COUNTABLY CATEGORICAL EXPANSIONS OF PROJECTIVE SPACES

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1. INTRODUCTION

A number of important problems in model theory ask what extra structure can be imposed upon a model M, while preserving various model-theoretic properties of M. For example, it has been conjectured that if extra structure is imposed upon an algebraically closed field F, then the resulting model F^+ no longer has finite Morley rank. In this paper, we shall discuss various open problems concerning ω-categorical structures of the form $M = \langle PG(\omega,q),R\rangle$. Here $PG(\omega,q)$ denotes an infinite dimensional projective space over the finite field $GF(q)$ and R is some extra relation. Our starting point is the observation that structures of this form provide an interesting test case for Lachlan's conjecture that a stable ω-categorical structure is ω-stable.

Theorem 1.1

Suppose that $M = \langle PG(\omega,q),R\rangle$ is ω-stable and ω-categorical. If $G = \text{Aut } M$ acts primitively on M, then M is strictly minimal.

Proof

By [8], M can be expressed as a union of finite algebraically closed subsets, $M = \bigcup_{i\in\omega} M_i$, such that

(i) $G_i = \text{Aut } M_i$ acts primitively on M_i;

(ii) G_i has the same number n_2 of orbits on the lines of M_i as G has on the lines of M. Let $M_i = \langle P_i, R_i\rangle$, where P_i is a subspace of dimension d_i. (Throughout this paper, we will be using vector space dimension; so that

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points are 1-dimensional, lines are 2-dimensional, etc.) We can suppose that $d_0 > 6$. By Hering [10], for each i either $n_2 = 1$ and $PSL(d_i,q) \leq G_i$, or else $n_2 = 2$, $PSp(d_i,q) \leq G_i$ and G_i preserves a nondegenerate symplectic polarity of P_i. (A statement of Hering's theorem can be found in Section 4). If the former occurs for all $i \in \omega$, then M is clearly strictly minimal. On the other hand, if $PSp(d_i,q) \leq G_i$ for all $i \in \omega$, then it is easily seen that G preserves a nondegenerate symplectic polarity of $PG(\omega,q)$, which contradicts the assumption that M is stable. □

Exercise 1.2

Find an elementary proof of this result, using the coordinatization theorem. (The proof of Hering's theorem makes use of the classification of the finite simple groups).

This paper is organized as follows. In Section 2, we shall discuss various conditions which imply that a 2-transitive stable ω-categorical structure has the form $\langle PG(\omega,q),R\rangle$. Sections 3 and 4 consider algebraic closure in such structures. Finally in section 5, we discuss projective space versions of results of Cameron [5], [6]. In particular, we will give a characterization of infinite dimensional symplectic spaces over finite fields.

If (G,Ω) is a permutation group and $X \subseteq \Omega$, then $G_{\{X\}}, G_{(X)}$ denote the setwise and pointwise stabilizers of X in G. The stabilizer of a point $\alpha \in \Omega$ is written as G_α.

A linear space is a structure $S = (\Omega,\mathcal{L})$, where $\mathcal{L} \subseteq P(\Omega)$, such that

(i) every pair of points $\alpha, \beta \in \Omega$ lie in a unique element of $\mathcal{L}$,

(ii) if $\ell, \ell' \in \mathcal{L}$ then $|\ell| = |\ell'| > 2$.

The elements of $\mathcal{L}$ are called lines.

If P is a projective space, then $P^{(k)}$ denotes the set of k-dimensional subspaces of P. $PG_2(\omega,q)$ is the linear space $(PG(\omega,q), PG^{(2)}(\omega,q))$. We use a similar notation for affine spaces.

2. TRIANGLE TRANSITIVE LINEAR SPACES

The results in this section give conditions under which a 2-transitive stable ω-categorical structure has the form $\langle PG(\omega,q),R\rangle$ or $\langle AG(\omega,q),R\rangle$.

Theorem 2.1

Let M be a stable ω-categorical structure and let $G = \mathrm{Aut}\ M$. Suppose that

(i) G acts 2-transitively on M;
(ii) if $\alpha \neq \beta \in M$, then $|\mathrm{acl}(\alpha,\beta)| > 2$;
(iii) $G_{\alpha,\beta}$ acts transitively on $M \setminus \mathrm{acl}(\alpha,\beta)$.

Let $\mathcal{D} = \{\mathrm{acl}(\alpha,\beta) \mid \alpha \neq \beta \in M\}$. Then $(M,\mathcal{D})$ isomorphic to $PG_2(\omega,q)$ for $q \geq 2$ or $AG_2(\omega,q)$ for $q \geq 3$.

Proof

Since M is 2-transitive and ω-categorical, any two points $\alpha \neq \beta \in M$ lie in a unique element of $\mathcal{D}$ and each element of $\mathcal{D}$ has the same finite cardinality. Thus $(M,\mathcal{D})$ is a linear space. By (iii), G acts transitively on the triangles of M. Let $\{\alpha_1,\alpha_2,\alpha_3\}$ be a triangle, and let P be the plane generated by $\{\alpha_1,\alpha_2,\alpha_3\}$. If $\{\beta_1,\beta_2,\beta_3\}$ is a second triangle of P, then there exists $\pi \in G$ such that $\beta_i^{\pi} = \alpha_i$ for $1 \leq i \leq 3$. Since $\{\alpha_1,\alpha_2,\alpha_3\} \subseteq P \cap P^{\pi}$, we have that $P^{\pi} = P$. Hence $H = \mathrm{Aut}\ P$ acts transitively on the triangles of P. By Kantor [11], P must be one of the following linear spaces:

(a) $PG(2,q)$ for some $q \geq 2$;

(b) $AG(2,q)$ for some $q \geq 3$;

(c) the unital U associated with $PSU(3,4)$.

To eliminate (c), we make use of the stability of M. Since M is 2-transitive, the unique type $p \in S_1(\varnothing)$ is stationary. By (iii), $tp(\sigma|\alpha,\beta)$ doesn't fork over $\varnothing$ for all $\sigma \in M \setminus \mathrm{acl}(\alpha,\beta)$. Hence $tp(\sigma|\mathrm{acl}\ (a,\beta))$ doesn't fork over $\varnothing$. It follows that if $\ell \in \mathcal{D}$, then $G_{(\ell)}$ acts transitively on $M \setminus \ell$. Arguing as in the first paragraph, for each line ℓ of P, $H_{(\ell)}$ acts transitively on $P \setminus \ell$. Now consider the unital U associated with $PSU(3,4)$. By O'Nan

[14], $H = \text{Aut } U = P\Gamma U(3,4)$. It is easily checked that H acts sharply transitively on the triangles of U. Since $H_{\alpha,\beta}$ acts nontrivially on the line ℓ containing $\alpha \neq \beta \in U$, $H_{(\ell)}$ isn't transitive on $U \setminus \ell$.

It is well known that if each plane P of $(M,\mathcal{D})$ is isomorphic to $PG(2,q)$, then $(M,\mathcal{D}) \simeq PG_2(\omega,q)$. By Buekenhout [2], if each plane is isomorphic to $AG(2,q)$ for some $q \geq 4$, then $(M,\mathcal{D}) \simeq AG_2(\omega,q)$. The analogue of Buekenhout's theorem is false for $q = 3$. (For example, see Young [20]). However, by [19], if $(M,\mathcal{D})$ is a triangle transitive linear space in which each plane is isomorphic to $AG(2,3)$, then $(M,\mathcal{D}) \simeq AG_2(\omega,3)$. □

There are two ways in which Theorem 2.1 should be strengthened.

Problem 2.2

Show that the conclusion still holds without the assumption that M is stable. In other words, prove that there is no ω-categorical triangle transitive linear space in which each plane is isomorphic to the unital U associated with $PSU(3,4)$.

Problem 2.3

Show that the conclusion still holds without hypothesis (iii).

The next result shows that to solve problem 2.3, it is enough to show that each plane is projective or affine.

Theorem 2.4

Let $S = (\Omega,\mathcal{D})$ be a 2-transitive stable ω-categorical linear space. If each plane of S is isomorphic to a finite projective or affine plane, then either $S \simeq PG_2(\omega,q)$ for $q \geq 2$ or $S \simeq AG_2(\omega,q)$ for $q \geq 3$.

Proof

By Teirlinck [16], either all planes of S are projective or all planes of S are affine. The only difficulty occurs when each plane P of S is an affine plane in which the lines have cardinality 3. (The results which we used in the proof of Theorem 2.1 do not assume that the planes are Desarguesian). In this case, $P \simeq AG(2,3)$. It is possible to coordinatise S by a commutative Moufang loop of exponent 3 [20]. First we make S into a quasigroup by defining a binary operation o as follows:

$$x \circ x = x$$

$$x \circ y = z \qquad \text{for } x \neq y \text{ if } \{x,y,z\} \text{ is a line.}$$

Now fix a point $e \in S$. Then the loop of S based at e, Q, is defined by

$$xy = (e \circ x) \circ (e \circ y).$$

Q is a commutative loop of exponent 3 with identity element e. Furthermore, Q satisfies the Moufang condition

$$(xy)(zx) = (x(yz))x.$$

Now $S \simeq AG(\omega,3)$ if and only if Q is a group, i.e., the associative law also holds. Suppose then that Q is not a group. Notice that any automorphism of S which fixes e induces an automorphism of Q. Hence Aut Q acts transitively on $Q \setminus e$.

A subloop H of Q is said to be normal if for all $x,y \in Q$

$$xH = Hx, \quad (Hx)y = H(xy), \; y(xH) = (yx)H.$$

By Bruck 11.1 [1], Q is not simple. Since Aut Q acts transitively on $Q \setminus e$, this implies that Q has no minimal nontrivial normal subloops. (See Bruck 8.1 [1]). Let $a \in Q \setminus e$ and let $N(a)$ be the smallest normal subloop containing a. Since Q is ω-categorical, $N(a)$ is a -definable. Let $e \neq M \lneqq N(a)$ be a normal subloop. Then if $b \in M \setminus e$, we have that $N(b) \lneqq N(a)$. Since a,b lie in the same Aut Q-orbit, this means that Q is unstable. Hence S is unstable. □

3. ALGEBRAIC CLOSURE

Throughout this section, we will suppose that $M = \langle PG(\omega,q), R\rangle$ satisfies the following conditions.

3.1 $G = \mathrm{Aut}\ M$ acts 2-transitively on M.

3.2 M is stable ω-categorical, but not ω-stable.

We begin by collecting together some elementary lemmas. If $A \subseteq M$, then $<A>$ denotes the subspace of $PG(\omega,q)$ generated by A.

Lemma 3.1

The unique type $p \in S_1\emptyset$ is stationary. □

Lemma 3.2

Let $\{a_i \mid 1 \le i \le n\}$ be a Morley sequence for $p \in S_1(\emptyset)$ and let $X = <a_i \mid 1 \le i \le n>$. Then $PSL(X) \le G_{\{X\}} / G_{(X)}$.

Proof

Let $a_n \neq \beta \in <a_1,...,a_n> \setminus <a_1,...,a_{n-1}>$. Then the line $<\beta,a_n>$ intersects the hyperplane $<a_1,...,a_{n-1}>$ in a point α. Suppose that $tp(\beta \mid <a_1,...,a_{n-1}>)$ forks over α. Then, by Shelah III 6.7 [15], $tp(a_n \mid <a_1,...,a_{n-1}>)$ forks over α, a contradiction. As G is 2-transitive and $tp(\beta \mid <a_1,...,a_{n-1}>)$ doesn't fork over α, it follows that $tp(\beta \mid <a_1,...,a_{n-1}>)$ doesn't fork over $\emptyset$.

Let $H = G_{\{X\}}/G_{(X)}$. For each $1 \le i \le n$, let $Y_i = <\{a_1,...,a_n\}\setminus\{a_i\}>$. By the previous paragraph, $H_{(Y_i)}$ acts transitively on $X \setminus Y_i$, i.e., $X \setminus Y_i$ is a Jordan subset for H. It follows that

$$\bigcup_{i\neq 1} (X \setminus Y_i) = X \setminus \bigcap_{i \neq 1} Y_i = X \setminus \{a_1\}$$

is a Jordan subset. (For example, see Neumann [13]). Similarly $X \setminus \{a_2\}$ is a Jordan subset, and so H acts 2-transitively on X. Clearly we can suppose that $n \ge 5$. The result now follows by Cameron and Kantor [7]. □

Hence, by passing to a suitable quotient geometry, we can suppose that M also satisfies the following condition.

3.3 G has more than one orbit on the set $PG^{(3)}(\omega,q)$ of planes of M. For $\alpha \neq \beta \in M$, define

$$\varphi(M; \alpha,\beta) = \{\gamma \in M \mid tp(\gamma \mid \alpha,\beta) \text{ forks over } \emptyset\}.$$

Then $<\alpha,\beta> \subsetneq \varphi(M; \alpha,\beta)$.

RM(-) denotes Morley rank.

Lemma 3.3

$RM(\varphi(M; \alpha,\beta)) = \infty$.

Proof

Suppose not. First suppose that $\varphi(M; \alpha,\beta)$ is finite. Then $\varphi(M; \alpha,\beta) = acl(\alpha,\beta) \supsetneq <\alpha,\beta>$, and $G_{\alpha,\beta}$ acts transitively on $M \setminus acl(\alpha,\beta)$. Let $\mathcal{L} = \{acl(\alpha,\beta) \mid \alpha \neq \beta \in M\}$. By Theorem 2.1, $(M,\mathcal{L})$ is isomorphic to $PG_2(\omega,r)$ or $AG_2(\omega,r)$ for some prime power $r \neq q$. If X is a finite algebraically closed subset of M, then X must be a subspace of both $PG(\omega,q)$ and $(M,\mathcal{L})$. Suppose that $(M,\mathcal{L}) \simeq PG_2(\omega,r)$. If $\ell \in \mathcal{L}$, then

$$|\ell| = r + 1 = q^n + q^{n-1} + ... + q + 1$$

for some $n \geq 2$. But this means that $r = q^n +...+ q$ is not a prime power, a contradiction. Hence $(M,\mathcal{L}) \simeq AG_2(\omega,r)$. So if $\ell \in \mathcal{L}$, then $|\ell| = r = q^{n+1} - 1/q-1$ for some $n \geq 2$. Let X be a finite algebraically closed subset of M, chosen so that

$$|X| = r^d = q^{m+1} - 1/q-1$$

for some $m + 1 > \max\{n+1,6\}$. By Zsigmondy [21], there exists a prime p such that $p|q^{m+1} - 1$ but p does not divide $q^i - 1$ for any $1 \leq i \leq m$. Again this contradicts the fact that r is a prime power.

Thus $0 < RM(\varphi(M; \alpha,\beta)) < \infty$. Let $L(\alpha,\beta) = \varphi(M;\alpha,\beta)$. By Shelah V 7.8 [15], $\bigcup_{\gamma \neq \delta \in L(\alpha,\beta)} L(\gamma,\delta)$ also has Morley rank less than ∞, and so $L(\alpha,\beta) = \bigcup_{\gamma \neq \delta \in L(\alpha,\beta)} L(\gamma,\delta)$. In particular, if $\gamma \neq \delta \in L(\alpha,\beta)$, then $L(\gamma,\delta) \subseteq L(\alpha,\beta)$. Since M is 2-transitive, we must have that $L(\gamma,\delta) = L(\alpha,\beta)$. Let $\mathcal{L} = \{L(\alpha,\beta) \mid \alpha \neq \beta \in M\}$. Then $(M,\mathcal{L})$ is a linear space, in which each line is infinite. Let α, β, γ be a noncollinear triple of $(M,\mathcal{L})$. Define a sequence of $\{\alpha,\beta,\gamma\}$-definable subsets of M inductively by

$$S_0 = L(\alpha,\beta) \cup L(\beta,\gamma) \cup L(\gamma,\alpha)$$
$$S_{n+1} = \bigcup_{a \neq b \in S_n} L(a,b).$$

There exists n such that $S_n = S_{n+1}$. Clearly $RM(S_n) < \infty$. But now it is easily checked that $(S_n, \{L(a,b) \mid a \neq b \in S_n\})$ is a pseudoplane, contradicting [8]. □

We can now easily prove

Theorem 3.4

There exists a finite dimensional subspace $X \subseteq M$ such that $X \subsetneq \mathrm{acl}\,(X)$.

My reason for recording this result is that it is conceivable that this already leads to a contradiction for the following problem still seems to be open.

Problem 3.5

Does there exist a (not necessarily stable) 2-transitive ω-categorical expansion of $PG(\omega,q)$ in which $X \subsetneq \mathrm{acl}\,(X)$ for some finite dimensional subspace X?

Proof of Theorem 3.4

Let Δ be the Booleanly closed set of formulas generated by $\{\varphi(x;\alpha,\beta) \mid \alpha \neq \beta \in M\}$. Choose $\pi(x;\bar{c}) \in \Delta$ such that

(i) $RM(\pi(x;\bar{c})) = \infty$;

(ii) $R_\varphi(\pi(x;\bar{c})) = k$ is minimal subject to (i) and $\deg_\varphi(\pi(x;\bar{c})) = 1$.

By Lachlan [12], we can also suppose that

(iii) $\pi(x;\bar{z})$ is normal with respect to $\varphi(x;\bar{y})$.

By Lemma 3.3, $\pi(M;\bar{c}) \neq M$. Let $\mathfrak{L}$ be the set of conjugates of $\pi(M;\bar{c})$. Then if $A \neq B \in \mathfrak{L}$, $RM(A \cap B) < \infty$.

Suppose that there exist $A,B \in \mathfrak{L}$ such that $0 < RM(A \cap B) < \infty$. Then we can replace $\mathfrak{L}$ by the set of conjugates of a normal strongly minimal subset of M.

Hence we can assume that if $A \neq B \in \mathfrak{L}$, then $A \cap B$ is finite. Let $n \in \mathbb{N}$ be such that if $A \neq B \in \mathfrak{L}$, then $|A \cap B| < n$. Since G acts primitively on M, there exist $A \neq B \in \mathfrak{L}$ with $A \cap B \neq \varnothing$. Let $\gamma \in A \cap B$. Define inductively $\alpha_i \in A$, $1 \le i \le n$, so that $\alpha_{i+1} \notin \langle \alpha_1,\ldots,\alpha_i, \gamma \rangle$. Then $\gamma \notin \langle \alpha_1,\ldots,\alpha_n \rangle$. Continuing in this fashion, we can find $\beta_i \in B$, $1 \le i \le n$, such that $\gamma \notin X = \langle \alpha_i, \beta_i \mid 1 \le i \le n \rangle$. But clearly $\gamma \in \mathrm{acl}(X)$.

□

The situation described in the second paragraph of the above proof seems extremely unlikely. More generally, the following problem may be manageable.

Problem 3.6

Prove that if M is a primitive stable ω-categorical structure with an infinite definable ω-stable subset, then M is ω-stable.

We can suppose that for each $A \in \mathfrak{L}$, $G_{\{A\}}$ acts transitively on A. Let $S \subset A$ be a subset of maximal cardinality, subject to the condition that S lies in infinitely many elements of $\mathfrak{L}$. (It is easily seen that each $\alpha \in M$ lies in infinitely many elements of $\mathfrak{L}$). Let $\Omega = \{S^\pi \mid \pi \in G\}$ and define an incidence relation I by

$$T \ I \ B \iff T \subset B$$

for $T \in \Omega$ and $B \in \mathfrak{L}$. Then $(\Omega,\mathfrak{L})$ is a pseudoplane. Thus Theorem 3.4 is just the observation that it is possible to define a pseudoplane geometry in M^{eq} which interacts nontrivially with the projective geometry on M.

Given the nature of our hypotheses on M, it seems best to continue working directly with $(M,\mathfrak{L})$. At this point, we would like to have that if $A \in \mathfrak{L}$, then $\langle A \rangle$ is a proper subspace of M. (Of course, this is true if A is strongly minimal). Unfortunately this does not follow immediately from general facts about forking.

Problem 3.7

Let $\mathfrak{L}$ be a uniformly definable family of infinite almost disjoint subsets of the (not necessarily stable) 2-transitive ω-categorical structure $M = \langle PG(\omega,q),R \rangle$. Is $\langle A \rangle$ a proper subspace of M for $A \in \mathfrak{L}$?

Suppose that $\langle A\rangle$ is indeed a proper subspace of M. By the stable descending chain condition for uniformly definable subspaces of M, there exists a definable subspace $S \leq \langle A\rangle$ such that the set S of conjugates of S is almost disjoint. Showing that such a family S cannot exist is a very interesting geometric problem. The special case when $(PG(\omega,q),\mathsf{S})$ is actually a linear space seems particularly attractive, although still very difficult.

4. DESIGNS OVER FINITE FIELDS

Again $M = \langle PG(\omega,q),R\rangle$ satisfies conditions 3.1 to 3.3. It seems useful to split the analysis of M into 3 cases, depending on the action of G_α on $\mathbb{P}_\alpha = PG(\omega,q)/\langle\alpha\rangle$. The possibilities are:

(A) G_α preserves a nontrivial equivalence relation on $\mathbb{P}_\alpha$ which has finite classes;

(B) G_α acts primitively on $\mathbb{P}_\alpha$;

(C) G_α preserves a nontrivial equivalence relation on $\mathbb{P}_\alpha$ which has infinite classes.

The cases are listed in what I believe to be the order of difficulty. In this section, case A will be discussed. This corresponds to the situation when $\ell \subsetneq \mathrm{acl}(\ell)$ for $\ell \in PG^{(2)}(\omega,q)$. Let $\mathcal{D} = \{\mathrm{acl}(\ell) \mid \ell \in PG^{(2)}(\omega,q)\}$. Then $(M,\mathcal{D})$ is a linear space. Let $\dim Q = m > 2$ for $Q \in \mathcal{D}$.

Definition 4.1

Given a finite field $\mathbf{F} = GF(q)$, a t-(n,m,λ) design over $\mathbf{F}$ is an incidence structure $\mathcal{D} = (P,\mathcal{D})$ which satisfies the following conditions:

(a) P is an n-dimensional projective space over $\mathbf{F}$.

(b) $\mathcal{D} \subseteq P^{(m)}$.

(c) Each $S \in P^{(t)}$ lies in exactly λ elements of $\mathcal{D}$.

If $\mathcal{D} = P^{(m)}$, then the design $\mathcal{D}$ is said to be trivial. We define $\mathrm{Aut}\,\mathcal{D} = \{\pi \in P\Gamma L(n,q) \mid \mathcal{D}^\pi = \mathcal{D}\}$.

Let X be a finite algebraically closed subspace of M with $\dim X = n > m$, and let $\mathcal{D}_X = \{Q \in \mathcal{D} \mid Q \subseteq X\}$. Then $(X,\mathcal{D}_X)$ is an example of a nontrivial 2-(n,m,1) design over GF(q). Despite the efforts of a number of finite combinatorialists, no such designs have been found. In fact, the only known examples of nontrivial 2-designs over finite fields are those in the infinite family described in [17]. For each of these designs, $\lambda = 7$.

Lemma 4.2

Let $(P,\mathcal{D})$ be a 2-(n,m,1) design over GF(q).

(i) Each point $\alpha \in P$ lies in $r = q^{n-1}-1/q^{m-1}-1$ elements of $\mathcal{D}$. Hence $m-1 \mid n-1$.

(ii) $|\mathcal{D}| = b = (q^n-1)(q^{n-1}-1)/(q^m-1)(q^{m-1}-1)$. □

While no examples of nontrivial $2-(n,m,1)$ designs are known, it has not been shown that no such designs exist for any pair of integers n,m for which $r,b \in \mathbb{N}$. My guess is that many such designs exist, but that their automorphism groups are extremely small. (This would account for the difficulty in finding examples. The usual method of constructing a design is to specify a small number of blocks $\{B_1,\dots,B_s\}$, one from each orbit of the automorphism group Γ. Then $\mathcal{D} = \{B_i^\pi \mid 1 \le i < s, \pi \in \Gamma\}$).

Problem 4.3

Show that for each finite field $F = GF(q)$, there exists an integer $N(q)$ such that: whenever $\mathcal{D} = (P,\mathcal{D})$ is a nontrivial 2-(n,m,1) design over F and $G = \mathrm{Aut}\mathcal{D}$, if $X \subseteq P$ satisfies $G_{\{X\}} / G_{(X)} \simeq \mathrm{Syn}(X)$, then $|X| < N(q)$.

Of course, this would eliminate case A. A special case of this conjecture can be deduced from Hering's theorem.

Theorem 4.4

Suppose that $\mathcal{D} = (P,\mathcal{D})$ is a nontrivial 2–(n,m,1) design over a finite field F. If $G = \mathrm{Aut}\mathcal{D}$ acts transitively on P, then G is soluble.

Hering's theorem classifies the groups G which act transitively on the set of nonzero vectors of a finite vector space V. We can suppose that V is an n-dimensional vector space over the prime field $GF(p)$, so that $G \leq GL(n,p)$. Let L be a subset of $Hom(V,V)$ maximal with respect to the condition that L is normalized by G, L contains the identity and L is a field with respect to the addition and multiplication in $Hom(V,V)$. (By Lemma 5.2 [9], L is unique unless $n = 2$, $p = 3$ and G is isomorphic to a quarterion group of order 8). There exist integers m and n^* such that $n = mn^*$, $|L| = p^m$ and n^* is the dimension of the vector space (V,L). Then $G \leq \Gamma L(V,L)$ must be one of the following types.

I. $SL(V,L) \leq G \leq \Gamma L(V,L)$.

II. There exists a nondegenerate skew-symmetric scalar product on (V,L) and G contains as a normal subgroup the group consisting of all isometries of the corresponding symplectic space.

III. $n^* = 6$, $p = 2$ and G contains a normal subgroup isomorphic to $G_2(2^m)$.

IV. There are finitely many exceptional groups. For our purposes, it is enough to know that in each of these cases $n = 2,4,6$. By Lemma 4.2, the corresponding projective spaces cannot carry nontrivial designs, and so these groups can safely be ignored.

This result also yields a classification of the groups $G \leq P\Gamma(n,q)$ which act transitively on the points of the n-dimensional projective space P over $GF(q)$. Let $G^* \leq \Gamma L(n,q)$ be the preimage of G under the homomorphism $\Gamma L(n,q) \to P\Gamma L(n,q)$. Regard G^* as a subgroup of $GL(N,p)$, where $q = p^t$ and $N = nt$. Let L be as above. Then $GF(q)$ is a subfield of L, say $L = GF(q^r)$. If $r > 1$, then G preserves a geometric r-spread $\mathfrak{L}$ of P. This means that $\mathfrak{L}$ is a collection of r-subspaces which form a partition of P satisfying:

If $Q_1,Q_2,Q_3 \in \mathfrak{L}$ and $Q_3 \cap \langle Q_1,Q_2\rangle \neq \emptyset$, then $Q_3 \leq \langle Q_1,Q_2\rangle$. Define an incidence structure $P(\mathfrak{L})$ as follows. The points are the elements of $\mathfrak{L}$ and the blocks are the sets of elements of $\mathfrak{L}$ contained in the subspaces $\langle Q_1,Q_2\rangle$ for $Q_1 \neq Q_2 \in \mathfrak{L}$. Then $P(\mathfrak{L})$ is an *n/r-dimensional* projective space over L. Furthermore, if G^* is of type II, then G preserves a symplectic polarity of $P(\mathfrak{L})$. A similar remark holds if G^* is of type III.

Proof of Theorem 4.4

First suppose that there exists r such that $n = rs$ and $PSL(s,q^r) \le G \le P\Gamma L(s,q^r)$. Clearly $s < n$, since otherwise G acts transitively on $P^{(m)}$, a contradiction. Hence G preserves a geometric r-spread $\mathfrak{L}$ for some $r \ge 2$. Suppose that $s > 1$. Let $X \in \mathfrak{L}$ and let $\ell \subseteq X$ be a line. Suppose that $\ell \subset Q \in \mathfrak{D}$ and that $\gamma \in Q \setminus X$. Then the orbit of γ under $G_{(X)}$ contains at least $(q^n - 1/q^r - 1) - 1$ points. Hence

$$q^m - 1/q-1 = |Q| > q^n - 1/q^r - 1.$$

Since $r \le n/2$ and $m \le n+1/2$, this is impossible. Hence each $X \in \mathfrak{L}$ is a subdesign, and so $m-1 \mid r-1$. Let $X_1 \ne X_2 \in \mathfrak{L}$ and let $Y = \langle X_1, X_2 \rangle$. If Y is a subdesign, then $m-1 | 2r-1$. But then $m-1 | (r-1, 2r-1) = 1$, a contradiction In particular, $s \ge 3$. Choose $\ell \subset Q \in \mathfrak{D}$ such that $\ell \subset Y$ and $Q \not\subseteq Y$. Let $\gamma \in Q \setminus Y$. Then the orbit of γ under $G_{(Y)}$ contains at least q^{2r} points. Hence

$$q^{2r} < q^m - 1/q-1 \le q^r - 1/q-1,$$

which is impossible. Hence $s = 1$ and G is soluble.

Next suppose that there exists $s \ge 4$ such that $PSp(s,q^r) \le G$, where $n = rs$. Arguing as above, we find that $r > 1$ and that each $X \in \mathfrak{L}$ is a subdesign. Let $X_1, X_2 \in \mathfrak{L}$ be chosen so that $Y = \langle X_1, X_2 \rangle$ is a totally isotropic line of $P(\mathfrak{L})$. There exists a line $\ell \subset Y$ such that $Q \not\subseteq Y$, where $\ell \subset Q \in \mathfrak{D}$. Let $\gamma \in Q \setminus Y$ and let $\gamma \in X_3 \in \mathfrak{L}$. Let $Z = \langle Y, X_3 \rangle$. Then Z is a plane of $P(\mathfrak{L})$ and $G_{(Y)}$ acts transitively on $\{X \in \mathfrak{L} \mid X \subseteq Z \setminus Y\}$. Hence the orbit of γ under $G_{(Y)}$ contains at least q^{2r} points, a contradiction.

Hence we can suppose that $q = 2^t$ and $G_2(2^{tr}) \le G$, where $n = 6r$. By Lemma 4.2, $r > 1$. Let $X \in \mathfrak{L}$ and let $\ell \subset X$ be a line. Suppose that $Q \not\subseteq X$, where $\ell \subset Q \in \mathfrak{D}$. Let $\gamma \in Q \setminus X$. By [7], the orbit of γ under $G_{(X)}$ contains at least $q^r(q^r + 1)$ points. Hence $q^m - 1/q - 1 > q^r(q^r + 1)$, and so $m \le 2r + 1 = n/3 + 1$, i.e., $n \le 3(m - 1)$. Since $m - 1 \mid n - 1$, this implies that $n - 1 = 2(m - 1)$. But then $n = 2m - 1$ is odd, a contradiction. Hence each $X \in \mathcal{L}$ is a subdesign. Let $\mathcal{g}$ be the generalized hexagon of order (q^r,q^r) on $P(\mathcal{L})$ which is preserved by $G_2(q^r)$. Let $Y = \langle X_1,X_2\rangle$ be a $\mathcal{g}$-line. Then we can suppose that there exists a line $\ell \subset Y$ such that $\ell \cap X_i \neq \emptyset$ for $i = 1,2$ and $Q \not\subseteq Y$, for $\ell \subset Q \in \mathcal{D}$. Let $\gamma \in Q \setminus Y$ and let $\gamma \in X_3 \in \mathcal{L}$. Since $G_2(q^r)$ acts transitively on the set of ordered ordinary hexagons of $\mathcal{g}$ [7], the orbit of γ under $G_{(X_1 \cup X_2)}$ contains at least q^{2r} points. Again, this is impossible. □

Corollary 4.5

Let $\mathcal{D}_3 = (P,\mathcal{D}_0)$ be the plane of $(M,\mathcal{D})$ generated by the Morley triple $\{\alpha,\beta,\gamma\}$. Then $H = \mathrm{Aut}\,\mathcal{D}_3$ does not act transitively on P.

Proof

Arguing as in Lemma 3.2, we see that if $\langle\alpha,\beta\rangle \subset Q \in \mathcal{D}_0$, then $H_{\{Q\}}$ acts 2-transitively on Q. By [7], either $\mathrm{Alt}(7) \leq H_{\{Q\}}/H_{(Q)}$ or $PSL(m,q) \leq H_{\{Q\}}/H_{(Q)}$. Since $m \geq 3$, H is not soluble. □

It seems likely that a more detailed study of $\mathcal{D}_3$ will allow us to eliminate case A. Let $\mathcal{L} = \{Q^\pi \mid \pi \in H\}$, where $\langle\alpha,\beta\rangle \subset Q \in \mathcal{D}_0$, and let $\Omega = \bigcup\{Q \mid Q \in \mathcal{L}\}$. The only candidates for the partial linear space $(\Omega,\mathcal{L})$ appear to be (possibly disjoint unions of) the flag-transitive generalized quadrangles, hexagons and octagons associated with various groups of Lie type. However, even these partial geometrics can be eliminated by looking at the actions of their automorphism groups on lines. I intend to say more on this matter in a later paper.

I end this section with an easy exercise. By Theorem 2.4 and the proof of Lemma 3.3, we already know that some of the planes of $(M,\mathcal{D})$ are neither projective nor affine. In fact, a stronger result holds.

Exercise 4.6

Prove that if $\mathcal{D}$ is a nontrivial 2-(n,m,1) design over a finite field, then $\mathcal{D}$ is neither a projective nor an affine plane.

5. EXTREMAL PROBLEMS

In this section, we consider projective space versions of theorems of Cameron [5],[6]. Let $G \leq P\Gamma L(\omega,q)$ have $n_k < \omega$ orbits on the set $PG^{(k)}(\omega,q)$ of k-subspaces of $PG(\omega,q)$ for each $k \in \mathbb{N}$. Fix an integer k such that $n_k > 1$, and let $\chi\colon PG^{(k)}(\omega,q) \to \{c_1,\ldots,c_{n_k}\}$ be a colouring which assigns distinct colours to the different G-orbits. The next result says roughly that G must locally preserve chains of subspaces. (Throughout, we adopt the convention that all r colours are used in an r-colouring).

Theorem 5.1 [18]

For all $r \geq 2$ and $t > k \geq 1$, there exists an integer $f(t,k,r)$ with the following property. Suppose that S is a projective space of GF(q) of dimension $m \geq f(t,k,r)$. If $S^{(k)}$ is r-coloured, then there exist

(i) subspaces $\emptyset \subseteq A \subsetneq B \subseteq C$ of S with $\dim C = t$, $\dim A = i$ and $\operatorname{codim}_C B = j$, where $0 < i + j \leq k$, and

(ii) distinct colours c_1, c_2 such that $T \in C^{(k)}$ is coloured c_2 if $A \subseteq T$ and $\dim(T \cap B) = k - j$, and is coloured c_1 otherwise. □

We would like to use this local information to understand the global structure of the ω-categorical structure $M = \langle PG(\omega,q),\chi\rangle$. Unfortunately, very little can be said in general. However, in certain extremal circumstances, it is possible to use this local information to identify M.

If $L \in PG^{(k+1)}(\omega,q)$, then the <u>colour scheme</u> of L is the n_k-tuple $(a_1,\ldots,a_{n_k})$, where a_i is the number of elements of $L^{(k)}$ which have colour c_i. The <u>colour scheme matrix</u> A of the colouring χ is the matrix whose rows are the distinct colour schemes.

Theorem 5.2 [18]

The colour scheme matrix A has rank n_k. Furthermore, the colours and colour schemes can be ordered in such a way that the first n_k rows of A form a lower triangular matrix with nonzero diagonal entries. $\square$

In particular, $n_k \leq n_{k+1}$. Suppose that $n_k = n_{k+1}$. Then A is a nonsingular lower triangular matrix, and two $(k+1)$-subspaces lie in the same G-orbit if and only if they have the same colour scheme. Looking at the last row of A, we see that there exists a colour, say blue, which lies in a unique colour scheme. So after amalgamating the other colours and applying Theorem 5.1 to the resulting 2-colouring, we obtain

Theorem 5.3 [18]

Suppose that $G \leq P\Gamma L(\omega,q)$ satisfies $1 < n_k = n_{k+1} < \omega$. Then there exists a G-invariant colouring φ: $PG^{(k)}(\omega,q) \to \{red,blue\}$ and integers i,j with $0 < i + j \leq k$ such that

(i) G acts transitively on blue k-spaces and on $(k+1)$-spaces which contain a blue k-space; and

(ii) if $C \in PG^{(k+1)}(\omega,q)$ contains a blue k-space then there are subspaces $\varnothing \subseteq A \subsetneq B \subseteq C$ with $\dim A = i$ and $\operatorname{codim}_C B = j$ such that $T \in C^{(k)}$ is blue if and only if $A \subseteq T$ and $\dim(B \cap T) = k - j$. $\square$

If k is small, this condition is strong enough to enable us to identify $M = \langle PG(\omega,q),\chi\rangle$. For example, the following is the main result of [18].

Theorem 5.4

Suppose that $q \neq 2$. If $G \leq P\Gamma L(\omega,q)$ acts transitively on the points of $PG(\omega,q)$ and satisfies $1 < n_2 = n_3 < \omega$, then one of the following cases holds.

(i) G preserves a symplectic polarity of $PG(\omega,q)$ and acts transitively on the sets of totally isotropic lines and nonisotropic lines.

(ii) G preserves a geometric 2-spread L of $PG(\omega,q)$ and acts transitively on the planes of the incidence structure $P(L) \simeq PG(\omega,q^2)$. □

This is proved by working through the various possibilities for the pair of integers (i,j), $0 < i + j \leq 2$, given in Theorem 5.3. To give a flavor of the type of reasoning involved, I will give the details for two of the more interesting cases.

Suppose that $i = 1$ and $j = 0$. So if the plane Q contains a blue line, then there exists a point $\alpha(Q) \in Q$ such that $\ell \in Q^{(2)}$ is blue if and only if $\alpha(Q) \in \ell$. Let S be a finite dimensional subspace which contains a blue line, and let $\mathcal{B}_S = \{\ell \in S^{(2)} \mid \ell$ is blue$\}$. Suppose that $\ell \in \mathcal{B}_S$ and $\alpha \in S \setminus \ell$. Considering the plane $Q = \langle\ell,\alpha\rangle$, we see that there is unique point $\beta \in \ell$ such that $\langle\alpha,\beta\rangle \in \mathcal{B}_S$. Thus the incidence structure $(S,\mathcal{B}_S)$ ia a (possibly degenerate) generalized quadrangle. By Buekenhout and Lefevre [3], if $\dim S > 6$ then there exists a point $\alpha \in S$ such that $\mathcal{B}_S = \{\ell \in S^{(2)} \mid \alpha \in \ell\}$. This implies that there is a point $\alpha \in PG(\omega,q)$ such that $\ell \in PG^{(2)}(\omega,q)$ is blue if and only if $\alpha \in \ell$, This contradicts the assumption that G acts transitively on points.

Suppose that $i = 0$ and $j = 2$. So if the plane Q contains a blue line, then there exists a point $\alpha(Q) \in Q$ such that $\ell \in Q^{(2)}$ is blue if and only if $\alpha(Q) \notin \ell$. Let S be a finite dimensional subspace, and consider the incidence structure $(S,\mathbb{R}_S)$, where $\mathbb{R}_S = \{\ell \in S^{(2)} \mid \ell$ is red$\}$. Let $\ell \in \mathbb{R}_S$ and $\alpha \in S \setminus \ell$. If the plane $Q = \langle\ell,\alpha\rangle$ contains a blue line, then $\alpha(Q) \in \ell$ and $\alpha(Q)$ is the unique point of ℓ which is collinear with α in $(S,\mathbb{R}_S)$. Otherwise, α is collinear with every point of ℓ. By definition, $(S,\mathbb{R}_S)$ is a Shult space. By Buekenhout and Shult [4], there exists a (possibly degenerate) symplectic form σ on the underlying vector space of S such that $\mathbb{R}_S$ is the set of lines which are totally isotropic with respect to σ. It follows that there is a symplectic form σ on $V(\omega,q)$ such that $\ell \in PG^{(2)}(\omega,q)$ is red if and only if ℓ is totally isotropic with respect to σ. Since G acts transitively on points, σ is nondegenerate. Thus G preserves a symplectic polarity of $PG(\omega,q)$ and acts transitively on the set of nonisotropic lines. The reader is referred to [18] for a proof that G also acts transitively on totally isotropic lines.

If $G \leq P\Gamma L(\omega,q)$ is the full automorphism group of either a symplectic polarity or a geometric 2-spread of $PG(\omega,q)$, then G satisfies $n_{2k} = n_{2k+1} = k+1$ for each $k \in \mathbb{N}$.

Problem 5.5

Suppose that $G \leq P\Gamma L(\omega,q)$ acts transitively on the points of $PG(\omega,q)$ and satisfies $1 < n_k = n_{k+1} < \omega$ for some $k \in \mathbb{N}$. Does G preserve either a symplectic polarity or a geometric 2-spread of $PG(\omega,q)$?

This problem is open even for $k = 3$. In this case, the hardest situation to analyze is when each 4-subspace contains at most one blue plane. When this occurs, if $Q \in PG^{(3)}(\omega,q)$ is blue and $\ell \in Q^{(2)}$, then $acl(\ell) = Q$. A similar difficulty arises for larger values of k. Thus problem 5.5 provides further motivation for the problem of understanding expansions of $PG(\omega,q)$ in which certain subspaces are no longer algebraically closed.

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