NON-ASSOCIATIVE RINGS OF FINITE MORLEY RANK

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INTRODUCTION

Stable associative rings have been investigated by Cherlin–Reineke [Ch–Re], by Baldwin–Rose [B–Ro] and by Felgner [Fe] too early in the history of stable algebraic structures to get the attention they deserve. Rose started to investigate stable non–associative rings [Ro] in the late 1970's but again his work did not get the attention of model theorists. Macintyre's classification of ω_1–categorical fields [Mac2] and its generalization to superstable fields [Ch–S] and to ω–stable division rings [Ch2] became important because, we think, of their importance in the study of stable groups. We believe in the near future stable rings (associative or not) and their stable modules will become an important research area in applied model theory. They arise naturally in the study of stable groups. Here we list 4 instances:

a) Zil'ber classified ω_1 – categorical associative rings of characteristic 0 as indecomposable algebras over an algebraically closed field of characteristic 0 [Zi2]. He announces in [Zi3] that the same methods classify also ω_1 – categorical nilpotent Lie algebras over $\mathbb{Q}$ and that using Campbell–Baker–Hausdorff formula (see e.g. [Jac 1]), one can deduce that ω_1 – categorical torsion–free nilpotent groups are algebraic groups over an algebraically closed field of characteristic 0.

b) Let R be an arbitrary ring (not necessarily associative, does not necessarily have a unit). Then on the set $G = R \times R$ we can define a group multiplication by

$$(x,y)\,(x_1,y_1) = (x + x_1,\, y + y_1 + xx_1).$$

This group can also be viewed as

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y \in R \right\}$$

with the obvious multiplication. It is easily checked that G is nilpotent of class ≤ 2 and is Abelian iff R is commutative. Since G is interpretable in R, it inherits

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stability properties of R. Thus to classify nilpotent groups, say of small Morley Rank, we should at least classify rings of small Morley Rank.

c) We can generalize the above example. Let R be any ring (not necessarily associative, does not necessarily have a unit) of finite Morley rank. Let M be a left R–module of finite Morley rank. Define

$$G = \left\{ \begin{pmatrix} 1 & r & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : r \in R, x, y \in M \right\}$$

with the obvious multiplication. Then G is a nilpotent of class 2 group of finite Morley rank.

If R is associative with an identity and M is an associative, unitary right R–module then $G = M \rtimes R^*$ i.e. the group

$$G = \left\{ \begin{pmatrix} 1 & x \\ 0 & r \end{pmatrix} : r \in R^*, x \in M \right\}$$

has also finite Morley rank. (Here R^* denotes the multiplicative group of invertible elements of R).

d) Associative, commutative rings with identity come naturally into scene when one studies solvable of class 2, centerless, connected groups of finite Morley rank (see [Ne 1]). Such rings of finite Morley rank have been classified by Cherlin and Reineke in [Ch–R]. But it happens that we are also interested in their modules simply because the above groups can be interpretably imbedded into a finite product of groups of the form $M \rtimes R^*$ where M is an R–module of finite Morley rank and R is a ring with the above properties.

The methods of this article are not original. The basic ideas are mainly Zil'ber's. We found analogues of the known results about groups for rings.

In §1 we set the basics to study the non–associative rings. The lemmas are so simple that the mere knowledge of the basic concepts like Morley rank and degree is enough to understand their proofs.

In §2 we discuss Jacobson density theorem for associative rings.

In §3 we show how one can use the density theorem in the study of non–associative rings. The ideas are due to Zil'ber [Zi2]. He applied them to associative rings. We generalize his methods to general rings. In the end of this section we apply the previous results to connected Lie rings of Morley rank 1. Ivo Herzog can prove parts i) and ii) of Lemma 13 by some other methods.

In §4 we study Lie rings of finite Morley rank. We prove the analogue of Zil'ber's theorem for solvable groups. Namely we show that in a connected, solvable, non–nilpotent Lie ring of finite Morley rank one can interpret an algebraically closed field. The method is almost exactly like Zil'ber's original proof. But the result is amazingly different: in the construction of the field K we find K^+ where Zil'ber finds K^*. In other words we find the logarithm of Zil'ber's construction! This may not be so shocking for logicians who know the exp–log correspondance between Lie groups and Lie algebras. We also notice that the construction of this field by means of Zil'ber's methods is also given by Jacobson's density theorem. Finally we notice (without proof) that the proof of [Ne2] can be mimicked to show that if L is solvable and connected then L' is nilpotent.

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§1. BASICS

Before starting to set the basics we would like to recall Zil'ber's indecomposability Theorem [Zi 1] (not in its full generality):

Zil'ber's indecomposability Theorem : Let G be a group of finite Morley rank. Let $(X_i)_{i\in I}$ be a set of connected subgroups of G. Then the group generated by $(X_i)_{i\in I}$ is definable and connected. In fact if H is this group then there are a finite number of $i_1, \ldots, i_n \in I$ such that

$$H = X_{i_1} \cdot \ldots \cdot X_{i_n} .$$

If G is Abelian and written additively then we write

$$H = X_{i_1} + \ldots + X_{i_n}.$$

Pillay's article "Model Theory, Stability Theory and Stable Groups" in this volume has a proof of the above Theorem. The reader can also find a proof in [Ne 3] or in [Th].

Unless otherwise stated R will always denote an arbitrary ring of finite Morley rank (not necessarily associative, does not necessarily have an identity). In particular R is a group under addition. Thus it has a connected component R° which is an additive subgroup of R. If I is a definable ideal of R we may also speak about I°, the connected component (as an additive subgroup) of I.

Lemma 1. If I is a left (or right or bi) ideal of R then so is I°.

Proof: For $x \in R$, xI° is a subgroup of R. Since I is a left ideal $xI^\circ \subseteq I$. By Zil'ber's indecomposability theorem the group generated by xI° and I° is connected, thus it is I°, so $xI^\circ \subseteq I^\circ$. □

It follows from Lemma 1 that R° is an ideal of R. We may therefore speak about a "connected" ring. From now on, unless otherwise stated, all rings will be connected. Notice that if $(R,+)$ is torsion–free or divisible then R is necessarily connected.

Lemma 2. If I is a finite left (resp. right) ideal then $RI = 0$ (resp. $IR = 0$).

Proof: Let $i \in I$. R being connected, Ri is a connected additive subgroup of R. But $Ri \subseteq I$, so Ri is finite. A connected finite group is 0. Thus $Ri = 0$. □

For $n \in \mathbb{N}$, define

$$R_n = \{x \in R \mid nx = 0\}.$$

R_n is a definable bi–ideal of R. If n and m are prime to each other, then $R_n \cap R_m = \{0\}$, so also $R_n R_m = 0$. If n divides m then $R_n \subseteq R_m$. For p, a prime number, define

$$R_{p^\infty} = \bigcup_{k\geq 1} R_{p^k}.$$

R_{p^∞} is a bi–ideal. But it is not necessarily definable. Corollary 4 will show that it is almost always definable.

We define $\text{ann}_R R = \{r \in R: Rr = 0\}$, $\text{ann} R_R = \{r \in R: rR = 0\}$. If R is not associative these are not necessarily ideals of R. But they are definable additive subgroups.

Lemma 3. There is an $n \in \mathbb{N}$ such that

$$p^n R_{p^\infty} \subseteq \text{ann}\, R_R \cap \text{ann}_R R.$$

Proof: Consider the additive group I generated by Rx, $x \in R_{p^\infty}$. By Zil'ber's indecomposability theorem

$$I = Rx_1 + \ldots + Rx_k$$

for some $x_1, \ldots, x_k \in R_{p^\infty}$. Let n be such that $x_1, \ldots, x_k \in R_{p^n}$. Then $I \subseteq R_{p^n}$. In particular for any $x \in R_{p^\infty}$, $Rx \subseteq R_{p^n}$, i.e. $p^n Rx = 0$ or $R p^n x = 0$, i.e. $p^n x \in \text{ann}\, R_R$. Similarly $p^n x \in \text{ann}_R R$. □

Corollary 4. If $\text{ann}\, R_R = 0$ (or $\text{ann}_R R = 0$) then R_{p^∞} is definable. In fact $R_{p^\infty} = R_{p^n}$ for some $n \in \mathbb{N}$.

Proof: By Lemma 3, if $x \in R_{p^\infty}$ then $p^n x = 0$, so $x \in R_{p^n}$. □

The next corollary states that the torsion part and the torsion–free part of a ring without annihilator direct sum definably.

Corollary 5. Let R be a connected ring with $\text{ann}\, R_R = 0$ then there are finitely many distinct primes $p_1, \ldots, p_k$, there is a definable <u>torsion–free</u> (as an additive group) ideal D such that

$$R = R_{p_1^\infty} \oplus \ldots \oplus R_{p_k^\infty} \oplus D.$$

Proof: Notice that by Corollary 4 each $R_{p_i^\infty}$ is definable and has bounded order.

By Macintyre [Mac 1] $R = D \oplus H$ as an additive group where D is divisible and H is of bounded order. If n is such that $nH = 0$ then $D = nR$. So D is definable. This also shows that D is a bi–ideal.

Let $r \in R$ have finite additive order. If $r = d + h$ for $d \in D$, $h \in H$ then $d = r - h \in D$ and has finite order if $d \neq 0$. So if $d \neq 0$, D would have an element d' of prime order p. But then $R_{p^\infty} \neq R_{p^n}$ for any n (because D is divisible). This contradicts Corollary 4. Thus $d = 0$, so $r \in H$. We showed that any element of finite order is in H. Thus $R_{p^\infty} \subseteq H$ for all p.

Clearly any element of finite order can be written as the sum of its primary parts. Thus

$$H = \bigoplus_{p \text{ prime}} R_{p^\infty}.$$

Since R is connected, no R_{p^∞} is finite. Since R has finite Morley rank we can have only finitely many R_{p^∞} involved in H. This proves the Corollary. □

Corollary 5 shows that the study of connected rings with $\mathrm{ann}\, R_R = 0$ can be reduced to the study of their primary parts. Let us underline the essence of Corollary 5:

Proposition 6. Let R be a connected ring with $\mathrm{ann}_R R=0$. Then $R = D \oplus H$ where D, H are definable ideals, D is torsion–free divisible, H has bounded order. □

Lemma 7. Let Σ be a set of connected definable additive subgroups S_i of R. Then the subring and the left (or right, or bi) ideal generated by Σ are definable and connected.

Proof: Define inductively

R_1 = Additive group generated by S_i's,

R_{n+1}= Additive group generated by R_n and xR_n, R_nx for $x \in R_n$.

By induction, using Zil'ber's indecomposability theorem, we see that R_n's are definable and connected. By definition $R_n \subseteq R_{n+1}$. Thus $\bigcup_{n\geq 1} R_n = R_n$ for some n.

R_n is clearly the subring generated by Σ. (R_n may not have all the constants of R, e.g. R_n may be without identity even if R has an identity).

To prove the lemma for left ideals, in the definition of R_{n+1} we omit R_nx's and let x range over R. □

It follows from Lemma 7 that if a connected ring R has an identity then every ideal is definable and connected. This is simply because an ideal I is generated by the connected subgroups Rx for $x \in R$. In particular such a ring is Noetherian and Artinian. Rose [Ro] proved that if R is an arbitrary stable ring then J(R), the Jacobson radical of R, is definable and R/J(R) is Artinian and Noetherian.

Corollary 8. R^n, $R^{(n)}$ are definable and connected.

Proof: By definition R^{n+1} is the ideal generated by $\{xy: \text{for } x \in R, y \in R^n\}$, i.e. the ideal generated by $(xR^n)_{x\in R}$.

Since for x fixed xR^n is a definable homomorphic image of R^n, by induction xR^n is a connected subgroup. Now apply Lemma 7.

For $R^{(n+1)}$ we consider the ideal generated by $(xR^{(n)})_{x\in R^{(n)}}$. The proof is the same. □

Corollary 9. If I is a minimal left ideal of R and if $I \not\subseteq \text{ann } R_R$ then I is definable and connected (also infinite by Lemma 2).

Proof: Let $a \in I \setminus \text{ann } R_R$. Then $0 \neq Ra \subseteq I$. Thus the left ideal generated by Ra is I which is definable and connected by Lemma 7. □

Remark: If $\omega = \Sigma\, r_n (r_{n-1} (\ldots(r_2r_1) \ldots)$ a formal sum of formal monomials ($r_i \in R$), let us denote by ωa (for $a \in R$) the element of R defined by

$$\Sigma\, r_n(\ldots(r_2(r_1a))\ldots).$$

Then Corollary 9 and the proof of Lemma 7 tell us that the minimal left ideal I is the set of ωa's (a fixed in $I - \{0\}$) for ω ranging over words whose "sum length" and "product length" are bounded by some natural number n. This will be made more precise in §3, Lemma 12.

Corollary 10. The ideal generated by $\{xy - yx: x,y\in R\}$ is definable and connected.

Proof: Let $[x,R] = \{xy - yx \mid y \in R\}$. $[x,R]$ is a definable homomorphic image of R (as an additive group). Thus it is definable and connected. Now the ideal in question is generated by $[x,R]$ for x ranging over R. Use Lemma 7. □

Remark: If C is the ideal generated $\{xy - yx: x,y \in R\}$ then R/C is a commutative ring of finite Morley Rank.

Corollary 11. If $\mathrm{ann}R_R = 0$ then R has minimal left ideals which are definable. Furthermore these minimal ideals I are generated by the set Ra for any fixed $a\in I - \{0\}$, i.e. they are principal ideals.

Proof: By Lemma 7 the left ideal generated by Ra is definable and connected. Choose a minimal such. By Corollary 9 and its proof it is a minimal ideal. □

§2. DENSITY THEOREM FOR ABELIAN GROUPS

In this section we will forget about the multiplicative structure of our ring. We will not assume any stability conditions either. Let R be an Abelian group written additively. An additive group M is said to be an R–module if there is a homomorphism ρ of Abelian groups:

$$\rho: R \to \mathrm{End}_{\mathbb{Z}}\,(M).$$

M is said to be a faithful module if ρ is injective. We can define a multiplication $rx \in M$ for $r\in R$, $x \in M$ via ρ:

$$rx = \rho(r)\,(x).$$

Then all the module–theoretical concepts can be defined.

Schur's Lemma: If M is an irreducible R–module then $\Delta = \{\varphi: M \to M:$ φ linear and $\varphi(rx) = r\varphi(x)$ for all $r \in R, x \in M\} = End_R M$ is a division ring.

Under the conditions of Schur's Lemma, M becomes a vector space over Δ and $\rho(r)$ is a Δ– linear map for $r \in R$. If M is also faithful then R imbeds naturally into $End_\Delta M$.

A subset S of $End_\Delta M$ is said to be dense if for any n, any $x_1,...,x_n \in M$ linearly independent over Δ and any $y_1,...,y_n \in M$ there is an $s \in S$ such that

$$s(x_i) = y_i \quad (i = 1, . . ,n).$$

Notice that if $\dim_\Delta M < \infty$ then a dense subset of $End_\Delta M$ is necessarily $End_\Delta M$.

Density theorem for primitive abelian groups: Let M be an irreducible faithful module for the Abelian group R. Define $\Delta = End_R M$. Then $R \leq End_\Delta M$ as an additive group and the ring S generated by R in $End_\Delta M$ is dense in $End_\Delta M$.

Proof: This is a rephrasing of Jacobson's density theorem (see [Jac 2] p. 28) that states the above conclusion in case $R = S$ is an associative ring.

Let S be the (associative) ring generated by R in $End_\Delta M$. Since $S \subseteq End_\Delta M$, M is a faithful S–module. Since $R \subseteq S$, M is also S–irreducible. So if $\Delta' = End_S M$, S is dense in $End_{\Delta'} M$ by the original Jacobson density theorem. But since $R \subseteq S$ we also have $End_S M \subseteq End_R M$. Since S is generated by R, $End_R M \subseteq End_S M$. Thus $End_R M = End_S M$, i.e. $\Delta = \Delta'$. So S is dense in $End_{\Delta'} M = End_\Delta M$. □

If R is a ring then for M to be an R–module we may need to add some more conditions on the R–action. For instance if R has an identity 1 then we want $\rho(1) = Id_M$. Or if R is an associative ring we want ρ to be a ring homomorphism. If R is a Lie ring so that it satisfies the Jacobi identity $((rs)t + (st)r + (tr)s = 0)$ then we impose to ρ the condition to be a Lie–homomorphism, i.e. $\rho(rs) = \rho(r)\,\rho(s) - \rho(s)\,\rho(r)$.

§3. APPLICATIONS OF THE DENSITY THEOREM

Suppose R is an Abelian group of finite Morley rank. Suppose we can interpret in R a faithful irreducible R–module M (M could be an $\bar{R}$–module where $\bar{R}$ is a group interpreted in R). Suppose also that the division ring $\Delta = End_R M$ is interpretable in R. Then Δ and M have finite Morley rank. It follows that if Δ is infinite then it is an algebraically closed field [Ch 2] and M is a finite dimensional vector space over Δ. Therefore by the density theorem the ring S generated by R in $End_\Delta M$ is $End_\Delta M$! In fact, as John Baldwin noticed, if the ring S generated by R in $End_\Delta M$ is interpretable in R (e.g. if R is already an associative ring) then we do not even need the non–finiteness of Δ to claim that $S = End_\Delta M$. Because in this case M will have finite dimension over Δ anyway: if $x_1,\ldots,x_n,\ldots$ is a Δ–base of M, let

$$S_k = \{s \in S \mid sx_1 = \ldots = sx_k = 0\}.$$

Clearly $S_k \geq S_{k+1}$. But also by the density theorem $S_k \neq S_{k+1}$. This contradicts the descending chain condition. (For this argument we only need stability, because the subgroups S_k are intersections of uniformly definable subgroups).

Therefore to use the density theorem we need the following steps:

1) Find an interpretable faithful irreducible R–module M (or may be $\bar{R}$–module).

2) Interpret $\Delta = End_R M$ in R.

3) Then we know that $R < End_\Delta M$ (as an additive subgroup) and the ring S generated by R in $End_\Delta M$ is dense in $End_\Delta M$. To show that $S = End_\Delta M$, prove that either Δ is infinite or S is interpretable in R.

For the first step: an obvious candidate for M, in case R is a ring, is a minimal left ideal. Then this ideal will be definable if $ann\, R_R = 0$ (Corollary 9). The minimality of M will ensure that it is an irreducible module. But we do not necessarily have the faithfulness. Then divide R by the annihilator of M:

$$ann_R M = \{r \in R \mid rM = 0\}.$$

Now $R/ann_R M$ is an additive group and M is an irreducible and faithful $R/ann_R\, M$–module. Notice that $ann_R\, M$ is not necessarily an ideal. But it is so if R is an associative or a Lie ring (see end of the section for the definition of Lie ring).

Now about the second step; it is astonishing that Δ is almost always interpretable in R.

Lemma 12. Let M be a minimal left ideal (necessarily definable) in a connected ring of finite Morley rank with $\mathrm{ann} R_R = 0$. Then $\Delta = \mathrm{End}_R M$ is an interpretable division ring (hence a field). If Char R = 0 or more generally if Δ is infinite then Δ is an algebraically closed field.

Proof: (The idea of the proof is from [Zi2]). We know by Corollary 9 that M is definable. We need to recall explicitly its definition. Let ω be a formal word in R of the form

$$\omega = \omega_1 + \ldots + \omega_n,$$
$$\omega_j = r_{j1} (r_{j2}(\ldots(r_{jk}\, r_{j,k+1}) \ldots)).$$

Here, k depends on j. Such words will be called special. For $a \in R$ and ω a special word, define an element $\omega(a)$ of R by

$$\omega(a) = \omega_1(a) + \ldots + \omega_n(a),$$
$$\omega_j(a) = r_{j1} (r_{j2}(\ldots(r_{jk}(r_{j,k+1} a))\ldots)).$$

Notice that $\omega(a) \in R$ and is not a formal word. Let us also define the length $\ell(\omega)$ of a special word ω by

$$\ell(\omega) = \ell(\omega_1) + \ \ldots + \ell(\omega_n),$$
$$\ell(\omega_j) = k + 1 \quad \text{(see the definition of } \omega_j).$$

Now we are ready to give the explicit definition of M. Let $a_o \in M-\{0\}$. Then for some integer n,

$$M = \{\omega(a_o) : \omega \text{ is a special word of length } \leq n \}.$$

This is the content of the proof of Lemma 7. Fix such an integer n.

We need one more definition before interpreting Δ. Let

$$J = \{a \in M: \forall\ \omega, \omega_1, \omega_2 \text{ special words of length } \leq n,$$
$$(\omega(a_o) = \omega_1(a_o) \rightarrow \omega(a) = \omega_1(a))$$
$$\& (\forall\, r \in R \quad r(\omega_1(a_o)) = \omega(a_o)) \rightarrow r(\omega_1(a)) = \omega(a))$$
$$\& \ (\omega_1(a_o) + \omega_2(a_o) = \omega(a_o) \rightarrow \omega_1(a) + \omega_2(a) = \omega(a))\}.$$

J is a definable subset of R and $a_o \in J$. Clearly J is an additive subgroup of R. Thus J will be infinite if charR = 0.

Now for $a \in J$ define $\gamma_a: M \rightarrow M$ by

$$\gamma_a(\omega(a_o)) = \omega(a).$$

The definition of J implies that γ_a is well defined and is in $End_R M$. Conversely if $\sigma \in End_R M$ then clearly $\sigma(a_o) \in J$ and $\sigma = \gamma_{\sigma(a_o)}$. Thus

$$\Delta = End_R M = \{\gamma_a: a \in J\}$$

is interpretable in R. □

Now we can carry out our third step:

Corollary 13. Let R be a connected ring of finite Morley rank with $\text{ann } R_R = 0$. Let M be a minimal left ideal of R (Corollary 11). Then

i) M is necessarily definable (Corollary 9).

ii) M is a faithful irreducible $R/\text{ann}_R M$–module. Let $\bar{R} = R/\text{ann}_R M$.

iii) $\Delta = End_{\bar{R}}(M) = End_R M$ is interpretable in R and is a field (algebraically closed if infinite).

iv) $R < End_\Delta M$ as an additive group and the ring S generated by R in $End_\Delta M$ is dense in $End_\Delta M$.

v) If either $\text{char } R = 0$ or S is interpretable in R or S is commutative then Δ is infinite and so M is a finite dimensional vector space over Δ. Hence $S = End_\Delta M$. Also if S is commutative then $\dim_\Delta M = 1$.

Proof: Everything is already proved except some parts of v). If S is interpretable we noticed in the beginning of this section that M must have finite dimension over Δ. So if Δ is finite then M is also finite. But then by Lemma 2 $M \subseteq \text{Ann} R_R = 0$, a contradiction.

If S is commutative then (since it is dense in $End_\Delta M$) it can easily be checked that $\dim_\Delta M = 1$. Again Δ is infinite. □

Conjecture: $\dim_\Delta M < \infty$ always (notation as above).

Let us give an illustration of this Corollary.

Recall that a Lie ring is an additive group L with a bilinear product (called bracket) $[x,y]$ such that for all $x,y,z \in L$

$[x,x] = 0$,

$[[x,y], z] + [[y,z], x] + [[z,x], y] = 0$ (Jacobi identity).

Sometimes we will omit the brackets and write xy for $[x,y]$.

Let L be a connected Lie ring of Morley rank 1. We would like to prove that L is Abelian, i.e.

$$[x,y] = 0$$

for all $x,y \in L$. But being unable to prove it, let us see what the Corollary gives us.

Define $Z(L) = \{x \in L : [x,y] = 0\}$ = center of L,

$Z_2(L) = \{x \in L : [x,y] \in Z(L)\}$.

By the Jacobi identity $Z(L), Z_2(L)$ are ideals. If $Z(L)$ is infinite then $Z(L) = L$ and so L is abelian. Suppose therefore that $Z(L)$ is finite. Let $\bar{L} = L/Z(L)$. Then $Z(\bar{L}) = Z_2(L)/Z(L)$. Suppose $Z_2(L)$ is infinite. Then $Z_2(L) = L$. Thus $L^2 \subseteq Z(L)$. But by Corollary 8, L^2 is connected. Thus $L^2 = 0, Z(L) = L$, a contradiction. Thus $Z_2(L)$ is finite. Then by Lemma 2, $Z_2(L) \subseteq Z(L)$. Thus $Z_2(L) = Z(L)$ and $\bar{L}$ is centerless.

We showed the following:

Lemma 14: If L is a connected non Abelian Lie ring of Morley rank 1 then $\bar{L} = L/Z(L)$ is a connected centerless Lie ring of Morley rank 1. □

Now we can apply the Corollary to $\bar{L}$. Assume $L = \bar{L}$ for the sake of notational simplicity. $\text{ann}\, L_L = 0$ (because $Z(L) = 0$), $M = L$ (because L has Morley rank 1, so is a minimal (definable) ideal). So we have parts i) and ii) of the following Lemma.

Lemma 15. If L is a centerless connected Lie ring of Morley rank 1 then

i) $L \subseteq \text{End}_\Delta L$ where $\Delta = \text{End}_L L$.

ii) The associative ring generated by L in $\text{End}_\Delta L$ is dense in $\text{End}_\Delta L$.

iii) L has no non–trivial ideals (i.e. L is simple).

iv) Δ is a finite field.

v) L has characteristic p for some prime $p \neq 0$.

Proof: We have already proved i) and ii). If I were a proper ideal of L then the ideal J generated by $\{[L,x]: x \in I\}$ would be definable by Lemma 7. Since this definable ideal J is in $I \subsetneq L$, J would be finite and hence central by Lemma 2. Thus $J = 0$. Then $Lx = 0$ all $x \in I$, i.e. $I \subseteq Z(L) = 0$. This proves iii).

Let $a \in L \setminus \{0\}$. Let $C_L(a) = \{x \in L \mid [x,a] = 0\}$. Since $\Delta = End_L L$, Δ acts on the finite (but non–zero) set $C_L(a)$. If Δ were infinite it would be an algebraically closed field and so it would be connected, then $\Delta a \subseteq C_L(a)$ would also be connected. But $C_L(a)$ is finite, so $\Delta a = 0$. Since $Id \in \Delta$, this is a contradiction. This proves iv).

If v) were not true then $nx \in C_L(x)$ for all $n \in \mathbb{N}$, so $C_L(x) = L$, $x = 0$. □

Let us make a weaker conjecture then the previous one:

Conjecture: Connected Lie rings of Morley rank 1 are Abelian.

Reineke proved (see [Re] or [Ch 1]) that connected groups of Morley rank 1 are Abelian. The proof is very easy but one cannot give the same proof for Lie rings. Cherlin and the author proved that if L is non–Abelian then $\dim_\Delta C_L(a) > 1$ for a generic element a of L.

Added to the last version: In view of Hrushovski's discovery of new strongly minimal sets the author of the above conjecture does not believe in it anymore, thus:

Conjecture: There is a non–Abelian connected Lie ring of Morley rank 1.

§ 4. SOLVABLE, NON–NILPOTENT LIE RINGS

Let us first recall the definitions of solvable and nilpotent rings. We defined the ideals R^n and $R^{(n)}$ in Corollary 8. A ring R is said to be solvable if $R^{(n)} = 0$ for some n. It is said to be nilpotent if $R^n = 0$ some n. Nilpotent implies solvable.

Let us also define the centers: $Z_0(R) = 0$,

$Z_{i+1}(R) = \{x \in R : xR \subseteq Z_i(R)\}$.

If $R = L$ is a Lie ring then by the Jacobi identity $Z_i(L)$ is an ideal of L. Clearly $Z_i(L) \subseteq Z_{i+1}(L)$. It is relatively easy to check that L is nilpotent iff $Z_m(L) = L$ for some m. We have $Z_1(L) = Z(L) = ann_L L$. Notice also that $Z(L/Z_i(L)) = Z_{i+1}(L)/Z_i(L)$.

We should also remind the reader that in a Lie ring we have $[x,x] = 0$. This implies $[x,y] = -[y,x]$. So that all left (or right) ideals are bi–ideals.

If $X, Y \subseteq L$ are any subsets, then the centralizer of X in Y is $C_Y(X) = \{y \in Y \mid [x,y] = 0 \text{ for all } x \in X\}$.

Theorem 16. Let L be a connected, solvable, non–nilpotent Lie ring of finite Morley rank. Then an algebraically closed field can be interpreted in L.

Proof: We will follow Zil'ber's steps (see [Zi 1] or [Ne 3] or [Th]). Since L is connected and not nilpotent it is infinite.

We first reduce the problem to the case where L is centerless:

Claim 1: Without loss of generality L is centerless.

Divide L by its centers until there isn't any left. The point is that we need to divide L only a finite number of times. Since L has finite Morley rank and is not nilpotent and since to divide by an infinite definable ideal decreases the Morley rank, at some point we can only divide by finite ideals Z_{i+1}/Z_i. Assume without loss of generality that $Z = Z(L)$ and $Z_2 = Z_2(L)$ are finite. Then by Lemma 2, $Z_2 \subseteq Z$. Thus $Z_2 = Z$. □

From now on we assume that L is centerless. By Lemma 2 this implies that L has no, non–zero, finite ideals.

Let A be a minimal ideal of L. A exists and is definable by Corollary 11 (ann $R_R = Z(L) = 0$). By Lemma 1, A is connected. By Corollary 8, A^2 is definable. Since L is solvable, so is A. Thus $A^2 \subsetneq A$. But then A^2 is finite, so $A^2 = 0$. This shows that A is Abelian, i.e. $[A,A] = 0$.

Let $C = C_L(A) = \{x \in L: [x,A] = 0\}$. Since A is an ideal of L, so is C. L being centerless, $C \subsetneq L$. L being connected L/C is an infinite Lie ring. Let H be a minimal definable ideal of L such that $C \subsetneq H \subseteq L$ and H/C is infinite. Since $[L,A] \subseteq A$, also $[H,A] \subseteq A$. Choose $B \subseteq A$, a minimal (definable) infinite ideal such that $[H,B] \subseteq B$. B is again connected.

Claim 2: $C_A(H) = 0$, $C_B(H) = 0$.

Since A and H are ideals, $C_A(H)$ is an ideal of L. By minimality of A, $C_A(H)$ is either A or finite. If it is A then $H \subseteq C_L(A)$, a contradiction. So it is finite. Then it is 0 because finite ideals of L are 0. Since $C_B(H) \subseteq C_A(H)$, the second equality follows from the first one. □

Claim 3: $H/C_L(A)$ is connected.
Let H° be the connected component of H. H° is still an ideal by Lemma 1. We have $C_L(A) \subseteq H^\circ + C_L(A) \subseteq H$. Since H/H° is finite so is $H/H^\circ + C_L(A)$. Thus $H^\circ + C_L(A)/C_L(A)$ is infinite. Hence by the choice of H, $H = H^\circ + C_L(A)$. But now

$$H/C_L(A) = H^\circ + C_L(A)/C_L(A) \approx H^\circ/H^\circ \cap C_L(A).$$

Since H° is connected, so is $H^\circ/H^\circ \cap C_L(A)$ and therefore also $H/C_L(A)$. □

Claim 4: $H/C_L(A)$ is abelian, i.e. $[H,H] \subseteq C_L(A)$.

By Claim 3 and Corollary 8 $(H/C_L(A))^2$ is connected. But it is also finite (because $H/C_L(A)$ is solvable as L is and it has no infinite definable ideals). Thus $(H/C_L(A))^2 = 0$, i.e. $H^2 \subseteq C_L(A)$. With the notation of Lie rings: $[H,H] \subseteq C_L(A)$.

□

Claim 5: If $h \in H$ then $C_B(h) = 0$ or B.

Claim 4 and Jacobi identity imply that $C_B(h)$ is an H–ideal. If $C_B(h)$ is not finite, then it is B by the minimality of B. If it is finite, since the connected ring $H/C_L(A)$ acts on it, as in Lemma 2, $H/C_L(A)$ annihilates $C_B(h)$. So also H annihilates $C_B(h)$. Thus $C_B(h) \subseteq C_B(H) = 0$. □

Claim 6: $H/C_H(B)$ is infinite.

$C_H(B) = C_L(B) \cap H \supseteq C_L(A) \cap H = C_L(A)$. Thus $H/C_H(B) = (H/C_L(A))/(C_H(B)/C_L(A))$. So $H/C_H(B)$ is connected by Claim 3. Now if $H/C_H(B)$ were finite then we would have $C_H(B) = H$, or $B \subseteq C_A(H)$, contradicting Claim 2.□

By Claim 6 the infinite Lie ring $H/C_H(B)$ acts on B by adjoint representation:

$$H/C_H(B) \longrightarrow \mathrm{End}\, B$$
$$\bar{h} \longrightarrow \mathrm{ad}\, h$$

where $(\mathrm{ad}\,h)(b) = [h, b]$ for $h \in H, b \in B$.

By Claim 5 adh is a 1–1 map if $h \neq 0$. Since B has finite Morley rank it is also onto. Thus we have an imbedding

$$H/C_H(B) \to \mathrm{Aut}\,B \cup \{0\}$$

Let R be the (associative) subring generated by the image of $H/C_H(B)$ in End B. Since $[H,H] \subseteq C_H(A) \subseteq C_H(B)$, R is a commutative ring. We will show that R is an algebraically closed field. By Claim 6, R is infinite.

Let $b \in B - \{0\}$ be a fixed element. As in the proof of Lemma 11 there is a natural number n for which

$$B = \{\omega(b) \mid \omega \text{ is a special word with entries in } H \text{ of length } \leq n\}.$$

This is because B is minimal H–normal so the H–ideal generated by [H,b] must be B.

This says that if $R_1 \subseteq R$ is the set of endomorphisms of "length $\leq n$" then $R_1(b) = B$.

Claim 7: Let $r \in R$, $c \in B - \{0\}$. If $r(c) = 0$ then $r = 0$.

Without loss of generality $b = c$. Then $0 = R_1(r(b)) = r(R_1(b)) = r(B)$. So $r = 0$. □

Claim 8: $R = R_1$, i.e. R is interpretable.

Let $r \in R$. Then $\exists r_1 \in R_1$ such that $r(b) = r_1(b)$. By claim 7, $r = r_1$. □

Now we are ready to show that R is a field. By Claim 7, R has no zero–divisors. By Claim 7 again, there is a 1–1 correspondance between the elements of B and the elements of R:

$$R \to B$$
$$r \to r(b).$$

This is a homomorphism which is 1–1 and onto. Thus the additive group of R and B are isomorphic. In particular R is connected. So if $r \in R-\{0\}$, $Rr = R = rR$. Therefore there is a $u \in R$ such that $ur = r$. Now if $s \in R$ then for some $t \in R$, $s = rt$. So

$$us = u(rt) = (ur)t = rt = s.$$

Thus u is an identity of R.

Let us show that inverses exist. If $r \in R-\{0\}$ and $b' = r(b)$, applying to b' what we have said about b ($b' \neq 0$) we get an $s \in R$ such that $s(b') = b$. So $sr(b) = b = u(b)$. Therefore $sr = u$.

Thus R is an infinite field which is interpretable in L. By Macintyre [Mac 2] it is algebraically closed. Theorem 16 is now proved. □

Let us have a closer look at our field R. We have seen that $(R,+) \simeq (B,+)$ as additive groups. This is also the case in the construction of a field in a solvable connected group. Also in the above case $\mathrm{ad}\, H \subseteq \mathrm{Aut}\,(B) \cup \{0\}$ is an additive subgroup of R and it generates R as a ring. On the other hand in the case of solvable connected groups $\mathrm{ad}\, H$ (H acting by conjugation) is a multiplicative group!

There is a conjecture (which is part of another conjecture called Zil'ber's conjecture) that states that an infinite field R of finite Morley rank cannot have a proper infinite definable subgroup (additive or multiplicative). If this conjecture is true for additive subgroups then $\mathrm{ad}\, H = R$ and $B = [H,b]$ for any $b \in B \setminus \{0\}$! Since the conjecture for additive subgroups is true in case $\mathrm{Char} R = 0$ we have

Corollary 17. (Notation as in the proof of Theorem 16) If $\mathrm{Char}\, L = 0$ then $H/C_H(B)$ and B are isomorphic as additive subgroups via

$$h \rightarrow [h,b]$$

for any fixed element $b \in B - \{0\}$. If we define a multiplication on $H/C_H(B)$ by

$$h_1 * h_2 = h \Leftrightarrow [h_1 [h_2,b]] = [h,b]$$

then $(H/C_H(B), +, *)$ is an algebraically closed field of characteristic 0. □

We know that if G is a connected solvable group of finite Morley rank then $G' = [G,G]$ is nilpotent ([Zi 4], [Ne 2]). We presume that one can mimic the proof in [Ne 2] to show the following:

Theorem 18. If L is a connected solvable Lie ring of finite Morley rank then $L^2 = [L,L]$ is nilpotent.

Now we will use the density theorem once more.

Theorem 19. Let L be a connected solvable centerless Lie Ring of finite Morley rank. Let $A \subseteq Z(L^2)$ be a minimal ideal (then A is infinite and is definable). Let

$\Delta = End_L(A)$. Then $dim_\Delta A = 1$ and the ring generated by $L/C_L(A)$ in $End_\Delta(A)$ is isomorphic to Δ.

Proof: We know by Corollary 13 that the ring generated by $L/C_L(A)$ in $End_\Delta A$ is dense in $End_\Delta A$. But since $A \subseteq Z(L^2)$ we have , $L^2 \subseteq C_L(A)$. So $L/C_L(A)$, hence S, are commutative. But a commutative ring can be dense in $End_\Delta A$ iff $dim_\Delta A = 1$. Thus $S = End_\Delta A \simeq \Delta$. □

The field S we get in this way is the field we got in Theorem 16. To convince yourself of this fact, trace back the definition of $\Delta = End_L(A)$ and notice that it is just the ring R of Theorem 16.

Bibliography

[B–Ro] J. Baldwin and B. Rose, "ω_0 – Categoricity and Stability of Rings" J. Algebra 45 (1977), 1–16.

[Ch 1] G. Cherlin, "Groups of small Morley Rank", Annals Math. logic, 17 (1979), 1–28.

[Ch 2] G. Cherlin, "Superstable Division Rings", Logic Colloquim '77. Proceedings ASL European Summer Meeting, Wroclaw, Poland. Ed. by A. Macintyre, L. Pacholski, J. Paris, North Holland Publishing Company, 1978.

[Ch–Re] G. Cherlin – J. Reineke, "Stability and Categoricity for Commutative Rings", Annals Math Logic 10 (1976), 376–399.

[Ch–S] G. Cherlin–S. Shelah, "Superstable Fields and Groups" Annals Math Logic 18 (1980), 227–270.

[Fe] U. Felgner, "ω_1–Kategorishe Theorien nicht–Kommutativen Ringe", Fund Math 82 (1975), 331–346.

[Jac 1] N. Jacobson, Lie Algebras , Denver Publications, (1962).

[Jac 2] N. Jacobson, Structure of Rings, AMS Colloquim Publications, vol. XXXVII (1964).

[Mac 1] A. Macintyre, "On ω_1–Categorical Theories of Abelian Groups", Fund. Math. 70 (1971), 253–270.

[Mac 2] A. Macintyre, "On ω_1–Categorical Theories of Fields", Fund. Math. 71 (1971), 1–25.

[Ne 1] A. Nesin, "On Solvable Groups of finite Morley Rank ". Submitted to Trans. of AMS.

[Ne 2] A. Nesin, "Solvable Groups of finite Morley Rank" to be published in J. Algebra (1988 +).

[Ne 3] A. Nesin, "Zil'ber's Results", Preprint.

[Re] J. Reineke, "Minimale Gruppen" S. Math. Logik Grundlagen Math 21 (1975), 357–379.

[Ro] B.I. Rose, "Model Theory of Alternative Rings", Notre Dame Journal of Formal Logic, vol XIX, number 2, April 1978, 215–243.

[Th] S. Thomas, Preprint.

[Zi 1] B. Zil'ber, "Groups and Rings with Categorical Theories" (in Russian) Fund. Math 95 (1977), 173–188.

[Zi 2] B. Zil'ber, "Rings with ω_1–Categorical theories", Algebra i Logika, vol 13, no 2, (1974), 168–187. (Translation exists).

[Zi 3] B. Zil'ber, "Uncountable Categorical nilpotent groups and Lie Algebras" Izvestiya VUZ. Matematika 26, no 5, (1982), 75.

[Zi 4] B. Zil'ber, "Groups with Categorical Theories" Abstract of papers presented at the Fourth All–Union Symposium on Group Theory, Novosibirsk, 1973. In Russian, Math inst. Sibirsk. Otdel. Aikad. Nauk SSSR, pp 63–68.