ueno fibration**. It is called **trivial** if B is a point.

Theorem (Kawamata Structure Theorem, [**K**]). *There exists a finite set $Z_1, \ldots, Z_n$ of subvarieties of X, each having nontrivial Ueno fibration, such that any nontrivial translated abelian subvariety of A contained in X is contained in one of the Z_i.*

The set $Z_1 \cup \cdots \cup Z_n$ is called the **Kawamata locus** of X.

Then Lang's conjecture is the analogue of the following statement, proved by Kawamata [**K**], using work of Ochiai [**O**]:

Theorem. *Let $f : \mathbb{C} \to X$ be a nontrivial holomorphic curve. Then the image of f is contained in the Kawamata locus of X.*

By the Kawamata Structure Theorem, this statement is equivalent to Bloch's conjecture, which asserts that the image of f is not Zariski-dense in X unless X itself is a translated abelian subvariety of A. (Bloch's conjecture was also proved, independently, by Green and Griffiths [**G-G**], also using Ochiai's work). Similarly, to prove Lang's conjecture, it would suffice to prove that $X(k)$ is not Zariski-dense unless X is a translated abelian subvariety of A.

For further details on these ideas, see ([**L 3**], Ch. 1 §6 and Ch. 8 §1). For more details on the connection with diophantine questions, see [**V 3**], especially Section 5.ABC for connections with the asymptotic Fermat conjecture.

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