# MODEL THEORY, STABILITY THEORY & STABLE GROUPS

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The aim of this chapter is to introduce the reader to the theory of stable groups not to give a rigorous exposition of the general theory. Thus we tend to proceed from the concrete to the abstract, with several examples and analyses of special cases along the way. On the other hand, getting to grips with stable groups presupposes some understanding of the point of view of model theory in general and stability theory in particular, and the first few sections are devoted to the latter.

## 1. MODEL THEORY

By a <u>relational structure</u> $\underline{M}$ we understand a set $M$ (called the universe or underlying set of $\underline{M}$) equipped with relations $R_i$ of arity $n_i < \omega$ say, for $i \in I$. Namely, for $i \in I$, $R_i$ is a subset of the Cartesian product $M^{n_i}$. Here $I$ and $<n_i : i \in I>$ depend on $\underline{M}$ and are called the <u>signature</u> of M. We also insist that $I$ always contains a distinguished element $i_=$ such that $R_{i_=}$ is the diagonal $\{(a,a) : a \in M\} \subseteq M^2$. Often the distinction between $M$ and $\underline{M}$ is blurred. The model theorist is interested in certain subsets of $M$ and of $M^n$ (the <u>definable</u> sets) which are obtained in a simple fashion from the $R_i$. So $\mathcal{D}(M)$ is a collection of subsets of $M^n$, $n < \omega$, which can be characterized as follows:

    (i) Every $R_i \in \mathcal{D}(M)$.

    (ii) If $n < \omega$, $X \in \mathcal{D}(M)$ is a subset of $M^n$ and $\pi$ is a permutation of $\{1,...,n\}$ then $\pi(X) = \{(a_{\pi(1)},...,a_{\pi(n)}) : (a_1,...,a_n) \in X\} \in \mathcal{D}(M)$.

    (iii) $\mathcal{D}(M)$ is closed under Boolean combinations, i.e. if $n < \omega$ and

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X, Y $\in \mathcal{D}(M)$ are subsets of $M^n$ then $X \cup Y$, $X \cap Y$, $M^n - X$ are all in $\mathcal{D}(M)$.

(iv) If $X \in \mathcal{D}(M)$ and $Y \in \mathcal{D}(M)$ then $X \times Y \in \mathcal{D}(M)$.

(v) If $X \in \mathcal{D}(M)$ is a subset of $M^{n+m}$, then the projection of $X$ on $M^n$ is in $\mathcal{D}(M)$.

(vi) If $X \in \mathcal{D}(M)$ is a subset of $M^{n+m}$ and $\bar{a} \in M^n$ then $X_{\bar{a}} = \{\bar{b} \in M^m : (\bar{a}, \bar{b}) \in X\}$ is in $\mathcal{D}(M)$.

(vii) Nothing else is in $\mathcal{D}(M)$.

We call $\mathcal{D}(M)$ the class of <u>definable sets</u> of M.


These definable sets can be defined (and usually are) <u>syntactically</u>. Associated to the relational structure M (in fact to its signature) is a <u>language</u> L(M) consisting of symbols: $P_i$ for each $i \in I$, "variables" $x_j$ for each $j < \omega$, and logical symbols $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\forall$ (for all) and $\exists$ (there exists). L(M)-formulas are constructed from these symbols as follows: if $x_j$ are variables then $P_i x_1 ... x_{n_i}$ is an (atomic) formula. If $\varphi, \psi$ are formulas and x is a variable then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg \varphi$, $(\exists x)\varphi$, $(\forall x)\varphi$ are all formulas. A variable x is said to be free in the formula $\varphi$ if some occurrence of x in $\varphi$ is not in the scope of any quantifier. We write $\varphi(x_1,...,x_n)$ to mean that $x_1,...,x_n$ are the free variables in the formula $\varphi$. We then define "$\varphi(x_1,...,x_n)$ is true of $(a_1,...,a_n)$ in <u>M</u>" (where $a_1,...,a_n \in M$) as follows:

If $\varphi$ is atomic, say $P y_1 ... y_n$, and for some permutation $\pi$ of $\{1,...,n\}$, $x_i = y_{\pi(i)}$ then $\varphi(x_1,...,x_n)$ is true of $(a_1,...,a_n)$ in M if $(a_{\sigma(1)},...,a_{\sigma(n)}) \in P$ where $\sigma = \pi^{-1}$.

If $\psi$ is $(\exists x_{n+1}) \varphi$ and $x_{n+1}$ is a free variable of $\varphi$, then $\psi(x_1,...,x_n)$ is true of $(a_1,...,a_n)$ in M if there is $a_{n+1} \in M$ such that $\varphi(x_1,...,x_n,x_{n+1})$ is true of $(a_1,...,a_{n+1})$ in M. Similarly for $\psi = (\forall x_{n+1}) \varphi$.

The clauses for $\wedge, \vee, \neg$ are obvious. We abbreviate "$\varphi(x_1,...,x_n)$ is true of $(a_1,...,a_n)$ in M" by the notation $\underline{M} \models \varphi(a_1,...,a_n)$. (Note this notation depends on our having listed the free variables in $\varphi$ in a certain order).

By abuse of everything, we can and will think of $\underline{M} \models \varphi(a_1,...,a_n)$ as saying that $\varphi$ is true when we substitute $a_i$ for $x_i$.

It is now routine to check that

**Fact 1.1.** If $X \subset M^n$, then $X \in \mathcal{D}(M)$ if and only if there are an L(M) formula $\varphi(x_1,...,x_n,y_1,...,y_m)$ and $b_1,...,b_m \in M$ such that

$$X = \{\bar{a} \in M^n : M \models \varphi(\bar{a},\bar{b})\}.$$

The syntactic approach to defining definable sets appears at first to be preferable as one can make the following definition.

**Definition 1.2.** $X \in \mathcal{D}(M)$, a subset of $M^n$, is said to be A-definable or defined over A (for $A \subset M$) if in Fact 1.1 we can choose $\varphi$ with $\bar{b} \subset A$.

**Example 1.3.** Let K be an <u>algebraically closed field</u>. We can consider K as a relational structure in the above sense by choosing $\{0\}$, $\{1\}$, and the graphs of addition and multiplication as the "distinguished" relations. Note that if $P_1,...,P_r$ are polynomials in n-variables over K, then the subset V of $K^n$ consisting of the simultaneous zero set of these polynomials is a <u>definable set</u>. These are called the <u>affine algebraic sets</u>. Finite Boolean combinations of such sets are called <u>constructible sets</u> of K, and either Tarski's "quantifier elimination theorem" (quantifier elimination in a language with function symbols for addition and multiplication) or Chevalley's theorem states
**Fact:** The constructible sets of K are precisely the definable sets of K.

For an affine algebraic set $X \subseteq K^n$ there is an algebraic-geometrical notion of X being defined over k (k a subfield of K) which may have some discrepancy with the model theoretic notion (Definition 2.2). Namely: let I(X) $\subseteq K[x_1,...,x_n]$ be the ideal of polynomials which vanish on X. According to the algebraic geometer X is defined over $k \subset K$ if I(X) can be generated as an ideal by polynomials in $k[x_1,...,x_n]$.

We do have (for $X \subseteq K^n$ affine algebraic and k subfield of K)
**Fact:** X is defined over k in the model theoretic sense iff X is defined over $k^{p^{-\infty}}$ in the sense of algebraic geometry (where $p = \text{char } K$).

So if k is perfect, or char k = 0, the notions agree.

**1.4.** The usual procedure in model theory is to start with a language $L$ and to consider various subclasses of $L$-structures. So $L$ will essentially be a signature as above, i.e. will consist of a set of relation symbols of specified arity and an $L$-structure will be a relational structure equipped with corresponding relations of the right arity. This enables us to compare $L$-structures in various respects. For instance, by an L-sentence we mean an L- formula which has no free variables. An $L$-structure $M$ is said to be a model of a set $\Gamma$ of $L$-sentences if for every $\sigma \in \Gamma$, $M \models \sigma$, i.e. every $\sigma \in \Gamma$ is true in $M$. A set of $L$-sentences $\Gamma$ is said to be consistent if it has a model. A consistent set of sentences $\Gamma$ is said to be a complete theory if for every $L$-sentence $\sigma$ either $\sigma \in \Gamma$ or $\neg\sigma \in \Gamma$, equivalently for some $M$, $\Gamma = \{\sigma: M \models \sigma\}$; in the latter case $\Gamma$ being called the theory of $M$. Two $L$-structures $M$ and $N$ are called elementarily equivalent if they have the same theory, equivalently they satisfy the same $L$-sentences. As an example, any two algebraically closed fields of the same characteristic, say p, are elementarily equivalent; in other words the set $T_{ACF_p}$ of sentences (in the language in Example 1.3 for example) true in all algebraically closed fields of characteristic p is a complete theory.

     A crucial tool in model theory is the compactness theorem: a set of sentences $\Gamma$ is consistent iff every finite subset of $\Gamma$ is consistent. This gives substance to the following important notion: Let $M, N$ be $L$-structures with $M$ a substructure of $N$ ($M \subseteq N$, with the obvious meaning). $M$ is said to be an elementary substructure of $N$, $M \prec N$, if for every formula $\varphi(\overline{x})$ of $L$ and $\overline{a} \in M$, we have $M \models \varphi(\overline{a})$ iff $N \models \varphi(\overline{a})$.

     Let us remark that if $M \prec N$ then any definable set $X \subseteq M^n$ in $M$ has a canonical extension to a definable set $X' \subseteq N^n$ in $N$. Namely, let $\varphi(\overline{x},\overline{y})$, $\overline{a} \subseteq M$ be such that $\varphi(\overline{x},\overline{a})$ defines $X$ in $M$. Then let $X' = \{\overline{x} \in N^n : N \models \varphi(\overline{x},\overline{a})\}$. Note that $X \subseteq X'$ and $X'$ does not depend on the particular choice of $\varphi$ and $\overline{a}$.

     The compactness theorem yields for any infinite $M$, elementary extensions $N$ of $M$ of arbitrarily large cardinality. Another consequence of

Tarski's quantifier elimination is that if $K_1 \subseteq K_2$ are algebraically closed fields then $K_1 \prec K_2$, noting the following characterisation : let $M_1 \subseteq M_2$, then $M_1 \prec M_2$ iff for any non-empty $M_1$-definable subset $X$ of $M_2$, $X \cap M_1 \neq \varnothing$.

## 1.5. Saturated models.

Let $\kappa$ be an infinite cardinal. The structure $N$ is said to be $\underline{\kappa\text{-}}$ $\underline{\text{saturated}}$ if for any $A \subset N$ with $|A| < \kappa$ and any collection $X_i, i \in I$ of

$A$-definable subsets of $N$ with the finite intersection property ( $\bigcap_{i \in J} X_i \neq \varnothing$ for

all finite $J \subseteq I$), we have $\bigcap_{i \in I} X_i \neq \varnothing$. Again the compactness theorem gives for

any $M$ and $\kappa$ some $\kappa$-saturated $N \succ M$.


It is worth noting that the definition above of $\kappa$-saturation would be equivalent if we allowed the $X_i$ to be $A$-definable subsets of $N^n$ for any $n \geq 1$ This apparently stronger fact follows by use of the $\underline{\text{existential quantifier}}$.

One can think of the property of $\kappa$-saturation of $N$ as meaning that for any $M \prec N$ with $|M| < \kappa$, any situation that can happen in some elementary extension of $M$ already happens in $N$. (In this sense $N$ is like a universal domain. In fact, what Weil calls a universal domain - an algebraically closed field of infinite transendence degree $\kappa$ over the prime field- is $\kappa$-saturated). Moreover if $M \equiv N$ and $|M| < \kappa$ then there is an elementary embedding (obvious meaning) of $M$ into $N$. $\underline{\text{It will be convenient to assume that any}}$ $\underline{\text{complete theory has models which are } \kappa\text{-saturated and of cardinality } \kappa, \text{ for}}$ $\underline{\text{arbitrarily large } \kappa}$. Such a model, $N$ say, will have homogeneity properties in addition to saturation properties, which are pointed out subsequently. (For stable theories the existence of such models is guaranteed. Otherwise, it depends on set theory).

Let us now fix such a model $N$ ($\kappa$-saturated of cardinality $\kappa$ for some large $\kappa$). $A, A', B$ etc. will denote subsets of $N$ of cardinality $<\kappa$, and $M$, $M', M_1,...$ elementary substructures of $N$ of cardinality $< \kappa$ (often called $\underline{\text{models}}$). We now introduce the important notion of a $\underline{\text{type}}$.

Let $A \subseteq N$. By a <u>complete n-type</u> over $A$ we mean a maximal consistent collection of A-definable subsets of $N^n$ (where <u>consistent</u> means having the finite intersection property). Alternatively, with some abuse of earlier notation, a complete n-type over $A$ is a maximal set $\Gamma$ of formulas of the form $\varphi(x_1,...,x_n,\bar{a})$ where $\bar{a} \subset A$ and for $\varphi_1,...,\varphi_m \in \Gamma, N \models$
$$\exists \bar{x} \left( \bigwedge_{i=1}^{m} \varphi_i (\bar{x})\right).$$

Let $b_1,...,b_n \in N$. By the type of $\bar{b}$ over $A$ (in $N$ if you wish), $tp(\bar{b}/A)$ is meant the collection of A-definable subsets of $N^n$ containing $\bar{b}$. $tp(\bar{b}/A)$ is clearly a complete n-type over $A$. Conversely, saturation of $N$ implies that every complete n-type $\Gamma$ over $A$ is the form $tp(\bar{b}/A)$, for some $\bar{b}$ $\in N^n$. $\bar{b}$ is said to realize $\Gamma$. The set of complete n-types over $A$ is denoted $S_n(A)$, and types themselves are usually denoted by $p$, $q$ etc.

The fact that $N$ is saturated in its own cardinality gives us a nice characterization: if $\bar{b}_1 \in N^n$, $\bar{b}_2 \in N^n$ then $tp(\bar{b}_1/A) = tp(\bar{b}_2/A)$ iff there is an automorphism $f$ of $N$ such that $f(\bar{b}_1) = \bar{b}_2$ and $f$ fixes $A$ pointwise. (Similarly for types of infinite tuples of cardinality $< \kappa$).

Saturation of $N$ also enables us to give the notion "definable over $N$" a "Galois theoretic" interpretation. Firstly, the compactness theorem yields: Let $X \subseteq N^n$ be definable, let $A \subset N$ and suppose that whether or not some $\bar{b} \in$ $N^n$ is in $X$ depends only on $tp(\bar{b}/A)$. Then $X$ is A-definable. In conjunction with the previous observation this shows that for definable $X \subseteq N^n$, $X$ is A-definable iff for every automorphism $f$ of $N$ which fixes $A$ pointwise, $f(X)$ $= X$.

## 1.6.  $N^{eq}$

It will be sometimes convenient (especially when dealing with groups) to work in a structure which is "closed under definable quotients". We can construct from $N$ such a universe, $N^{eq}$, which is "essentially" the same as $N$. Informally, $N^{eq}$ is the disjoint union of a collection of universes, one of which is $N$, and each being picked out by a new predicate. Each new universe is identified, by means of a new function symbol, with the set of classes of a

$\emptyset$–definable equivalence relation on $N^m$ for some $m < \omega$. Moreover, all $\emptyset$-definable equivalence relations on $N^m$, $m < \omega$ are accounted for by these new universes. More formally, if $L$ is the original language then $L_{eq}$ will be $L$ augmented by a unary predicate symbol $P_E$ and a function symbol $f_E$ for every $\emptyset$-definable equivalence relation $E$. $N^{eq}$ will be the $L_{eq}$ structure whose underlying set is the disjoint union of the interpretations of the various $P_E$'s (which we can call $N_E$). $N_=$ is precisely $N$ with its original $L$-structure. If $E$ is an equivalence relation on $N^m$, then the interpretation of $f_E$ is a surjective map $N^m \to N_E$ whose fibres are the $E$-classes. This whole construction is of course a function of $Th(N)$, so for any $M \equiv N$, we can obtain in the same way $M^{eq}$. We however make the additional stipulation that in the models of $T^{eq}$ ($= Th(N^{eq})$), every element should satisfy one of the predicates $P_E$ (So $T^{eq}$ is a "many sorted" theory). This can be formally accomplished by requiring that every $L_{eq}$ formula we consider must state for each of its variables the predicate $P_E$ in which the variable lies. We think of the $P_E$'s as picking out certain sorts.

$N^{eq}$ has the following properties:

**1.7.**    (i) A subset $X$ of $N^m$ definable in $N^{eq}$ is definable in $N$.

(ii) Any automorphism of $N$ has a unique extension to an automorphism of $N^{eq}$.

(iii) For any definable subset $X$ of $(N^{eq})^n$ there is an element $a_X \in N^{eq}$ such that an automorphism $f$ of $N^{eq}$ fixes $X$ setwise iff it fixes $a_X$.

An element $b$ is said to be <u>algebraic</u> over $A$ ($A, b \subset N$ or even $N^{eq}$) if $b$ lies in some finite $A$-definable set. $b$ is <u>definable</u> over $A$ if $\{b\}$ is $A$-definable.

$acl\,(A) = \{b: b$ is algebraic over $A\}$,

$dcl\,(A) = \{b: b$ definable over $A\}$.

A definable set $X$ is said to be <u>almost over</u> $A$ ($X, A$ in $N$ or $N^{eq}$) if $X$ has finitely many images under $A$-automorphisms of $N$.

**Fact 1.8.**    $X$ is almost over $A$ iff $X$ is acl(A)-definable in $N^{eq}$.

## 2. ω-STABILITY

Stability is an hypothesis on the "complexity" of the family of definable sets in a model. Generally we talk of stability, superstability, ω-stability (or total transcendence) of a <u>complete theory T</u>, which translates into certain rank or dimension functions on the definable sets of a saturated model $N$ of $T$ being everywhere defined. Probably the easiest such rank to define and understand is Morley rank, RM.

**Definition 2.1.** Let $n < \omega$, $X \subseteq N^n$ a definable set. $RM_n(X)$ is defined as follows:

(i)     $RM_n(X) \geq 0$ if $X \neq \varnothing$.

$RM_n(X) \geq \delta$ if $RM_n(X) \geq \alpha$ for all $\alpha < \delta$ ($\delta$ limit).

$RM_n(X) \geq \alpha + 1$ if there are pairwise disjoint definable subsets $X_i \subset N^n$ for $i < \omega$ such that $RM_n(X \cap X_i) \geq \alpha$ for all $i < \omega$. If $RM_n(X) = \alpha$ some $\alpha$, we say $RM_n(X)$ is defined. Otherwise (i.e. if $RM_n(X) \geq \alpha$, for all $\alpha$) we put $RM_n(X) = \infty$.

(ii)    If $p \in S_n(A)$, $A \subset N$, we put $RM_n(p) = \min \{RM_n(X): X \in p\}$.

## Remarks and Definitions 2.2.

(i)     Let $M$ be an arbitrary structure. Let $N$ be a $\kappa$-saturated elementary extension of $M$ of cardinality $\kappa$, where $\kappa > |M| +$ cardinality of $L(M)$. Let $X \subseteq M^n$ be definable in $M$. Let $X' \subseteq N^n$ be the canonical extension of $X$ to a set definable in $N$ (as in 1.4). Then we define $RM_n(X) = RM_n(X')$.

(ii)    Let $T$ be a complete theory and let $N$ be a model of $T$ ($N$ saturated in its own cardinality $\kappa$, for large $\kappa$). We say $T$ is <u>totally transcendental</u> if for every definable $X \subseteq N^n$, $RM_n(X)$ is defined (actually it is enough to demand this for $X \subseteq N$).

(iii)    If the language of $T$ is countable and $T$ is totally transcendental, we say that $T$ is $\omega$-stable. (This notation will be explained later: basically here it means that for countable models $M$ of $T$, $S_1(M)$ is countable). We may, by abuse of language, still use the expression $\omega$-stable to mean totally transcendental, even if $T$ is not countable.

(iv)    We usually drop the subscript n from $RM_n$ when the arity of $X$ is either clear from the context or unimportant.

**Fact 2.3.**

(i)    If $RM(X) = \alpha$ and $X$ is $A$-definable then there is a complete type $p \in S(A)$, with $RM(p) = \alpha$ and $X \in p$.

(ii)    If $RM(X) = \alpha$ then there is a <u>greatest</u> $k < \omega$ for which there are pairwise disjoint $X_i$ for $i = 0,1,...,k-1$ such that $RM(X \wedge X_i) = \alpha \ \forall i < k$. $k$ is called the <u>Morley degree</u> of $X$. Similarly one can define the Morley degree of a complete type $p \in S(A)$.

Let us remark that if we work in $N^{eq}$ then for any sort $S$ ($S$ is one of the $P_E$), we can define $RM_S(X)$ for $X$ a definable set of elements of sort $S$. It will then be the case that if $RM(X)$ is defined for all $X \subseteq N$ then also $RM_S(X)$ is defined for every sort $S$ in $N^{eq}$ (i.e. $T$ totally transcendental $\Rightarrow$ $T^{eq}$ is totally transcendental).

## Example 2.4.   Strongly minimal sets

Let $X \subseteq N^n$ be a definable set in (saturated) $N$. $X$ is said to be <u>strongly minimal</u> if for every definable $Y \subseteq N^n$ either $X \cap Y$ or $X - Y$ is finite.

Similarly, working in $N^{eq}$ we can speak of a definable set $X$ in sort $S$ being strongly minimal. The complete theory $T$ is said to be strongly minimal if the universe of a saturated model $N$ of $T$ is strongly minimal (Note: it does not make sense to speak of $T^{eq}$ being strongly minimal). As a case study we will show what Morley rank means in the case of strongly minimal theories (and also strongly minimal sets).

We first make a trivial remark:

**Remark:** Let $b, a_1,...,a_k \in N$, $A \subset N$ with $b \in acl(a_1,...,a_k,A)$. Then there is an A-definable set $X \subseteq N^{k+1}$ such that $(a_1,...,a_k, b) \in X$ and such that whenever $(a_1',...,a_k',b') \in X$ then $b' \in acl(a_1',...,a_k',A)$. We will say that the set $\{a_1,...,a_k\}$ is <u>algebraically</u> <u>independent</u> over A if for all i, $a_i \notin acl(a_1,...,a_{i-1},a_{i+1},...,a_k,A)$.

   (i) of the next fact is easy, but (ii) is less so.

**Fact 2.5.**  Let N be strongly minimal, $A \subset N$.

   (i)     Let $a_1,a_2 \in N$, $a_1,a_2 \notin acl(A)$. Then $tp(a_1/A) = tp(a_2/A)$.

   (ii)    Let $a_1,...,a_k \in N$. Then all maximal algebraically independent over A subsets of $\{a_1,...,a_k\}$ have the same size m. We call this number $dim(a_1,...,a_k/A)$. This being clearly a function of $tp(\bar{a}/A)$ (where $\bar{a} = (a_1,...,a_k)$) we write also $m = dim(tp(\bar{a}/A))$.

On the other hand, let $X \subseteq N^k$ be defined over A. We define $dim X$ to be max $\{dim(\bar{a}/A) : \bar{a} \in X\}$. This does not depend on A. Namely suppose $dim X$ as defined above is equal to r. Let $B \supset A$. Let $dim(\bar{a}/A) = r$ where $\bar{a} = (a_1,...,a_r,...,a_k) \in X$ and without loss of generality $\{a_1,...,a_r\}$ is algebraically independent over A. Let $\{b_1,...,b_r\}$ be algebraically independent over B. By Fact 2.5 (i) applied repeatedly, $tp(a_1,...,a_r/A) = tp(b_1,...,b_r/A)$. Thus by saturation of N we can extend $(b_1,...,b_r)$ to a sequence $\bar{b} \in N^k$ such that $tp(\bar{b}/A) = tp(\bar{a}/A)$. In particular $\bar{b} \in X$ and clearly $dim(\bar{b}/B) = r = max \{dim(\bar{b}'/B): \bar{b}' \in X\}$

We aim to show that for strongly minimal N, Morley rank equals dimension for types or definable sets of k-tuples, $k < \omega$.

**Lemma 2.6.** (N strongly minimal). Let $X \subset N^k$ be definable. Then $dim X \geq r+1$ iff there are pairwise disjoint definable $X_i \subseteq X$ for $i \in \omega$ such that $dim X_i \geq r$ for all r.

**Proof:** Note first that every nonempty definable set has dimension $\geq 0$. Suppose now $\dim X \geq r + 1$. Suppose $X$ to be A-definable and let $(a_1, \ldots, a_{r+1}, \ldots, a_k) = \bar{a} \in X$ with $\dim(\bar{a}/A) \geq r + 1$. Without loss of generality $\{a_1, \ldots, a_{r+1}\}$ is algebraically independent over $A$. Let $\{b_1^i : i < \omega\}$ be algebraically independent over $A$ (by saturation of $N$). By Fact 2.5. $\mathrm{tp}(b_1^i/A) = \mathrm{tp}(a_1/A) \; \forall \, i < \omega$, thus by "homogenity" of $N$ there are $\bar{b}^{\,i} = (b_1^i, \ldots, b_k^i)$ for $i < \omega$ such that $\mathrm{tp}(\bar{b}^{\,i}/A) = \mathrm{tp}(\bar{a}/A)$. For $i < \omega$ let $X_i = \{\bar{x} : \bar{x} \in X \text{ and } x_1 = b_1^i\}$. Then $X_i \subset X$, the sets $X_i$ are pairwise disjoint and $\dim X_i \geq r$ (for the latter, note $X_i$ is $A \cup \{b_1^i\}$-definable, $\bar{b}^{\,i} \in X_i$ and $\{b_2^i, \ldots, b_{r+1}^i\}$ is algebraically independent over $A \cup \{b_1^i\}$). So this shows left to right.

  Conversely, suppose that $X_i \subseteq X$ for $i \in \omega$, with the $X_i$ pairwise disjoint and $\dim X_i \geq r$. Suppose $X$ and all the $X_i$ to be B-definable. For each $i$ let $\bar{b}^{\,i} = (b_1^i, \ldots, b_k^i)$ be in $X_i$ with $\dim(\bar{b}^{\,i}/B) \geq r$. As there are infinitely many $i$, we can assume that for each $i$, $\{b_1^i, \ldots, b_r^i\}$ is algebraically independent over $B$. By repeated application of Fact 25 (i), $\mathrm{tp}(b_1^i, \ldots, b_r^i/B) = \mathrm{tp}(b_1^j, \ldots, b_r^j/B)$ for $i, j < \omega$. So by section 1.4 for each $i, j < \omega$ there is a B-automorphism taking $(b_1^i, \ldots, b_r^i)$ to $(b_1^j, \ldots, b_r^j)$. As every B-automorphism leaves each $X_i$ as well as $X$ setwise invariant, we can assume that there are $b_1, \ldots, b_r$ such that for all $i < \omega$ and $j \leq r$, $b_j^i = b_j$ (namely the $\bar{b}^i$ have same first $r$ coordinates).

**Claim:** $Y = \{\bar{c} \in N^{k-r} : (b_1, \ldots, b_r, \bar{c}) \in X\}$ is infinite. For otherwise, as $X_i \subseteq X$ for $i < \omega$, there would be $\bar{c}$ with $(b_1, \ldots, b_r, \bar{c}) \in X_i \cap X_j$ for some $i \neq j$, contradicting pairwise disjointness.

  By the claim and saturation of $N$ we can find $\bar{c} \in Y$ and some coordinate of $\bar{c}$, say $c_1$ such that $c_1 \notin \mathrm{acl}(b_1, \ldots, b_r, B)$. But then

$\dim X \geq r + 1$. This completes the proof of Lemma 2.6.                    $\square$

**Corollary 2.7.** (N strongly minimal).

      (i)    Let $X \subseteq N^k$ be definable, then $\dim X = RM(X)$.

      (ii)   Let $a_1,...,a_k \in N$, $A \subset N$, Then $\dim(a_1,...,a_k/A) = RM(tp(a_1,...,a_k/A))$.

**Proof:** (i) follows from Lemma 2.6 and the definition of Morley rank. (ii) follows by noting that if $p = tp(a_1,...,a_k/A)$ then $\dim p = \min \{\dim X : X \in p\} = RM(p)$.                    $\square$

**Example 2.8.** Let $K$ be a saturated algebraically closed field (with no additional structure beyond the field structure).

**Lemma 2.9.** (i) $K$ is strongly minimal.

      (ii) If $k \subset K$ and $a_1,...,a_n$ are in $K$, then $a_1,...,a_n$ are algebraically independent over $k$ in the sense of model theory iff they are algebraically independent over $k$ in the sense of field theory.

**Proof:** (i) We already remarked in Example 1.3 that any definable set $X \subset K$ is constructible, i.e. a finite Boolean combination of algebraic subsets of $K$. Noting that an algebraic subset of $K$ must be finite (or all of $K$), we see that $X$ is finite or cofinite, whereby $K$ is strongly minimal.

      (ii) We use again the fact (quantifier elimination) that every formula $\varphi(\overline{x})$ is equivalent in $K$ to a quantifier free formula $\psi(\overline{x})$ in a language with function symbols for $+, \cdot$ and constant symbols for $0, 1$. It easily follows that if $a_1,...,a_n$ are algebraically dependent over $k$ in the model-theoretic sense then $a_1,...,a_n$ satisfy a nontrival polynomial relation over $k$.                    $\square$

      Let now $V = V(\overline{a}/k) = \{ \overline{b} \in K^n : f(\overline{b}) = 0 \text{ whenever } f(\overline{a}) = 0 \text{ for } f \in k[\overline{x}]\}$. By definition, the <u>algebraic-geometrical dimension</u> of this affine algebraic set $V$ is the transcendence degree of $k(\overline{a})$ over $k$.

**Proposition 2.10.** (With the above notation)

$RM(tp(\bar{a}/k)) = RM(V) =$ algebraic geometrical dimension of $V$.

**Proof:** By part (ii) of the Lemma

(a) transcendence degree of $k(\bar{a})$ over $k = \dim(\bar{a}/k)$.

By Corollary 2.7 (ii)

(b) $RMtp(\bar{a}/k) = \dim(\bar{a}/k)$.

By part (ii) of the Lemma again, $\dim(\bar{b}/K) \le \dim(\bar{a}/K)$ for all $\bar{b} \in V$ and thus

(c) $\dim(V) = \dim(\bar{a}/k)$.

The Proposition now follows from (a), (b) and (c).               □

## 3. STABILITY

Although much of this volume will concentrate on $\omega$-stable groups, and even $\omega$-stable groups of finite Morley rank, it is worth saying something about stable theories in general. Stability is a property of certain theories (the stable ones) which is considerably weaker than $\omega$-stability. It might be considered as "local $\omega$-stability", and we subsequently introduce it in this way. For now, we can take a definition of stability as: $T$ is stable if there is no formula $\varphi(\bar{x}, \bar{y})$ and tuples $\bar{a}_i, \bar{b}_j$ $(i,j < \omega)$ in a model $M$ of $T$ with $M \models \varphi(\bar{a}_i, \bar{b}_j)$ iff $i \le j$; or equivalently there is no $\psi(\bar{x}_1, \bar{x}_2)$ and $\bar{a}_i$ $(i \le \omega)$ with $M \models \psi(\bar{a}_i, \bar{a}_j)$ iff $i \le j$. ("One cannot define an order").

Under the sole assumption of stability, a good notion of independence can be defined: for a model $N$ of stable $T$, $\bar{a} \subset N$, $A \subset B \subset N$, we will make sense of " $\bar{a}$ is free from $B$ over $A$", " $\bar{a}$ and $B$ are independent over $A$". This will depend only on the formulas true of $\bar{a}, A, B$ and so we will also say "tp($\bar{a}/B$) does not fork over $A$". For $T$ $\omega$-stable this will agree with "$RM(\bar{a}/B) = RM(\bar{a}/A)$". In general, the following will be true:

(i)     $\bar{a}$ is free from $B$ over $A$ iff $\bar{a}$ is free from $B_0 \cup A$ over $A$ for every finite $B_0 \subset B$.

(ii)    For any $\bar{a}$, B there is $B_0 \subset B$ $|B_0| \leq |T|$, with $\bar{a}$ free from B over $B_0$.

(iii)   If $A \subset B \subset C$, then $\bar{a}$ is free from C over A iff $\bar{a}$ is free from C over B and $\bar{a}$ is free from B over A.

(iv)    $\bar{a}$ is free from B over A iff for all $\bar{b} \subset B$, $\bar{b}$ is free from $\bar{a} \cup A$ over A.

(v)     For given $p \in \bar{\phantom{}}(A)$ and $B \supset A$ there are at most $2^{|T|}$ and at least one $q \in S(B)$ such that $q \supseteq p$ and q does not fork over A.

(vi)    $\bar{a}$ is free from B over M (M a model $M \subset B$) iff tp($\bar{a}$/B) is <u>definable</u> over M i.e. for every $\psi(\bar{x}, \bar{y}) \in L$ there is $\delta(\bar{y})$ with parameters in M such that for all $\bar{b} \in B$, $N \models \delta(\bar{b})$ iff $N \models \varphi(\bar{a},\bar{b})$.

We will also say $\bar{a}$ is free from A over A, whereby, by (vi) we see, every $p \in S(M)$ is definable.

There are a number of ways of introducing forking and proving its properties, the original being due to Shelah [ S] and another influential treatment being due to Lascar and Poizat [L.P]. See also [P], [H.H], [R].

One of the more efficient procedures appears in the introduction of Hrushovski's thesis which is apparently the content of a course given at Berkeley by Harrington. For the interested reader we outline this approach giving selective proofs.

We work in a saturated model $\mathbb{C}$ of T. Let $\Delta(\bar{x})$ be a (usually finite) collection of L–formulas, say $\{\delta_i(\bar{x}, \bar{y}_i) : i \in I\}$ where $\bar{x} = (x_1,...,x_n)$. By a $\Delta$-type we mean a consistent (small) collection of formulas of the form $\delta(\bar{x}, \bar{a}), \neg \delta(\bar{x}, \bar{a})$, for $\delta(\bar{x}, \bar{y}) \in \Delta$ and $\bar{a} \subset \mathbb{C}$. A <u>complete</u> $\Delta$-type over $A \subset \mathbb{C}$ is a $\Delta$-type, all of whose formulas have parameters in A and which is maximal (consistent) such. Sometimes we identify a complete $\Delta$-type over A with its closure under conjunctions and disjunctions. $S_\Delta(A)$ denotes the complete $\Delta$-types over A.

**Definition 3.1.** A $\Delta$-defining schema over $A$ is a map that assigns to each $\delta(\bar{x}, \bar{y}) \in \Delta$ a formula $\psi_\delta(\bar{y})$ over $A$ such that for any $B \supset A$ (or for some $B \supset A$ which is a model), the following set

$\{\delta(\bar{x},\bar{b}) : \bar{b} \subset B, \delta \in \Delta, \models \psi_\delta(\bar{b})\} \cup \{\neg\delta(\bar{x}, \bar{b}) : \bar{b} \subset B, \delta \in \Delta, \models \neg \psi_\delta(\bar{b})\}$

is a complete $\Delta$-type over $B$. If we let $d$ denote the map $\delta(\bar{x}, \bar{y}) \to \psi_\delta(\bar{y})$ then we call the above complete $\Delta$-type $d(B)$. If $\Delta = L$, we just talk about a defining schema (over $A$).

We can observe immediately:

**Lemma 3.2.** (T stable). Let $p(\bar{x}) \in S(A)$, $q(\bar{y}) \in S(A)$ and $\varphi(\bar{x}, \bar{y}) \in L$. Let $\Delta_1(\bar{x})$ contain $\varphi(\bar{x}, \bar{y})$ and $\Delta_2(\bar{y})$ contains $\varphi(\bar{x}, \bar{y})$. Let $d_1$ be a $\Delta_1$-defining schema over $A$ such that for some (any) $M \supset A$, $p(\bar{x}) \cup d_1(M)$ is consistent. Let $d_2$ be a $\Delta_2$-defining scheme over $A$ such that for (any) $M \supset A$, $q(\bar{y}) \cup d_2(M)$ is consistent. Let $\psi_1(\bar{y})$ be $d_1(\varphi(\bar{x}, \bar{y}))$, and $\psi_2(\bar{x})$ be $d_2(\varphi(\bar{x},\bar{y}))$. Then $\psi_1(\bar{y}) \in q(\bar{y})$ if and only if $\psi_2(\bar{x}) \in p(\bar{x})$.

**Proof:** Suppose not and let $M \supset A$ be saturated. Without loss of generality, $\psi_2(\bar{x}) \in p(\bar{x})$ but $\neg \psi_1(\bar{y}) \in q(\bar{y})$. We define $\bar{a}_i, \bar{b}_i$ in M, inductively as follows: $\bar{a}_n$ realizes $p(\bar{x}) \cup d_1 (A \cup \{\bar{a}_0, \bar{b}_0,...,\bar{a}_{n-1}, \bar{b}_{n-1}\})$, and $\bar{b}_n$ realizes $q(\bar{y}) \cup d_2 (A \cup \{\bar{a}_0,\bar{b}_0,...,\bar{a}_{n-1},\bar{b}_{n-1},\bar{a}_n\})$. It is easy to check that $M \models \varphi(\bar{a}_i,\bar{b}_j)$ iff $i < j$, contradicting stability.  $\square$

The aim now is to show that for every $p(\bar{x}) \in S(A)$, where $A$ is algebraically closed in $\mathbb{C}^{eq}$ and finite $\Delta(\bar{x})$, there is a $\Delta$-defining schema $d$ over $A$ such that $p(\bar{x}) \cup d(M)$ is consistent, for all $M \supseteq A$. For this we introduce $\Delta$-rank. So fix finite $\Delta(\bar{x})$. Let $\varphi(\bar{x})$ be a formula maybe with parameters.

**Definition 3.3.**
(i) $R_\Delta(\varphi(\bar{x})) \geq 0$ if $\models \exists \bar{x} \varphi(\bar{x})$.

(ii) $R_\Delta(\phi(\overline{x})) \geq \alpha + 1$ if for every $m < \omega$ there are finite pairwise contradictory $\Delta$-types $q_1,...,q_m$ (i.e. for $i \neq j$ there is a formula in $q_i$ whose negation is in $q_j$) such that $R_\Delta(\phi(\overline{x}) \wedge \wedge q_i) \geq \alpha$, for $i = 1,...,m$.

(iii) For limit $\delta$, $R_\Delta(\phi(\overline{x})) \geq \delta$ if $R_\Delta(\phi(\overline{x})) \geq \alpha$ all $\alpha < \delta$.

(iv) If $\Phi(\overline{x})$ is a set of formulas then $R_\Delta(\Phi) = \min\{R_\Delta(\phi(\overline{x})): \phi$ is finite conjunction of members of $\Phi\}$.

(v) Suppose $R_\Delta(\Phi(\overline{x})) = \alpha$ and $m < \omega$ is greatest such that there are $q_1,...,q_m$ as in (ii). We call $m$ the $\Delta$-multiplicity of $\phi(\overline{x})$ and we denote it by $m_\Delta(\phi(\overline{x}))$.

**Fact 3.4.** T is stable implies that for all $\phi(\overline{x})$ and finite $\Delta(\overline{x})$, $R_\Delta(\phi(\overline{x}))$ is defined (i.e. has an ordinal value).

Let us assume from now on T to be stable. We fix <u>finite</u> $\Delta(\overline{x})$, and we denote $(R_\Delta(\phi(\overline{x}), m_\Delta(\phi(\overline{x}))$ by $R\text{-}m_\Delta(\phi(\overline{x}))$. We equip these pairs with the lexicographic ordering.

**Fact   3.5.**

(i)      $R_\Delta(\phi(\overline{x}) \vee \psi(\overline{x})) = \max\{R_\Delta(\phi(\overline{x})), R_\Delta(\psi(\overline{x}))\}$.

(ii)     Let $m > 1$, then $R\text{-}m_\Delta(\phi(\overline{x})) \geq (\alpha,m)$ iff there are $m_1,m_2$ with $m_1 + m_2 = m$, $\delta(\overline{x},\overline{y}) \in \Delta$, $\overline{a} \in \mathbb{C}$ such that $R\text{-}m_\Delta(\phi(\overline{x}) \wedge \delta(\overline{x},\overline{a})) \geq (\alpha,m_1)$ and $R\text{-}m_\Delta(\phi(\overline{x}) \wedge \neg\, \delta(\overline{x},\overline{a})) \geq (\alpha,m_2)$.

(iii)    Let $\Phi(\overline{x})$ be a consistent collection of formulas over A. Then there is a complete $\Delta$-type $q(\overline{x})$ over A such that $R_\Delta(\Phi) = R_\Delta(\Phi \cup q)$.

**Lemma 3.6.**  Let $R\text{-}m_\Delta(\phi(\overline{x})) = (\alpha,m)$. Let $\delta(\overline{x},\overline{y}) \in \Delta$. Then there is a formula $\psi(\overline{y})$ (with parameters from among those in $\phi$) such that for any $\overline{b}$
$\mathbb{C} \models \phi(\overline{b})$ iff $R\text{-}m_\Delta(\phi(\overline{x}) \wedge \delta(\overline{x},\overline{b})) = (\alpha,m)$.

**Outline of proof:** First, using Fact 3.5 (ii), the finiteness of $\Delta$ and induction one shows that for any formula $\varphi(\overline{x},\overline{y})$ and pair $(\alpha,m)$ there is a collection $\Gamma(\overline{y})$ of L-formulas such that $\mathbb{C} \models \Gamma(\overline{b})$ iff $R\text{-}m_\Delta(\varphi(\overline{x},\overline{b})) \geq (\alpha,m)$. Now, suppose $R\text{-}m_\Delta(\varphi(\overline{x})) = (\alpha,m)$. By Fact 3.5 (ii) "$R\text{-}m_\Delta(\varphi(\overline{x}) \wedge \delta(\overline{x},\overline{y})) \geq (\alpha,m)$" and "$R\text{-}m_\Delta(\varphi(\overline{x}) \wedge \neg \delta(\overline{x},\overline{y})) \geq (\alpha,1)$" are inconsistent. By compactness there is a formula $\psi(\overline{y})$ (with parameters among those in $\varphi(\overline{x})$) such that $\psi(\overline{y})$ is equivalent to $R_\Delta(\varphi(\overline{x}) \wedge \neg \delta(\overline{x},\overline{y})) < \alpha$. Thus $\mathbb{C} \models \psi(\overline{b})$ iff $R\text{-}m_\Delta(\varphi(\overline{x}) \wedge \delta(\overline{x},\overline{b})) \models (\alpha,m)$.                         $\square$

**Lemma 3.7.** $R_\Delta(\varphi(\overline{x})) < \omega$ for all $\varphi(\overline{x})$.

**Proof:** If not there is $\varphi(\overline{x})$ with $R\text{-}m_\Delta(\varphi(\overline{x})) = (\omega,1)$. By fact 3.5 (ii) and the finiteness of $\Delta$ there is $\delta(\overline{x},\overline{y}) \in \Delta$ such that for arbitrarily large $r < \omega$ there is $\overline{b}$ such that $R_\Delta(\varphi(\overline{x}) \wedge \delta(\overline{x},\overline{b})) \geq r$, $R_\Delta(\varphi(\overline{x}) \wedge \neg \delta(\overline{x},\overline{b})) \geq r$. But, as in proof of 3.6, $R_\Delta(\varphi(\overline{x}) \wedge \delta(\overline{x},\overline{b})) \geq r$ is equivalent to $\Gamma_r(\overline{b})$ for some collection $\Gamma_r(\overline{x})$ of formulas, and similarly for "$R_\Delta(\varphi(\overline{x}) \wedge \neg \delta(\overline{x},\overline{b})) \geq r$". There is clearly $\overline{b}$ such that $R_\Delta(\varphi(\overline{x}) \wedge \delta(\overline{x},\overline{b})) \geq \omega$ and $R_\Delta(\varphi(\overline{x}) \wedge \neg \delta(\overline{x},\overline{b})) \geq \omega$, contradicting $R\text{-}m_\Delta(\varphi(\overline{x})) = (\omega,1)$.                         $\square$

Let $p(\overline{x}) \in S(M)$. We say $p$ is <u>definable</u> if for every $\varphi(\overline{x},\overline{y}) \in L$ there is $\psi(\overline{y})$ over $M$ with : for $\overline{b} \subset M$, $\varphi(\overline{x},\overline{b}) \in p$ iff $M \models \psi(\overline{b})$. If the $\psi(\overline{y})$ are all over $A \subseteq M$ we say $p$ is definable over $A$. Note this means that there is a defining schema over $A$, $d$ say, such that $d(M) = p$.

**Proposition 3.8.** (i) Every $p(\overline{x}) \in S(M)$ is definable.

(ii) Let $p(\overline{x}) \in S(A)$ where $A$ is algebraically closed (in $\mathbb{C}^{eq}$). Then for any finite $\Delta(\overline{x})$ there is a $\Delta$-defining schema $d$ over $A$ such that $p(\overline{x}) \cup d(M)$ is consistent ($M$ any model containing $A$).

**Proof:** (i) Let $p(\overline{x}) \in S(M)$. Let $\varphi(\overline{x},\overline{y}) \in L$. Let $\Delta(\overline{x}) = \{\varphi(\overline{x},\overline{y})\}$. Let $\chi(\overline{x}) \in p$ be such that $R\text{-}m_\Delta(\chi(\overline{x})) = (k,m)$ is least possible (in fact $m = 1$). So for $\overline{b} \in M$, $\varphi(\overline{x},\overline{b}) \in p$ iff $R\text{-}m_\Delta(\chi(\overline{x}) \wedge \varphi(\overline{x},\overline{b})) = (k,m)$ which, by

Lemma 3.6, is equivalent to $\models \psi(\bar{b})$ for a formula $\psi(\bar{y})$ with parameters among those in $\chi$.

(ii) Let $p(\bar{x}) \in S(A)$. Let $M \supset A$ be saturated. By Fact 3.5 (iii) there is a <u>complete $\Delta$-type</u> $q(\bar{x})$ over $M$ with $R_\Delta(p(\bar{x})) = R_\Delta(p(\bar{x}) \cup q(\bar{x}))$. By the same proof as in (i) there is a $\Delta$-defining schema over $M$, d say, such that $q = d(M)$. We will show that d is over A (i.e. q is definable over A). Let f be an automorphism of M which fixes A pointwise. So $f(p \cup q) = p \cup f(q)$ and $R_\Delta(p) = R_\Delta(p \cup f(q))$. So by definition of $R_\Delta$, q has only finitely many images under such A-automorphisms of M. Thus for each $\delta(\bar{x},\bar{y}) \in \Delta$, the defining formula $d(\delta(\bar{x},\bar{y}))$ has only finitely many images under A-automorphisms of M, i.e. $d(\delta(\bar{x},\bar{y}))$ is almost over A. But A is algebraically closed in $\mathbb{C}^{eq}$ so $d(\delta(\bar{x},\bar{y}))$ is over A. This shows that d is a $\Delta$-defining schema over A.                    $\square$

**Proposition 3.9.** Let $p(\bar{x}) \in S(A)$, A algebraically closed. Let $d_1, d_2$ be $\Delta$-defining schema over A such that for $M \supset A$, $p(\bar{x}) \cup d_1(M)$ is consistent and $p(\bar{x}) \cup d_2(M)$ is consistent. Then $d_1(M) = d_2(M)$ (all $M \supset A$).

**Proof:** Let $\varphi(\bar{x},\bar{y}) \in \Delta$. Let $\delta_1(\bar{y}) = d_1(\varphi(\bar{x},\bar{y}))$ and $\delta_2(\bar{y}) = d_2(\varphi(\bar{x},\bar{y}))$. We must show that $M \models (\forall \bar{y})(\delta_1(\bar{y}) \leftrightarrow \delta_2(\bar{y}))$. Let $\bar{b} \subset M$ with $\ell(\bar{b}) = \ell(\bar{y})$. Let $q(\bar{y}) = tp(\bar{b}/A)$. Let $\Delta'(y)$ be a finite set of L-formulas containing $\varphi(\bar{x},\bar{y})$. By Proposition 3.8 (ii) there is a $\Delta'$-defining schema $d_3$ over A such that $q(\bar{y}) \cup d_3(M)$ is consistent. Let $\delta_3(\bar{x})$ be $d_3(\varphi(\bar{x},\bar{y}))$. By Lemma 3.2, $\delta_1(\bar{y}) \in q(\bar{y})$ iff $\delta_3(\bar{x}) \in p(\bar{x})$ iff $\delta_2(\bar{y}) \in q(\bar{y})$. In particular $\models \delta_1(\bar{b}) \leftrightarrow \delta_2(\bar{b})$, which proves what is required.                    $\square$

**Corollary 3.10.** Let $p(\bar{x}) \in S(A)$ (A algebraically closed). Then there is a <u>unique</u> defining schema d over A such that for all $M \supset A$, $p(\bar{x}) \cup d(M)$ is consistent (i.e. $p(\bar{x}) = d(A)$).

**Proof:** By 3.8 (ii) and 3.9.                    $\square$

**Note:** Corollary 3.10 says that for any $p \in S(A)$ (A algebraically closed) and $M \supset A$ there is a unique $q \in S(M)$ such that $q \supset p$ and $q$ is definable over $A$.

**Definition 3.11.** For $p \in S(A)$ (A algebraically closed), we denote by $d_p$ the unique defining schema over $A$ given by 3.10. Let $q \in S(B)$ and $A \subset B$ (A, B not necessarily algebraically closed). We say $q$ does not fork over $A$ if for some (equivalently $\underline{any}$) extension $q_1$ of over $acl(A)$, $d_{q_1}$ is over $acl(A)$.

**Remark 3.12.** So $q \in S(B)$ does not fork over $A$ iff for some (any) extension $q_1$ of $q$ over $acl(B)$, $d_{q_1}$ and $d_{q_1 \lceil acl(A)}$ are equivalent. This is by the uniqueness part of 3.10. It is also the case that $q \in S(B)$ does not fork over $A$ iff $R_\Delta(q) = R_\Delta(q\lceil A)$.

  Properties (i) - (vi) mentioned in the introduction to this section are now more or less immediate. Note that property (iv) (symmetry) follows from Lemma 3.2. We should also add

**Corollary 3.13.** For any $p \in S(A)$ and $B \supset A$ there is $q \in S(B)$ $q \supset p$, $q$ does not fork over $A$ ($q$ is called a nonforking extension of $p$).
  And we summarize:

**Corollay 3.14.** (i) Let $p(\overline{x}) \in S(M)$ and $A \subset M$. $p$ does not fork over $A$ iff $p$ is definable over $acl(A)$.
  (ii) Let $p(\overline{x}) \in S(A)$, $A$ algebraically closed. For any $B \supset A$, $p$ has a $\underline{unique}$ nonforking extension to $B$.

**Proposition 3.15.** Let $p(\overline{x}) \in S(M)$, $A \subset M$ and suppose $p$ does not fork over $A$. Let $\varphi(\overline{x}, \overline{b}) \in p(\overline{x})$ ($\overline{b} \subset M$). Then for every model $M_1 \supset A$, there is $\overline{a}' \in M_1$, for which $\models \varphi(\overline{a}', \overline{b})$ (i.e. $p$ is $\underline{almost}$ finitely satisfiable in $M_1$).

**Proof:** We may assume $A$ to be algebraically closed (in $\mathbb{C}^{eq}$) (as $A \subset M_1$ iff acl $A \subset M_1$). Now let $M_1 \supset A$ and let $p'$ be a nonforking extension of $p$ over $M \cup M_1$ (by Corollary 3.13). So clearly $p'$ does not fork over $M_1$. Let $\bar{a}$ realize $p'$ (so $\models \varphi(\bar{a}, \bar{b})$). By symmetry (iv) $\text{tp}(\bar{b}/M_1 \cup \bar{a})$ does not fork over $M_1$. So $\text{tp}(\bar{b}/M_1 \cup \bar{a}) = d(M_1 \cup \bar{a})$ for some defining schema over $M_1$. Let $\psi(\bar{x})$ be $d(\varphi(\bar{x}, \bar{y}))$ ($\psi(\bar{x})$ has parameters $M_1$). Note $\models \exists \bar{x} \, \psi(\bar{x})$ (namely $\bar{a}$). Thus there is $\bar{a}' \in M_1 \models \psi(\bar{a}')$, so $\models \varphi(\bar{a}', \bar{b})$.                    $\square$

We say $p(\bar{x}) \in S(A)$ is <u>stationary</u> if $p$ has a unique nonforking extension over any $B \supset A$.

**Lemma 3.16.** $p \in S(A)$ is stationary if and only if for some $M \supset A$ and some $q \in S(M)$ containing $p, q$ is definable over $A$.

**Proof:** Let $M \supset A$ be saturated (homogeneous). Let $q(\bar{x}) \in S(M)$ be a nonforking extension of $p$. So we know $q = d(M)$, where $d$ is a defining schema over acl($A$). Now for any $A$-automorphism $f$ of $M$, $f(q) = f(d)(M)$ still extends $p$, so is clearly also a nonforking extension of $p$. By hypothesis $f(q) = q$, i.e. $f(d) = d$. It clearly follows that $d$ is over $A$.

Conversely, suppose the right hand side to be true. Let $q \in S(M)$, $q \supseteq p, q$ definable over $A$. So $q = d(M)$ for some defining schema $d$ over $A$. By the uniqueness in Corollary 3.10 $d = d_p$, which implies that $p$ is stationary.                    $\square$

**Lemma 3.17.** $\text{tp}(\bar{a}/A)$ is stationary iff $\text{dcl}(A, \bar{a}) \cap \text{acl}(A) = \text{dcl}(A)$ (where acl, dcl are computed in $\mathbb{C}^{eq}$).

**Proof:** Assume $\text{tp}(\bar{a}/A)$ to be stationary. Let $c \in \text{dcl}(A, \bar{a})$. Easily $\text{tp}(c/A)$ is stationary. Thus if $c \in \text{acl}(A)$ then $c \in \text{dcl}(A)$.

On the other hand, suppose the right hand side is satisfied. Let $M \supseteq A \cup \bar{a}$ be saturated and let $q(\bar{x}) \in S(M)$ be a nonforking extension of $\text{tp}(\bar{a}/\text{acl}A)$. Let $q(\bar{x}) = d(M)$ where $d$ is a defining schema over aclA. Let $f$

be an $(A \cup \bar{a})$ -automorphism of M. So $f(aclA) = aclA$ (as a <u>set</u>), and so $f(q) = q$, so $f(d) = d$. Thus d is over $A \cup \bar{a}$ i.e. for every $\varphi(\bar{x},\bar{y}) \in L$, $d\varphi(\bar{x},\bar{y}) \in dcl(A \cup \bar{a}) \cap aclA$. By the condition, $d(\varphi(\bar{x},\bar{y})) \in dclA$, i.e. d is over A. By 3.10, p is stationary.                                    □

We now examine what some of these notions mean in the context of ω-stable theories and algebraically closed fields.

**Lemma 3.18.** Let T be ω-stable. Then T is stable. If $p \in S(A), p \subset q \in S(B)$, then q is a nonforking extension of p iff $RM(q) = RM(p)$. p is stationary iff Morley degree of p is 1.

**Proof:** $RM(\varphi(\bar{x})) < \infty$ clearly implies $R_\Delta(\varphi(\bar{x})) < \infty$ for all finite Δ. So T is stable.

It is easy to see that $RM(tp(\bar{a}/A)) = RM(tp(\bar{a}/aclA))$. So we may assume A to be algebraically closed. Also we may assume B is a big model M say. Let $q \in S(M), RM(q) = RM(p)$. Any conjugate of q under an A-automorphism of M clearly has the same property, so there are only finitely many such q. Thus q is definable over $aclA = A$. Hence q does not fork over A. The converse is similar. Clearly p has Morley degree 1 iff p has a <u>unique</u> extension to M with the same Morley rank, which by the above means that p is stationary.                                    □

In the next observation we will use the fact, pointed out by Poizat in [Po1] that algebraically closed fields admit so-called elimination of imaginaries. This means that if K is an algebraically closed field and $a \in K^{eq}$ then there is some k-tuple $(b_1,...,b_k)$ from K much that $\bar{b}$ and a are interdefinable (over $\varnothing$) in $K^{eq}$. We also use the fact that if K is an algebraically closed field of <u>characteristic 0</u> and $A \subset K$, then the definable closure in K of A equals the subfield of K generated by A (i.e. the <u>rational</u> closure of A). (By quantifier elimination).

**Corollary 3.19.** (of Lemma 3.17). Let $K$ be an algebraically closed field of characteristic 0, $\overline{a} \subset K$ and $k$ a subfield of $K$. Then $tp(\overline{a}/k)$ is stationary iff $k$ is algebraically closed in $k(\overline{a})$ (i.e. $k(\overline{a})$ is a regular extension of $k$).

**Proof:** By the above remarks $dcl(k \cup \overline{a})$ in $K^{eq}$ is interdefinable with $k(\overline{a})$ (the rational closure of $k \cup \overline{a}$). Similarly, $acl(k)$ in $K^{eq}$ is interdefinable with $\overline{k}$ (the algebraic closure of $k$ in $K$). Thus the condition in Lemma 3.17 translates into: $k$ is algebraically closed in $k(\overline{a})$, proving the Corollary.    $\square$

Let now $K$ be a saturated algebraically closed field (of arbitrary characteristic). Fix $n < \omega$. We have already remarked that the <u>affine algebraic subsets</u> of $K^n$ are the subsets of $K^n$ defined by $P_1(\overline{x}) = 0 \wedge .... \wedge P_k(\overline{x}) = 0$, where $P_i \in K[\overline{x}]$. We point out that these are the <u>closed</u> sets for a certain Noetherian topology on $K^n$, the Zariski topology. It is first easy to see that a finite union of affine algebraic sets is also an affine algebraic set. On the other hand, if $V_i \subseteq K^n$, $i < \omega$, are affine algebraic sets then the ideal of $K[\overline{x}]$ generated by all the polynominals defining all the sets $V_i$, is generated by finitely many such polynomials (as $K[\overline{x}]$ is a Noetherian ring) and thus the intersection of the $V_i$ is a finite subintersection. This shows that we have the DCC on affine algebraic sets. So we call the affine algebraic subsets of $K^n$, the Zariski closed subsets of $K^n$ and we see that this equips $K^n$ with a Noetherian topology (the Zariski topology). A Zariski closed set $V$ is said to be <u>irreducible</u> if we cannot write $V$ as $V_1 \cup V_2$, where $V_i \subsetneq V$ ($i = 1,2$) are also Zariski closed. It is standard to show that any Zariski closed $V \subseteq K^n$ can be written <u>uniquely</u> as a union of irreducible Zariski closed subsets.

**Proposition 3.20.** Let $k \subset K$, $k$ perfect. Let $\overline{a} \in K^n$ and let $V = V(\overline{a}/k)$. Then the number of irreducible components of $V$ = Morley degree of $V$ = Morley degree of $tp(\overline{a}/k)$.

**Proof:** (Remark: We take $k$ perfect, so that "defined over $k$" has the same meaning in model theory as in geometry). Let $p = tp(\overline{a}/k)$. Let $p_1,...,p_r$ be

the nonforking extensions of $p$ to $K$. (i.e. $p_i \in S_n(K)$, $p_i$ does not fork over $k$). So $r = $ Morley degree of $p$. For each $i = 1,...,r$ let $V_i = V(p_i) = $ smallest Zariski closed set in $p_i$. Clearly $V_i$ is irreducible (as $K$ is a model) and $V_i \subseteq V$. By quantifier elimination $p_i$ is "determined" by $V_i$. So if $f$ is a $k$-automorphism of $K$ then for any $i$, $f(V_i) = V_j$ for some $j$. Thus each $V_i$ has only finitely many conjugates over $k$. Fix say $V_1$ and let $V_1 = V_{i_1}$, $V_{i_2},...,V_{i_s}$, be the $k$-conjugates of $V_1$. Then $V_{i_1} \cup ... \cup V_{i_s}$ is defined over $k$, is in $p = tp(\bar{a}/k)$ and so equals $V$. By the irreducibility of each $V_i$ this shows that $V_1 \cup ... \cup V_r = V$ and that the $V_i$ are the irreducible components of $V$. This suffices to prove the proposition.     □

**Corollary 3.21.** (Again $k$ perfect). Let $V \subset K^n$ be an irreducible Zariski closed set, defined over $k$. Then there is $\bar{a} \in k^n$ with $V = V(\bar{a}/k)$. Also if $V_1 \subsetneqq V$ is Zariski closed then $RM(V_1) < RM(V)$.

**Proof:** By irreducibility of $V$ and compactness there is $\bar{a} \in K^n$ with $V = V(\bar{a}/k)$. Suppose $V_1 \subsetneqq V$, $RM(V_1) = RM(V)$. Let $p'$ be (by Prop 3.20) the nonforking extension of $p$ to $K$. So, as Morley degree of $V = 1$, $V_1 \in p'$. But then $V_2 = V(p')$ is contained in $V_1$ and so is a proper irreducible component of $V$, and by the proof of 3.20 we obtain a contradiction.     □

**Corollary 3.22.** ($k$ perfect). Let $V \subseteq K^n$ be irreducible and defined over $k$. Let $\bar{k}$ be the algebraic closure of $k$ (in $K$) and let $V(\bar{k})$ be $V \cap \bar{k}^n$. Then $V(\bar{k})$ in Zariski dense in $V$.

**Proof:** By 3.21 there is an $\bar{a} \in K^n$ with $V = V(\bar{a}/k)$. Let $p = tp(\bar{a}/k)$ and $p'$ the nonforking extension of $p$ to $K$. Let $X \subseteq V$ be Zariski open in $V$. So by Corollary 3.21, $RM(X) = RM(V)$, so $X \in p'$ does not fork over $k$ and $\bar{k}$ is an elementary substructure of $K$ containing $k$. By Proposition 3.15 $X \cap \bar{k}^n$ ($= X \cap V(\bar{k})$) is nonempty.     □

Let me finally in this section mention <u>superstability</u> a property stronger than stability but weaker than $\omega$-stability. Again it can be defined by means of a <u>rank</u>. We define the rank $R^\infty$ on definable subsets $X$ of a very saturated model $N$, (the crucial clause being: $R^\infty(X) \geq \alpha + 1$ if there are for all $\lambda$, $X_i$ $(i < \lambda)$, all defined by an instance of the <u>same</u> formula such that

(i) $R^\infty(X \wedge X_i) \geq \alpha$ and

(ii) the $X_i$ are m-inconsistent for some $m < \omega$, i.e. for distinct $i_1,...,i_m$ we have $X_{i_1} \wedge ... \wedge X_{i_m} = \varnothing$.

T is <u>superstable</u> if $R^\infty(X)$ is defined for all $A$. A rather different kind of rank, the U-rank (of Lascar) can be defined for complete types $p \in S(A)$ in a stable theory: $U(p) \geq \alpha + 1$ if $p$ has a forking extension $q$ such that $U(q) \geq \alpha$. It turns out that T is superstable if and only if $U(p) < \infty$ for all $p$.

Both ranks $R^\infty$ and $U$ reflect forking in a superstable theory: namely for $R = R^\infty$ or $U$ and $p \subset q$ complete types, $R(p) = R(q)$ just if $q$ is a nonforking extension of $p$.

## 4. $\omega$-STABLE GROUPS

A <u>stable group</u> is a group $(G,\cdot,...)$ equipped with possibly additional structure such that the theory of this structure is stable. Similarly for $\omega$-stable groups, superstable groups. One could also (and we do) consider a stable group $G$ as a group definable in a stable structure $M$; namely both the universe of the group $G$ and the group operation are definable in $M$. This is the case of say affine algebraic groups over an algebraically closed fields $K$, such groups being definable (in $K$) subgroups of $GL_n(K)$. The two points of view amount to the same thing. For suppose $G$ to be defined in the stable structure $M$; where $\varphi(\overline{x},\overline{a})$ defines the universe of $G$ and $f(\overline{x},\overline{y},\overline{a})$ defines the group operation on $G$ ($\overline{a} \subset M$). For each relation on $G^n$ defined in $M$ by a formula with parameter $\overline{a}$, introduce a new relation symbol. Let $\underline{G}$ be $G$ equipped with its multiplication and all there relations. By virtue of definablility of types in $M$, every definable in $M$ subset of $G^n$ is definable also in the structure $\underline{G}$. $\underline{G}$ inherits all the stability theoretic properties of $M$ (stability,

superstability, $\omega$-stability and even $\aleph_1$-category). Similarly for reducts of $\underline{G}$ (e.g. the pure group $(G,\cdot)$), except that $\aleph_1$-category may no longer hold: as pointed out in a paper of Baldwin in this volume $GL_2(\mathbb{C})$ is not $\aleph_1$-categorical.

Examples of stable groups are: Abelian groups (as pure groups), modules, affine algebraic groups over algebraically closed fields (which are $\omega$-stable of finite Morley rank), algebraic matrix groups over any stable ring, Abelian varieties (equipped with their induced structure from the underlying field).

We will specialize first to $\omega$-stable groups, where the proofs of basic properities (generic types etc.) are somewhat easier, and then say a few words about general stable groups.

So let $G$ be a group (with additional structure). At times we want to consider $G$ as an elementary substructure of a saturated $G_1$, and sometimes $G$ is itself taken to be saturated. The main new fact given to us by working with a group rather than an arbitrary structure is "homogeneity" - in the sense that for every $a, b \in G$ there is a <u>definable</u> bijection of $G$ with itself taking $a$ to $b$ (e.g. right multiplication by $a^{-1}b$). It is clear that any definable bijection of a structure $M$ also acts on the definable sets and preserves "everything" of a definable nature, in particular Morley rank, degree etc. So this is true in particular of left and right multiplication by elements of $G$. Note that if $X \subset G$ is defined by $\varphi(\overline{x},\overline{b})$ then $a\cdot X$ is defined by $\varphi(a^{-1}x,\overline{b})$. $G$ also acts on the 1-types $p \in S_1(G)$: if $a \in G_1 > G$ realizes $p$ and $b \in G$, then $bp = $ tp$(ba/G)$ and $pb = $ tp$(ab/G)$. Similarly $p^{-1} = $ tp$(a^{-1}/G)$. So
$$\text{RM } (p) = \text{RM}(p^{-1}) = \text{RM}(bp) = \text{RM}(pb). \quad (*)$$

**Proposition 4.1.** Let $G$ be $\omega$-stable. Then $G$ has the DCC on definable subgroups.

**Proof:** Let $H_1 \lneq H_2 < G$. ($H_1, H_2$ definable subgroups of $G$). If $H_1$ has infinite index in $H_2$ then clearly $\text{RM}(H_1) < \text{RM}(H_2)$. If $H_1$ has finite index in $H_2$ then $\text{RM}(H_1) = \text{RM}(H_2)$ but Morley degree $(H_1) <$ Morley degree

(H$_2$). As every definable subset of G has ordinal valued Morley rank and integer valued Morley degree, there cannot be an infinite descending chain of definable subgroups.                                                   □

**Corollary 4.2.** If G is ω-stable, then G has a <u>smallest</u> definable subgroup of finite index. We call this G°, the <u>connected component</u> of G.          □

　　　Note that G° is <u>Ø-definable</u> because (i) G° is definable and (ii) G° can be described without reference to any parameters.

**Definition 4.3.** Let G be ω-stable. p ∈ S$_1$(G) is said to be a generic type of G if RM(p) = RM(G) (= RM("x = x") in Th(G)).

**Remark 4.4.** (G ω-stable)

　　　(i) There are only finitely many generic p ∈ S$_1$(G) (and there is at least one.

　　　(ii) if p is generic so is p$^{-1}$ (by *).

　　　(iii) G acts (by left or right translation) on the generic types of S$_1$(G) (by *).

　　　In fact:

**Lemma 4.5.** G acts (by left or right translation) transitively and definably on the set of generics of S$_1$(G).

**Proof:** First, let p, q ∈ S$_1$(G) be both generic types, and let a, b ∈ G$_1$ > G realize p, q respectively such that a and b are <u>independent</u> over G. Let c = ab$^{-1}$.

**Claim:** RM(c/G ∪ b) = RM(a/G ∪ b).

This is because φ(x,$\bar{d}$) ∈ tp(c/G ∪ b) iff φ(xb$^{-1}$,$\bar{d}$) ∈ tp(a/G ∪ b) and φ(x,$\bar{d}$) ∈ tp(a/G ∪ b) iff φ(xb,$\bar{d}$) ∈ tp (c/G ∪ b) and φ(x,$\bar{d}$), φ(xb,$\bar{d}$) have the same Morley rank.

　　　Let α = RM(G). We know that RM(a/G ∪ b) = α. Thus

$RM(c/G \cup b) = \alpha$. So $RM(c/G) = \alpha$. By 3.18 c and b are independent over G and so by 3.15 $tp(c/G \cup b)$ is finitely satisfiable in G. Now let $\varphi(x) \in p = tp(a/G)$ have Morley rank $\alpha$ and degree 1. (So p is the unique type in $S_1(G)$ containing $\varphi(x)$ and with Morley rank $\alpha$). We clearly have $\models \varphi(cb)$. So there is $c' \in G$ such that $\models \varphi(c'b)$. But as $tp(c'b/G)$ has Morley rank $\alpha$, $tp(c'b/G) = p$. So $p = c'q$. This proves transitivity of the action.

Let $P = \{p_1,...,p_n\}$ be the generic types of G. To say that G acts definably on P we mean (in this specal case) for each i,j there is a formula $\varphi_{ij}(x)$ with parameters in G such that $G \models \varphi_{ij}(a)$ iff $ap_i = p_j$. So fix such i,j. Clearly $RM\, p_i = RM\, p_j = \alpha$. Let $\varphi(x,\bar{a}) \in p_i$ with $RM(\varphi(x,\bar{a})) = \alpha$ and degree of $\varphi(x,\bar{a}) = 1$. Clearly $ap_i = p_j$ just if $\varphi(a^{-1}x,\bar{a}) \in p_j$. But $p_j$ is definable (Prop 3.8), so there is $\psi(z,\bar{a},\bar{c})$ $(\bar{a},\bar{c} \subset G)$ such that for all $a \in G$. $\varphi(a^{-1}x, \bar{a}) \in p_j$ iff $G \models \psi(a,\bar{a},\bar{c})$. This is enough.                      □

**Corollary 4.6.** (G $\omega$-stable). The Morley degree of G ( = number of generic $p \in S_1(G)$) = index of $G^\circ$ in G.

**Proof:** Let $\{p_1,...,p_n\}$ be the set of generics in $S_1(G)$. Let $K = \{a \in G : ap_i = p_i\ \forall i = 1,...,n\}$. By Lemma 4.5, K is definable, and clearly has finite index in G and moreover

$$n \leq |G/K| \leq |G/G^\circ|.$$

On the other hand, each coset of $G^\circ$ in G has Morley rank $\alpha$ and so gives rise to at least one generic type, whereby $|G/G^\circ| \leq n$. Thus we have equality, proving the Corollary.                      □

**Proposition 4.7.** Let G be $\omega$-stable with $RM(G) = \alpha$. Let $X \subseteq G$ be definable. Then $RM(X) = \alpha$ iff finitely many translates (left or right) of X cover G.

**Proof:** As finitely many translates of $G^\circ$ cover G, we may assume that $G = G^\circ$ (i.e. G is connected), and so by 4.6, G has a unique generic type. The right to left direction of the proposition is easy (as all translates of X have

same Morley rank). For the left to right direction: Let $RM(X) = \alpha$. We will show that for all $q \in S_1(G)$ there is an element $c \in G$ such that the definable set $cX$ is in $q$. Fix $q \in S_1(G)$ and a realization $b$ of $q$. Let $a$ realize the generic type of $G$ such that $a$ and $b$ are independent over $G$. As in the proof of Lemma 4.5, $RM(ab/G) = \alpha$ ( $= RM(G)$). As $G$ has Morley degree 1 and $RM(X) = \alpha$, $ab \in X$. By independence and 3.15. $\exists\, a' \in G$ such that $a'b \in X$. i.e. $(a')^{-1}\cdot X \in q$. Put $c = (a')^{-1}$.

By compactness, finitely many translates of $X$ cover $G$.            □

So we can define a generic formula (or definable set) in $G$ to be one finitely many left translates of which cover $G$.

**Corollary 4.8.** ($G$ $\omega$-stable) $p \in S_1(G)$ is generic iff $p$ contains only generic formulas.

For <u>stable</u> groups $G$, Corollary 4.8 can be taken as a <u>definition</u> of generic type.

**Proposition 4.9.** Let $G$ be $\omega$-stable and <u>connected</u>. Let $X \subset G$ be generic. Then $X \cdot X = G$.

**Proof:** Let $a \in G$. Let $b \in G_1 > G$ realize the generic type of $G$. Let $\varphi(x)$ be the formula defining $X$. Now $tp(b^{-1}/G)$ and so also $tp(b^{-1}a/G)$ are generic. Thus $\models \varphi(b^{-1}a)$ and of course $\models \varphi(b)$. Thus $G_1 \models \exists x\, \exists y\, (\varphi(x) \wedge \varphi(y) \wedge a = xy)$. The same is true in $G$ (as $a \in G$). So $X \cdot X = G$.            □

Borel proved a useful fact about algebraic groups (over algebraically closed fields) which is the following: Let $G$ be an algebraic group. Let $X_i$ $i \in I$ be a family of <u>constructible</u> subsets of $G$, each containing the identity element $e$ such that the Zariski closure $\overline{X}_i$ of each $X_i$ is <u>irreducible</u>. Then the subgroup $H$ of $G$ generated by the $X_i$ is (Zariski)-closed (i.e. constructible), <u>connected</u> and $H = X_{i_1}^{\epsilon_1} \dots X_{i_n}^{\epsilon_n}$ fo some $i_1,\dots,i_n \in I$, where $\epsilon_j = \pm\, 1$.

The proof is so direct that it is worth giving: One first observes that for all $i_1,...,i_k \in I$ and $\in_1,...,\in_k = \pm 1$ the Zariski closure of $X_{i_1}^{\in_1},...,X_{i_k}^{\in_k}$ is irreducible. Thus there is $X = X_{i_1}^{\in_1}...X_{i_k}^{\in_k}$ such that $\bar{X}$ is greatest. Easily $\bar{X}$ is a subgroup of $G$. As $\dim X = \dim \bar{X}$ and $\bar{X}$ is connected, by Proposition 2.9 even $X \cdot X = \bar{X}$.

Zil'ber remarkably proved a generalization of this result to $\omega$-stable groups of finite Morley rank. Hrushovski in a paper in this volume proves the result in an even more general context. Here we give Zil'ber's proof. The problem of course is that in the general situation of $\omega$-stable groups we have no geometry (at least a priori), so no notion of irreducible. Zil'ber finds a substitute for this: he calls definable $X \subset G$ indecomposable if for any definable subgroup $H$ of $G$, either $|X/H| = 1$ or $|X/H|$ is infinite.

**Proposition 4.10.** Let $G$ be $\omega$-stable with finite Morley rank. For $i \in I$ let $X_i$ be an indecomposable definable subset of $G$ containing the identity element $e$. Let $H$ be the subgroup of $G$ generated by the subsets $X_i$. Then $H$ is definable, connected and is equal to $X_{i_1}...X_{i_k}$ for some $i_1,...,i_k \in I$.

**Proof:** As $RM(G)$ is finite, we can choose $i_1,...,i_k \in I$ such that $X = X_{i_1}...X_{i_k}$ has maximum possible Morley rank, say $m$. Let $p \in S_1(G)$, $X \in p$, $RM(p) = m$. Let $H = Fix(p)$ ($= \{a \in G : ap = p\}$) which is definable as in the proof of 4.5 ($H$ is clearly a subgroup of $G$).

**Claim (i):** $X_i \subset H$ for all $i \in I$ (so $H$ contains the subgroup generated by the $X_i$).

**Proof:** Fix $i \in I$. as $e \in X_i \cap H$ and $X_i$ is indecomposable if $X_i \not\subset H$ then there would be $a_j \in X_i$ for $j < \omega$ such that $a_j \neq a_{j'} \mod H$ for $j \neq j'$. As $H = Fix\ p$, it follows that $a_j p \neq a_{j'} p$ for $j \neq j'$. As $X_i X \in a_j p$ for all $j < \omega$, it

follows that $RM(X_iX) \geq m + 1$, contradicting the choice of X. Thus Claim (i) is established.

**Claim (ii):** H is connected and p is the generic type of H.

**Proof:** Note that by Claim (i) $X \subset H$ and thus "$x \in H$" $\in$ p. As in the proof of 4.7, we can find $c \in H$ such that cp is a generic of H. By definition of H (as Fix(p)) we see that p is already a generic of H. By Lemma 4.5, it is the unique generic of H and thus by 4.6, H is connected. Thus we see that $RM(H) = m$, $RM(X) = m$ and H is connected. By 4.9. H = X·X.                                                                                  □

Macintyre [M] proved that an ω-stable field is algebraically closed. The model theoretic ingredients of this are:

**Lemma 4.11:** Let K be an ω-stable infinite field. Then K has Morley degree 1 (so by 4.6 is connected both additively and multiplicatively).

**Proof:** Suppose A is a proper additive subgroup of K of finite index. So $\cap \{kA : k \in K\}$ is an <u>ideal</u> I of K. But by 4.1 $I = k_1A \cap ... \cap k_nA$ for some $k_1,...,k_n \in k$, so I has finite index in K. Since K does not have nontrivial ideals, I = K. So A = K. Thus by 4.6, K has Morley degree 1.
                                                                                  □

**Lemma 4.12.** Let G be a connected ω-stable group and f: $G \rightarrow G$ a definable endomorphism with finite kernel. Then f is surjective.

**Proof:** G/Kerf is definably isomorphic to lmf, the latter being a definable subgroup of G. As Kerf is finite, properties of Morley rank imply that $RM(G) = RM(lmf)$. By connectedness of G, lmf = G.                    □

Now, by considering the maps $x \rightarrow x^n$ (endomorphism of K* with finite kernel) and, if charK = p, the maps $x \rightarrow x^p - x$ (endomorphism of K with finite kernel) we see from the previous two lemmas that K is perfect,

$x^n - a = 0$ has a root in $K$ $\forall a \in K$, $\forall n < \omega$, and if char $K = p$, $x^p - x - a = 0$ has a root in $K$ $\forall a \in K$. Moreover this is also true for every finite extension of $K$ (as any finite extension of $K$ is interpretable in $K$ and thus also $\omega$-stable). Now Galois theory implies that $K$ is algebraically closed.

One of the important applications of Zil'ber's indecomposability theorem is to find a <u>field</u> in certain algebraic situations of finite Morley rank. The following proposition summarizes the essential points.

Before stating and proving this we make a few explanatory remarks: First let $G$ be an $\omega$-stable group, $A \subset G$ and $a \in G$; we say that $a$ is generic over $A$ if $tp(a/A)$ has a nonforking extension $p \in S_1(G)$ which is generic (Note that if $p \in S_1(G)$ is generic then $p$ does not fork over $\varnothing$ so $\forall A \subset G$ $p \lceil A$ is "generic").

Secondly let $G$, $A$ be definable groups in an $\omega$-stable structure such that $G$ <u>acts definably</u> on $A$, as a group of automorphisms of $A$ (e.g. $A$ is a normal subgroup of $G$ and $G$ acts by conjugation). We call $X \subset A$ $G$-invariant if $X$ is fixed setwise by $G$.

**Fact 4.13:** Let $X \subset A$ be $G$-invariant. Then $X$ is <u>indecomposable</u> if and only if for every definable $G$-invariant subgroup $H$ of $A$ we have $|X/H| = 1$ or $\infty$.

**Proof:** Let $H$ be an arbitrary definable subgroup of $A$. By the DCC $\bigcap_{s \in G} H^s = H^{s_1} \cap ... \cap H^{s_n}$ for some $s_1,...,s_n \in G$. If $X$ is $G$-invariant and $|X/H| < \omega$ then clearly $|X/(H^{s_1} \cap ... \cap H^{s_n})| < \omega$ and note that $H^{s_1} \cap ... \cap H^{s_n}$ is $G$-invariant.                                                             □

**Proposition 4.14.** Let $G$, $A$ be (infinite) definable Abelian groups in a structure $M$ of finite Morley rank and assume that $G$ acts definably and faithfully on $A$. Suppose moreover that $A$ has no infinite proper $G$-invariant subgroup. Then there is in $M$ a definable field $R$ such that the additive group of $R$ is definably isomorphic to $A$, $G$ definably embeds into the

multiplicative group of R, and the action of G on A corresponds to multiplication in R.

**Proof:** We first note that A must be connected. Now let a be a generic of A over $\emptyset$. We claim that $G \cdot a$ is infinite. For otherwise $G^\circ \cdot a$ is finite, so $G^\circ \cdot a = \{a\}$ (as $G^\circ$ is connected), but then as every element of A is a sum of generics, $G^\circ b = b \; \forall \, b \in A$ so $G^\circ = \{1\}$, contradicting G being infinite. Now $G \cdot a$ is clearly G-invariant. By Fact 4.13, $Ga \cup \{0\}$ is an indecomposable subset of A. By Proposition 4.10, there is an integer $n < \omega$ such that for every $b \in A \; \exists \, g_1,...,g_n \in G$ such that $b = g_1 \cdot a + ... + g_n \cdot a$ (\*\*).
Let R be the subring of End(A) generated by G. So R is commutative and by (\*\*) every element r of R is determined by its action on a; in fact every $r \in R$ is of the form $g_1 + ... + g_n$ for some $g_1,...,g_n \in G$. It easily follows that R and its action on A are definable, using a as a parameter. We claim that R is a field. We must show that every nonzero element of R has an inverse in R. Let $r \in R \; r \neq 0$. Now $\operatorname{Ker} r$ is a G-invariant subgroup of A, so must be finite. By Morley rank considerations r is surjective. Let therefore $b \in A$ be such that $r \cdot b = a$ where a is the generic element of A chosen above. By (\*\*) there is $s \in R$ such that $sa = b$. So $rs(a) = a$. By (\*\*) again $rs = 1$. This shows that R is a field. Clearly $G \subset R$, and the map $r \rightarrow ra$ is an additive isomorphism between R and A.

  $\square$

**Corollary 4.15.** Let G be a connected $\omega$-stable group of finite Morley rank which is solvable but non nilpotent. Then G interprets an infinite field.

**Proof:** First a remark: If G is connected and Z(G) is finite then G/Z(G) is centreless. For if $a \in G$ is such that a is central mod Z(G) then $a^{-1} \cdot a^G \subset Z(G)$, so $a^G$ is finite, so $C_G(a)$ has finite index in G. Thus $C_G(a) = G$ and $a \in Z(G)$.

  Now suppose G to be solvable, non nilpotent and connected of finite Morley rank. If $Z_n$ is the upper central series, then since G has finite Morley

rank, $Z_{n+1}/Z_n$ is finite for some n. But then $G/Z_{n+1}$ is centerless by the above remark. Clearly $G \neq Z_{n+1}$ (as G is nonnilpotent). So working now with $G/Z_{n+1}$ (which remains solvable) we may assume G to be centreless (and still connected). Let A be a minimal infinite definable normal subgroup of G. Now A is connected and solvable, and by Prop 4.10 the derived group $A'$ is definable and connected. Thus $A' = \{e\}$, A is abelian. As G has no center, $C_G(A) \neq G$. So $G/C_G(A)$ is infinite and acts faithfully on A. Let H be an infinite abelian definable subgroup of $G/C_G(A)$ (by $\omega$-stability [Ch]) and let B be a minimal infinite H-invariant subgroup of A. H and B satisfy the conditions of Prop 4.14, thus an infinite field is interpreted.          □

    Before the next application of 4.14, we need to know something about automorphisms of fields of finite Morley rank.

**Fact 4.16.** Let K be a (infinite) field of finite Morley rank. Then K has no infinite definable proper subfield.

**Proof:** Let L be a definable infinite subfield. So L is $\omega$-stable and thus algebraically closed. K could not therefore be a finite extension of L. But then $L^n$ can be definably embedded in K for all $n < \omega$, whereby easily K must have infinite Morley rank.          □

**Lemma 4.17.** Let K be a (infinite) field of finite Morley rank. Let $\alpha$ be a nontrivial definable field automorphism of K. Then

$$\text{(i) } \alpha \text{ has infinite order}$$
$$\text{(ii) } \alpha \text{ is acl}(\varnothing)\text{-definable.}$$

**Proof:** (i) If $\alpha$ had finite order then the fixed field of $\alpha$ would be infinite and definable, contradicting the previous fact.

    (ii) If $\alpha$ were not acl$(\varnothing)$-definable then we could find $\beta$, a definable field automorphism, with $\alpha \neq \beta$ and stp$(\alpha) = $ stp$(\beta)$. But then $\alpha$ and $\beta$ must agree on the algebraic closure of the prime subfield of K, which is

infinite. So the fixed field of $\beta^{-1}\alpha$ is infinite, but $\beta^{-1}\alpha \neq$ id, again
contradicting the previous fact.


**Corollary 4.18.** A field $K$ of finite Morley rank cannot have a definable
group $G$ of automorphisms.

The second major application of 4.14 is Nesin's theorem:


**Proposition 4.19.** Let $G$ be solvable, connected of finite Morley rank.
Then $G'$ (the commutator subgroup) is nilpotent.


**Proof:** We suppose not and obtain a contradiction. As in the proof of 4.15,
we may assume $G'$ to be <u>centreless</u>. (Note $G'$ is definable, connected by
4.10). Let $A_1$ be a minimal infinite normal (in $G'$) definable subgroup of $G'$.
By induction we can choose $A_1$ in the center of $(G')'$. As $G'$ is solvable, $A_1$
is abelian. Let $A$ be the subgroup generated by the $A_1^g$, $g \in G$. Then $A < G'$
is abelian, normal in $G$ and in fact $A = A_1 \oplus A_1^{g_1} \oplus ... \oplus A_1^{g_k}$ some
$g_1,...,g_k \in G$ (using minimality of $A_1$, the fact that $G'$ is centreless and
finiteness of Morley rank). The proof now follows a series of steps: Let $R$ be
the ring of endomorphisms of $A$ generated by $G'$ (acting on $A$ by
conjugation) and let $I$ be the ideal in $R$ consisting of those $r \in R$ which
annihilate $A_1$.

**(1)** $R/I$ is an infinite field $K$, which is precisely the ring of endomorphisms
of $A_1$ generated by $G'/C_{G'}(A_1)$. The latter, as well as its action on $A_1$ is
definable, by 4.14. (Note $C_{G'}(A_1) \neq G'$ as $G'$ is centreless). Similarly
$I^g = \{r \in R : r(A_1^g) = 0\}$ is a maximal ideal of $R$ for all $g \in G$.


**(2)** There are only finitely many ideals $I^g$ of $R$ for $g \in G$.


**Proof:** Suppose not; let $m > RM(A)$, and suppose $I^{g_1},...,I^{g_m}$ are distinct
ideals. Let $B_i = A_1^{g_i}$, let $b_i \in B_i$ such that $b_1 +...+ b_m = 0$. Let $r \in R$ be
such that $r \equiv 1 \mod I^{g_1}$, $r \in I^{g_i}$ for $i = 2,...,m$ (as the $I^g$ are maximal

ideals). So $r(b_1+...+b_m) = rb_1 = 0$, so $b_1 = 0$. Similarly $b_i = 0$ $\forall i$. Thus the subgroups $B_i$ direct sum which implies that $RM(A) \geq m$, contradiction.

**(3)** $I = I^g$ for all $g \in G$.

**Proof:** Let by (2) $I_1, I_2, ..., I_n$ be the distinct conjugates of $I$. For all $m < \omega$ let $R_m$ be those members of $R$ that can be expressed as $h_1 + ... + h_m$ for $h_i \in G'$. Thus there is an $m$ such that the $I_j \cap R_m$ are pairwise distinct. But $G$ acts transitively and definably on the $I_j \cap R_m$. As $G$ is connected, there is only one. So (3) is proved.

**(4)** $R = K$ and is action on $A$ (making $A$ into a $K$-vector space) is definable.

**Proof:** As $A$ is generated by the $A^g$ and $I = I^g$ $\forall g$, it follows that $I = 0$. Thus "$R = K$" and the action of $r \in R$ on $A$ is determined by its action on $A_1$. More precisely: given that $I = 0$ it follows that the action of an element $r \in R$ on $A$ is determined by its action on $A_1$. But the action of $R$ on $A_1$ is precisely that of $K$. Now $K$ is definable: every element $k$ of $K$ can be represented by $h_1 + ... + h_n$, for $h_i \in G'$ and fixed $n \in \omega$ (by 4.14). Multiplication and addition of such elements are definable using a parameter from $A$, as is the relation $h_1 + ... + h_n = h'_1 + ... + h'_n$. The action of an element of $R$ represented by such $h_1 + ... + h_n$ on $A$ is precisely $(h_1 + ... + h_n) \cdot a = h_1 \cdot a + ... + h_n \cdot a$. We now identify $K$ with $R$.

It is easily checked that $G$ acts as a group of automorphisms of $K$ by $(h_1 + ... + h_n)^g$ and thus by Corollary 4.18 we have

**(5)** for every $k \in K$, $g \in G$ $k^g = k$.

Finally

**(6)** The action of $G$ on $A$ (by conjugation) is $K$-linear.

**Proof:** Let $k \in K$, $a \in A$, $g \in G$. Then clearly $(k \cdot (a))^g = k^g \cdot a^g = k \cdot a^g$, using (5).

By (6), $G'$ acts on the $K$-vector space $A$ as matrices with determinant 1. But $G'$ also acts as scalar multiplication. Thus $G'$ acts trivially on $A$ i.e. $A \subset Z(G')$ which contradicts everything.                    □

A result whose proof uses similar ideas to the above is:

**Proposition 4.20.** Let $G$ be solvable centreless connected of finite Morley rank. Let $A$ be the socle of $G$ (= group generated by minimal normal definable subgroups). Then the ring $R$ of endomorphisms of $A$ generated by the action of $G$ on $A$ by conjugation is definable, as well as the <u>action</u> of $R$ on $A$.

**Proof:** As $G'$ is nilpotent (by 4.19) it is clear that every minimal normal definable subgroup of $G$ is in $Z(G')$. Moreover $A = A_1 \oplus A_2 \oplus ... \oplus A_n$ where each $A_i$ is minimal normal definable. Then $R$ is generated by $G/C_G(A)$, which is Abelian (as $C_G(A) \supset G'$), and so $R$ is commutative. Let $I_i$ = the ideal of $R$ which annihilates $A_i$. As in 4.19, $R/I_i$ is a field and is "identical" to the ring of endomorphisms of $A_i$ generated by $G/C_G(A_i)$, which is definable by 4.15. Rewrite $A$ as $B_1 \oplus ... \oplus B_k$ where $B_j = A_{j_1} \oplus ... \oplus A_{jm_j}$, the ideals corresponding to the $A_{ji}$ are the same and for $j_i \neq j_2$ the ideals corresponding to the $A_{j_1 i}$, $A_{j_2 i}$ are different. Rewrite $I_j$ as the annihilator of $A_j$ (=annihilator of $B_j$). As in the proof of 4.19, $B_j$ is a $R/I_j$ vector space, where $R/I_j$ and its action on $B_j$ are definable. As the $I_j$ are maximal ideals of $R$, for each $j$ there is $r_j \in R$ which is 1 mod $I_j$ and 0 mod $I_k$ for $k \neq j$. Writing $K_j = R/I_j$, it follows that for any $s_1 \in K_1,...,$ $s_k \in K_k$ there is $r \in R$ such that for $a = b_1 + ... + b_k \in A$, $r \cdot a = s_1 \cdot b_1 + ... + s_k \cdot b_k$. Thus the action of $R$ on $A$ is the product of the actions of the $K_j$ on $B_j$, and so is definable.                    □

The greatest and in some sense correct level of generality of stable group theory is of course stable groups. In place of the DCC on definable subgroups, one has the weaker DCC on intersections of uniformly definable subgroups.

**Proposition 4.21.** Let G be a stable group, and $\varphi(x,\bar{y})$ a formula. Let $H_i$ $(i \in I)$ be subgroups of G, each defined by an instance of $\varphi$. Then $\bigcap_{i \in I} H_i = H_{i_1} \cap ... \cap H_{i_n}$ for some $i_1,...,i_n \in I$.

**Proof:** Let $\Delta(x)$ a a finite collection of formulas including $\varphi(z \cdot x, \bar{y})$. If by way of contradiction, we had an infinite descending chain of finite intersections of the $H_i$; say $K_1 \supsetneqq K_2 \supsetneqq K_3 ...$, then for each i we would clearly have $R_\Delta(K_{i+1}) < R_\Delta(K_i)$ or $m_\Delta(K_{i+1}) < m_\Delta(K_i)$, which would be a contradiction to 3.2. $\square$

Generic types in the general stable context can be defined using the notion of generic formula, introduced earlier: roughly p is generic iff it only contains generic formulas. This is equivalent to : if G is $|T|^+$-saturated and $p \in S_1(G)$, then p is a generic type of G if and only if ap does not fork over $\varnothing$ for all $a \in G$. A detailed exposition of the theory of generic types in the general stable situation appears in Victor Harnik's paper in this volume, so we will not go into any further details.

If G is stable and very saturated, we will call a subgroup H of G infinitely definable if H is defined by a collection of at most $|T|$ formulas. A theorem of Poizat asserts that the formulas can be taken to define subgroups of G. On the other hand, if H is the intersection of some arbitrary number of definable subgroups of G, then by 4.21, H is actually the intersection of at most $|T|$ many definable subgroups. In general infinitely definable subgroups arise in the stable context where definable subgroups appear in the $\omega$-stable context. For example, if G is saturated, $p \in S_1(G)$, then Fix p is an infinitely definable subgroup of G, by virtue of definability of types.

Superstable groups are of course in between. On the one hand by looking at the rank $R^\infty$ mentioned at the end of Section 3, we see that superstable groups have no infinite descending chains of definable subgroups, each of infinite index in the proceding one. On the other hand, a superstable group may not contain a smallest definable subgroup of finite index. So infinitely definable subgroups only come into the picture when we want to obtain connected components: If $G$ is a $|T|^+$–saturated superstable group then the connected component $G^\circ$ of $G$ is the intersection of all definable subgroups of $G$ of finite index. $R^\infty(G^\circ) = R^\infty(G)$. An important fact about superstable groups $G$ is that the <u>U-rank</u> of $G$ is well defined. There are types $p \in S_1(G)$ of maximum U–rank; these are precisely the generic types of $G$. An important topic that we have not mentioned is the study of groups of finite Morley rank from the point of view that they should resemble algebraic groups over algebraically closed fields. As this is clearly false for abelian groups (consider $Z_{p^\infty}$), some hypothesis of non-abelianness should be imposed. The major work was done by Cherlin [Ch], where he showed:

**(1)** A Morley rank 1 group is abelian-by-finite (Actually this is due to Reineke; Cherlin noted that any (infinite) $\omega$-stable group has an infinite definable abelian subgroup).

**(2)** If $G$ is connected, of Morley rank 2, then $G$ is solvable. If also $G$ is centreless then $G$ is the semidirect product of the additive and multiplicative groups of an algebraically closed field.

**(3)** If $G$ is connected of Morley rank 3 and $G$ is nonsolvable centreless and $G$ has a definable subgroup of Morley rank 2, then $G = PSL_2(K)$ for $K$ an algebraically closed field.

The attempt to eliminate (i.e. prove) the condition that $G$ has definable subgroup of rank 2 has led to an important line of research ([N1], [B.P], [Co]).

**Notes for section 4.** 4.1 is due to Macintyre [M] 4.6 is due to Cherlin [Ch] and Zilber [Z]. Our treatment of the material is influenced by Poizat's book [Po2]. Prop. 4.10 is due to Zilber [Z] as are 4.14, 4.15. Proposition 4.19, 4.20 are due to Nesin [N2]. 4.21 is Baldwin-Saxl [B.S]. The theory of stable groups in its full generality is due to Poizat.

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