Holomorphic C* Actions and Vector Fields on Projective Varieties

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In this series of talks, I will discuss two ways of relating the topology of a smooth projective variety X (over $\mathbb{C}$) with the fixed point set of a one dimensional group of automorphisms (either $\mathbb{C} = G_a$ or $\mathbb{C}^* = G_m$) on X . These ideas are summarized in the following diagrams:

$$(1) \quad \left\{\begin{array}{l}\text{Fixed point set } X^{\mathbb{C}^*}\\ \text{of a } \mathbb{C}^* \text{ action on } X\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{Integral homology}\\ \text{groups } H_\bullet(X,\mathbb{Z})\end{array}\right\}$$

$$(2) \quad \left\{\begin{array}{l}\text{Zeros of a holo-}\\ \text{morphic vector field}\\ \text{on } X \text{ with isolated}\\ \text{zeros}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{Complex cohomology}\\ \text{ring } H^\bullet(X,\mathbb{C})\end{array}\right\}$$

If X admits a $\mathbb{C}^*$ action with $X^{\mathbb{C}^*}$ finite and nontrivial, then X also has a holomorphic vector field with isolated zeros. The connection between the diagrams (1) and (2) is not clear, however, and seems to be one of the basic open questions in this area (c.f. §2.5).

This paper is divided into two parts, the first four chapters deal with $\mathbb{C}^*$ actions, and the next five with holomorphic vector fields. I have tried to keep the presentation on a nontechnical level. Several examples but very few proofs have been included. A few unsolved problems have also been mentioned.

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1. $\mathbb{C}^*$ ACTIONS ON PROJECTIVE VARIETIES

A good place to begin a discussion of $\mathbb{C}^*$ actions is with the fact that a holomorphic representation of $\mathbb{C}^*$ on a finite dimensional complex vector space V, say $\rho : \mathbb{C}^* \to GL(V)$, induces a holomorphic action of $\mathbb{C}^*$ on V, that is a holomorphic map $\mu : \mathbb{C}^* \times V \to V$ such that $\mu(1,v) \equiv v$ and $\mu(\lambda_1\lambda_2,v) = \mu(\lambda_1,\mu(\lambda_2,v))$. (We shall often write $\lambda\cdot v$ for $\mu(\lambda,v)$ when speaking of a $\mathbb{C}^*$ action.) The fact that ρ is a linear representation means that each $\lambda \in \mathbb{C}^*$ preserves lines through the origin in V, so μ descends to give a holomorphic action of $\mathbb{C}^*$ on $\mathbb{P}(V)$, $\tilde{\mu} : \mathbb{C}^* \times \mathbb{P}(V) \to \mathbb{P}(V)$.

A basic result about finite-dimensional representations of $\mathbb{C}^*$ says that V decomposes uniquely into a direct sum of weight spaces V_k, $k \in \mathbb{Z}$, i.e. $V = \oplus V_k(k \in \mathbb{Z})$, where $v \in V_k$ if and only if $\mu(\lambda,v) = \lambda^k v$ for all $\lambda \in \mathbb{C}^*$ The $k \in \mathbb{Z}$ such that $V_k \neq \{0\}$ are called the weights of the $\mathbb{C}^*$ action on V.

By a holomorphic $\mathbb{C}^*$ action on a complex projective variety X, we mean a holomorphic map $\mu : \mathbb{C}^* \times X \to X$ satisfying the properties mentioned above. It is well known that any holomorphic action of $\mathbb{C}^*$ on $\mathbb{C}\mathbb{P}^n$ arises through a one parameter subgroup $\lambda : \mathbb{C}^* \to \mathbb{P}GL(n,\mathbb{C})$, hence up to projective transformation a $\mathbb{C}^*$ action on $\mathbb{CP}^n$ is of the form

$$(1.1) \qquad \lambda\cdot[Z_0,Z_1,\ldots,Z_n] = [\lambda^{a_0}Z_0,\lambda^{a_1}Z_1,\ldots,\lambda^{a_n}Z_n]$$

where $a_0,a_1,\ldots,a_n \in \mathbb{Z}$.

One frequently encounters the situation in which X is an invariant subvariety of a $\mathbb{CP}^n$ with respect to $\mathbb{C}^*$ action of the form (1.1) on $\mathbb{CP}^n$. In this case the natural map $\mathbb{C}^* \times X \to X$ defines a $\mathbb{C}^*$ action on X .

Example 1. The variety $V(Z_0^{15} + Z_1^4 Z_2 Z_3^{10} + Z_1 Z_2^7 Z_3^7)$ in $\mathbb{CP}^3$ with action $\lambda\cdot[Z_0,Z_1,Z_2,Z_3] = [\lambda^3 Z_0,\lambda^{10}Z_1,\lambda^5 Z_2,Z_3]$.

Example 2. <u>Grassmannians</u>. Any $\mathbb{C}^*$ action on $\mathbb{C}^n$ permutes k-planes through the origin, hence defines a $\mathbb{C}^*$ action on the Grassmannian $G_k(\mathbb{C}^n)$. It is not hard to see that the image of $G_k(\mathbb{C}^n)$ in $\mathbb{P}(\wedge^k\mathbb{C}^n)$ under the Plucker imbedding is $\mathbb{C}^*$ invariant with respect to the $\mathbb{C}^*$ action on $\mathbb{P}(\wedge^k\mathbb{C}^n)$ given by the k^{th} exterior power representation $\lambda \to \wedge^k\lambda$.

Notice that any $\mathbb{C}^*$ action on $\mathbb{CP}^n$ has fixed points, i.e. points x so that $\lambda\cdot x \equiv x$. Indeed the connected components of the fixed point set are the linear subspaces of $\mathbb{CP}^n$ which correspond to eigenspaces of the induced linear action on $\mathbb{C}^{n+1}$. Clearly any closed invariant subset K of $\mathbb{CP}^n$ has fixed points: namely if $x \in K$ then $\lim_{\lambda \to 0} \lambda \cdot x$ and $\lim_{\lambda \to \infty} \lambda \cdot x$ are both fixed points in K. For convenience we set

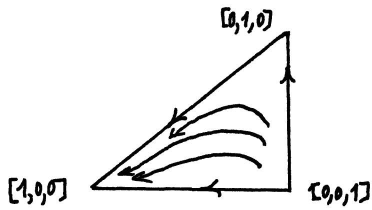
$$x_0 = \lim_{\lambda \to 0} \lambda \cdot x \quad \text{and} \quad x_\infty = \lim_{\lambda \to \infty} \lambda \cdot x$$

What is suggested by this construct is to consider the connected components of the fixed point set $X^{\mathbb{C}^*}$ (suppose these are labelled $X_1, \ldots, X_r$) and for each such component X_i its "plus and minus cells" X_i^+ and X_i^-, namely

$$X_i^+ = \{x \in X : x_0 \in X_i\}, \quad \text{and} \quad X_i^- = \{x \in X : x_\infty \in X_i\}$$

These "cells" turn out to be the fundamental objects that lead to connections between the topology of X and the topology of $X^{\mathbb{C}^*}$. We will frequently refer to them simply as B-B cells after A. Bialynicki-Birula who first proved the main structure theorem for them [B-B] (which will be discussed in §2).

Example 3. Let $\mathbb{C}^*$ act on $\mathbb{CP}^2$ by $\lambda \cdot [Z_0, Z_1, Z_2] = [Z_0, \lambda Z_1, \lambda^2 Z_2]$. Clearly the fixed points are $[1,0,0]$, $[0,1,0]$, and $[0,0,1]$. Then $[1,0,0]^+ = \mathbb{CP}^2 - V(Z_0)$, $[0,1,0]^+ = V(Z_0) - \{[0,0,1]\}$ and $[0,0,1]^+ = [0,0,1]$. In each case the plus cell is an affine space.



Example 4. Consider the action on $X = G_2(\mathbb{C}^4)$ induced by the action $\lambda\cdot(z_0,z_1,z_2,z_3) = (\lambda^{a_0}z_0,\lambda^{a_1}z_1,\lambda^{a_2}z_2,\lambda^{a_3}z_3)$ on $\mathbb{C}^4$ where $a_0 > a_1 > a_2 > a_3$. For a pair of independent vectors $u,v \in \mathbb{C}^4$, let $\langle u,v\rangle$ denote the 2-plane they span. We will compute $\langle e_1,e_3\rangle^+$, where $\{e_i : 0 \le i \le 3\}$ denotes the standard basis of $\mathbb{C}^4$. It suffices to consider $\lim_{\lambda\to 0} \lambda\cdot V$ for 2-planes of the form $V = \langle \alpha_0e_0 + \alpha_1e_1, \beta_0e_0 + \beta_1e_1 + \beta_2e_2 + \beta_3e_3\rangle$ where $\alpha_1\beta_3 \neq 0$. Now

$$\lambda\cdot V = \langle \lambda^{a_0}\alpha_0e_0 + \lambda^{a_1}\alpha_1e_1, \textstyle\sum_{i=0}^{3}\lambda^{a_i}\beta_ie_i\rangle$$

$$= \langle \lambda^{(a_0-a_1)}\alpha_0e_0 + \alpha_1e_1, \textstyle\sum_{i=0}^{3}\lambda^{(a_i-a_3)}\beta_ie_i\rangle$$

Since $a_0 > a_1 > a_2 > a_3$, it follows that $\lim_{\lambda\to 0}\lambda\cdot V = \langle\alpha_1e_1,\beta_3e_3\rangle = \langle e_1,e_3\rangle$. To give an invariant characterization of $\langle e_1,e_3\rangle^+$, we recall the definition of Schubert cycles in $G_2(\mathbb{C}^4)$. (See also [KL]). If b_1,b_2 are integers so that $1 \le b_1 < b_2 \le 4$, set

$$\Omega(b_1,b_2) = \{V \in G_2(\mathbb{C}^4) : \dim_{\mathbb{C}}(V \cap \mathbb{C}^{b_i}) \ge i\}$$

where $\mathbb{C}^1 \subset \mathbb{C}^2 \subset \mathbb{C}^3 \subset \mathbb{C}^4$ is the standard flag in $\mathbb{C}^4$. Note that if $\alpha_1 \neq 0$, then $V \not\in \Omega(1,4)$ and if $\beta_3 \neq 0$, then $V \not\in \Omega(2,3)$. Thus the above calculation shows that $\langle e_1,e_3 \rangle^+ = \Omega(2,4) - \Omega(1,4) - \Omega(2,3)$. It follows that $\overline{\langle e_1,e_3 \rangle^+} = \Omega(2,4)$. The Schubert cycles $\Omega(b_1,b_2)$ on $G_2(\mathbb{C}^4)$ are ordered by inclusion via the lexicographic order on the (b_1,b_2). This is shown in the diagram below. By a similar argument, $\langle e_i,e_j \rangle^+ = \Omega(i+1, j+1)$ if $0 \le i < j \le 3$.

$$
\begin{array}{lccccc}
\text{dimension} & 0 & 1 & 2 & 3 & 4 \\
 & & & \Omega(1,4) & & \\
 & \Omega(1,2)=\mathbb{C}^2 \prec & \Omega(1,3) & \nearrow \quad \searrow & \Omega(2,4) \prec & \Omega(3,4)=G_2(\mathbb{C}^4) \\
 & & & \searrow \quad \nearrow & & \\
 & & & \Omega(2,3) & &
\end{array}
$$

A useful feature of this diagram (sometimes called the Hasse diagram) is that one can see a free homology basis of any $\Omega(i,j)$: namely itself and the Schubert cycles that precede it in the diagram.

2. THE B-B DECOMPOSITION

The structure theorem of Bialynicki-Birula [B-B] describes the structure of the plus and minus cells on X when X is a complete, smooth variety with G_m action over an arbitrary algebraically closed field. Carrell and Sommese [CS_1] and Fujiki [Fu] showed that this theorem goes through without change to compact Kaehler manifolds.

Recently, however, Sommese has found an example of a compact Moishezon manifold X with a $\mathbb{C}^*$ action for which the B-B decomposition $X = \cup X_j^+$ exists but not all of the canonical maps $X_j^+ \to X_j$, $x \to x_0$, are continuous [S_2]. For a smooth projective variety X with fixed point components $X_1, \ldots,$ X_r the theorem says the following:

THEOREM 1. (i) For each $i = 1,\ldots,r$, the natural map $p_i : X_i^+ \to X_i$, $x \to x_0$, is the projection of a holomorphic fibre bundle whose fibres are all $\mathbb{C}^*$ equivariantly isomorphic to a fixed $\mathbb{C}^{m_j}$.

(ii) In fact, if $x \in X_j$, then $p_i^{-1}(x)$ is $\mathbb{C}^*$ equivariantly isomorphic to $T_x(X)/T_x(X_i)$ with $\mathbb{C}^*$ action induced by the representation $\lambda \mapsto d\lambda_x$ of C^* in $GL(T_x(X))$. ($d\lambda_x$ denotes the differential of the map $y \to \lambda \cdot y$ at x.)

(iii) X_i^+ is a Zariski open subset of its Zariski closure. Hence $\overline{X_i^+}$ (the topological closure) is a closed subvariety of X containing X_i^+ as a Zariski open.

(iv) There exists a unique component, say X_1, of $X^{\mathbb{C}^*}$ so that X_1^+ is Zariski open in X. X_1 is called the <u>source</u> of X.

A completely analogous result holds for the minus decomposition of X. The distinguished component X_i so that X_i^- is Zariski open in X is called the sink of X. We will always label the sink as X_r.

COROLLARY. Suppose either the source or sink of X is rational. Then X is rational i.e. X is birationally equivalent to $\mathbb{CP}^n$.

For a proof see $[CS_1]$. In the case, say, of an isolated source x, then X is a compactification of the vector space $N_x^+(\{x\})$.

An important, but easy to establish, fact is that if X is a smooth invariant subvariety of $\mathbb{CP}^n$, then there exists a Morse function f on X that has the property of increasing on the $\mathbb{R}^+$ orbits in X. In fact, let V denote infinitesimal isometry associated to $S^1 \subset \mathbb{C}^*$, and let Ω denote the Fubini-Study metric on $\mathbb{CP}^n$. One finds f by solving the equation $i(V)\Omega = dF$ on $\mathbb{CP}^n$ and then restricting F to X. Let

$$F[Z_0,\ldots,Z_n] = \sum a_i|Z_i|^2/\sum|Z_i|^2$$

as long as coordinates $[Z_0,\ldots,Z_n]$ have been chosen so that $\lambda\cdot[Z_0,\ldots,Z_n] = [\lambda^{a_0}Z_0,\ldots,\lambda^{a_n}Z_n]$. The following are not hard to verify using the contraction identity $i(V)\Omega = dF$:

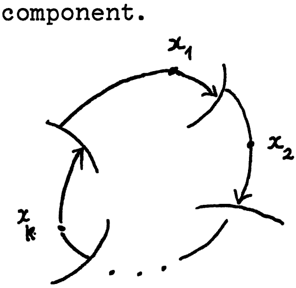
(i) $f = F|X$ is a Morse function on X whose critical submanifolds are $X_1, \ldots, X_r$;

(ii) f is strictly increasing on the $\mathbb{R}^+$ orbits of nonfixed points;

(iii) if X is not contained in a hyperplane of $\mathbb{CP}^n$, then $X_1 = X \cap (\text{source of } \mathbb{CP}^n)$ and $X_r = X \cap (\text{sink of } \mathbb{CP}^n)$; and

(iv) the Morse index of f on X_i is $\dim_{\mathbb{R}} N_x^-(X_i)$, $x \in X_i$, where $N_x^-(X_i)$ denotes the subspace of $T_x(X)$ generated by vectors of negative weight (it is actually a subspace of the normal space to X_i at x).

In the compact Kaehler case, assuming $X^{\mathbb{C}^*} \neq \emptyset$, there is a Morse function satisfying (i),(ii), and (iv) due to Frankel [Fr] and Matsushima. Its importance here is in guaranteeing that there is no sequence of points $x_1, \ldots, x_k$ in $X - X^{\mathbb{C}^*}$ so that $(x_i)_\infty$ and $(x_{i+1})_0$ lie in the same component for $i = 1, \ldots, k-1$ and $(x_1)_0$ and $(x_k)_\infty$ also lie in the same component.



Examples of such "quasi-cycles" are known in the non Kaehler case (see [Ju] and [S_2]) .

The Frankel-Matsushima Morse function is applied in a different manner in [At] .

Example 5. (G/B) . Let G be a semi-simple algebraic group, B a Borel subgroup, H a fixed maximal torus in B and $W = N_G(H)/C_G(H)$ be the Weyl group of H in G . It is well known (see e.g. [H]) that G/B is a smooth projective variety and that H acts holomorphically on G/B by left translation: $\mu(h,gB) = (hg)B$. Moreover $(G/B)^H = \{gB : g \in N_G(H)\}$ and gB depends only on $\bar{g} \in W$. Thus the correspondence $\bar{g} \to gB$ sets up a one to one correspondence between W and the fixed point set $(G/B)^H$, and we may unambiguously refer to wB . A one-parameter subgroup $\lambda : \mathbb{C}^* \to H$ is called <u>regular</u> if $(G/B)^{\mathbb{C}^*} = (G/B)^H$ under the action $\mu(t,gB) = (\lambda(t)g)B$ of $\mathbb{C}^*$.

By a theorem of Konarski [Kon] , the plus decomposition of G/B associated to λ is B invariant provided the source of λ is eB . This can be used to identify the associated plus decomposition and the Bruhat decomposition. In fact, for each $w \in W$,

(1.2) $(wB)^+ = B(wB)$ (the B orbit of $wB \in G/B$)

To see this note that $BwB \subset (wB)^+$ by Konarski's result. Since the plus cells $(wB)^+$ are disjoint and the Bruhat cells $B(wB)$ cover G/B, the proof of (1.2) is complete.

Another treatment of the Bruhat decomposition using $\mathbb{C}^*$ actions appears in [A_1].

We now turn our attention to possibly singular projective varieties X invariantly imbedded in a $\mathbb{CP}^n$. For example we can now consider actions on Schubert cycles and, more generally, on the generalized Schubert varieties $\overline{X_i^+}$ which are closures of the plus cells. Although the B-B decomposition is no longer always locally trivial, one can single out a natural class of actions (which always exist in Schubert varieties) on which the B-B decomposition is still nice enough. To do so, suppose X is endowed with an analytic Whitney stratification whose strata are $\mathbb{C}^*$ invariant. (For example, the canonical Whitney stratification of X is always invariant [W]). The Whitney stratification on X is called <u>singularity preserving as</u> $\lambda \to 0$ (resp. <u>singularity preserving as</u> $\lambda \to \infty$) if, for any stratum A, $x \in A$ implies $x_0 \in A$ (resp. $x_\infty \in A$). Intuitively, this means that x_0 is just as singular as x is.

Example 6. Let Y denote the cone in $\mathbb{CP}^3$ over a smooth algebraic curve $X \subset \mathbb{CP}^2$ with vertex $x = [0,0,0,1] \in \mathbb{CP}^3 - \mathbb{CP}^2$. The natural action of $\mathbb{C}^*$

on Y induced by the action $\mu(\lambda,[Z_0,Z_1,Z_2,Z_3]) = [Z_0,Z_1,Z_2,\lambda Z_3]$ on $\mathbb{CP}^3$ has source X and sink $\{x\}$. Y can be stratified with strata $\{x\}$ and $Y - \{x\}$ and this renders the action singularity preserving as $\lambda \to 0$. Since $\{x\}$ is an isolated singular point on Y, the action is not singularity preserving as $\lambda \to \infty$ for any Whitney stratification of Y. Note that although the cells $Y - \{x\}$ and $\{x\}$ of the plus decomposition are locally trivial affine space bundles, the minus cell $x^- = Y - X$ is not.

The next theorem partially answers the question of what structure a singular invariant subvariety must have. The proof will appear in [CG]

THEOREM 2. If X is a $\mathbb{C}^*$ invariant subvariety $\mathbb{CP}^n$ whose $\mathbb{C}^*$ action is singularity preserving as $\lambda \to 0$ with respect to some invariant Whitney stratification of X, then for each connected component X_j of $X^{\mathbb{C}^*}$, the natural projection $p_j : X_j^+ \to X_j$ renders X_j^+ a topologically locally trivial affine space bundle. The fibres are biregularly (and equivariantly) isomorphic to some $\mathbb{C}^{m_j}$ (depending only on X_j^+).

3. THE HOMOLOGY BASIS THEOREM

Recall that the classical Basissatz of Schubert calculus [KL] says that the Schubert cycles form a homology basis for $G_k(\mathbb{C}^n)$. To be precise, fix a flag $\mathbb{C}^1 \subset \mathbb{C}^2 \subset \ldots \subset \mathbb{C}^n$ in $\mathbb{C}^n$. Then for any increasing k-tuple $(a_1,\ldots,a_k)$ of integers so that $1 \le a_1 < a_2 < \ldots < a_k \le n$, set

$$(1.3) \qquad \Omega(a_1,\ldots,a_k) = \{V \in G_k(\mathbb{C}^n) : \dim_{\mathbb{C}}(V \cap \mathbb{C}^{a_i}) \ge i\}$$

The $\Omega(a_1,\ldots,a_k)$ are projective varieties called Schubert cycles (or Schubert varieties) whose associated homology classes in $H_*(G_k(\mathbb{C}^n),\mathbb{Z})$ we denote by $[\Omega(a_1,\ldots,a_k)]$. The Basissatz says: For each m with $0 \le m \le k(n-k)$, the $[\Omega(a_1,\ldots,a_k)]$ with $\sum_{j=1}^{k}(a_j - j) = m$ form a basis of $H_{2m}(G_k(\mathbb{C}^n),\mathbb{Z})$.

Even showing that $\Omega(a_1,\ldots,a_k)$ is a projective variety is somewhat complicated (see e.g. [KL]). However, by a calculation similar to that in Example 4, there exists a $\mathbb{C}^*$ action on $G_k(\mathbb{C}^n)$ so that $\overline{X_j^+} = \Omega(a_1,\ldots,a_k)$ for some component X_j. Consequently, by the theorem of Bialynicki-Birula, $\Omega(a_1,\ldots,a_k)$ is automatically a subvariety of $G_k(\mathbb{C}^n)$.

A more interesting fact, however, is that there exists an analog of the Basissatz for any smooth (and many singular) projective variety with $\mathbb{C}^*$ action in which the $\overline{X_j^+}$ play a role similar to the role played by the Schubert cycles (with respect to a fixed flag) in $G_k(\mathbb{C}^n)$. In fact, if $X^{\mathbb{C}^*}$ is isolated, the $\overline{X_j^+}$ form a homology basis of $H_{\cdot}(X,\mathbb{Z})$. For this reason, we sometimes refer to the $\overline{X_j^+}$ (and $\overline{X_j^-}$) as generalized Schubert varieties. Before stating this generalization of the Basissatz, let us mention that using the Frankel-Matsushima Morse function f, Frankel showed in [Fr] (see also [Kob]) that

(i) $b_k(X) = \sum_j b_{k-\lambda_j}(X_j)$ where $\lambda_j = \dim_{\mathbb{R}} N_x^-(X_j) = \mathrm{Ind}(f|X_j$

(ii) X has torsion if and only if $X^{\mathbb{C}^*}$ does.

THEOREM 3 [CS_2]. Let X be a smooth projective variety with $\mathbb{C}^*$ action having fixed point components $X_1,\ldots,X_r$. Let m_j (resp. n_j) denote the fibre dimension over $\mathbb{C}$ of $p_j : X_j^+ \to X_j$ (resp. $q_j : X_j^- \to X_j$). Then there exist canonical plus and minus isomorphisms

$$\pi_k : \oplus_j H_{k-2m_j}(X_j,\mathbb{Z}) \to H_k(X,\mathbb{Z}) \tag{1.4}$$

and

$$\mu_k : \oplus_j H_{k-2n_j}(X_j,\mathbb{Z}) \to H_k(X,\mathbb{Z}) \tag{1.5}$$

By dualizing these isomorphisms to cohomology over $\mathbb{C}$ and using the Hodge decomposition $H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^p(X,\Omega^q)$ one obtains the following result.

COROLLARY [CS_2]. The plus and minus isomorphisms induce isomorphisms

$$(1.6) \qquad \pi^* : H^p(X,\Omega^q) \to \oplus_j H^{p-m_j}(X_j,\Omega^{q-m_j})$$

and

$$(1.7) \qquad \mu^* : H^p(X,\Omega^q) \to \oplus_j H^{p-n_j}(X_j,\Omega^{q-n_j})$$

By taking dimensions (over $\mathbb{C}$) we get

$$h^{p,q}(X) = \sum_j h^{p-m_j,q-m_j}(X_j)$$

$$= \sum_j h^{p-n_j,q-n_j}(X_j)$$

which is a result obtained by several authors: independently by Luzstig and Wright [Wr] for isolated fixed points via Morse theory and independently by Fujiki [Fu] and Iversen using mixed Hodge structure.

There are several consequences that relate the source and the sink to each other and to X .

(a) $H^0(X,\Omega^q) \cong H^0(X_1,\Omega^q) \cong H^0(X_r,\Omega^q)$

(b) $\pi_1(X) \cong \pi_1(X_1) \cong \pi_1(X_r)$

(c) there exist exact sequences

$$0 \to K^+ \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_1) \to 0$$
$$0 \to K^- \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_r) \to 0$$

where K^+ (resp. K^-) is the $\mathbb{Z}$-module of divisors in X generated by the $\overline{X_i^+}$ (resp. $\overline{X_i^-}$) which are divisors in X.

Another relationship between X and $X^{\mathbb{C}^*}$ is

(d) $\operatorname{Index}(X) = \sum_j \operatorname{Index}(X_j)$

The proofs of (a) - (d) are contained in [CS_2]. (d) is also proved in [Fu].

4. A GENERALIZATION OF THE HOMOLOGY BASIS THEOREM

One can ask whether the homology basis theorem is also true for singular invariant subvarieties in $\mathbb{CP}^n$. The answer is, not surprisingly, no in general. However, for actions which we call "good", the answer is yes. Among the spaces with a good action are the generalized Schubert varieties $\overline{X_j^+}$ in a smooth X which are themselves unions of plus cells in X (i.e. there exist $i_1,\ldots,i_k$ so that $\overline{X_j^+} = X_{i_1}^+ \cup \ldots \cup X_{i_k}^+$) due to the fact that the plus cells in $\overline{X_j^+}$ are $X_{i_1}^+,\ldots,X_{i_k}^+$ and the fact that, since X is smooth, the $X_{i_k}^+$ are locally trivial affine space bundles. The strategy for extending the (plus) homology basis theorem is to single out a class of actions with plus cells being locally trivial affine space bundles for which a plus homomorphism with natural properties can be defined. The proof then uses the Thom isomorphism. It seems to us that the class of good actions does not give the optimal generalization.

For any component X_j of $X^{\mathbb{C}^*}$, let Γ_j denote the closure of the graph of $p_j : X_j^+ \to X_j$ in $X \times X_j$, and let $g_j : \Gamma_j \to X_j$ be the projection.

DEFINITION. An action $\mathbb{C}^* \times X \to X$ is <u>good as $\lambda \to 0$</u> if, for each connected component X_j of $X^{\mathbb{C}^*}$, the following conditions hold:

(i) the projection $p_j : X_j^+ \to X_j$ is a topologically locally trivial affine space bundle, and

(ii) X_j has an analytic Whitney stratification such that for each stratum A,

$$g_j^{-1}(\bar{A}) = \text{closure}\{(p_j(x),x) \in X_j \times X | x \in A^+\}$$

where $A^+ = \{x \in X : x_0 \in A\}$.

The condition (ii) means one can unambiguously write Γ_A for $g_j^{-1}(\bar{A}) \subset \Gamma_j$. It is easy to construct a space X with a point x_0 in the source X_1 of X having the property that $g_1^{-1}(x_0) \supsetneq \text{closure}\{(x_0,x) : x \in x_0^+\}$. Let $Y = \mathbb{CP}^1 \times \mathbb{CP}^2$ with the action $\lambda\cdot([z_0,z_1];[w_0,w_1,w_2]) = ([z_0,\lambda z_1];[w_0,w_1,w_2])$, and let X be Y with the point

([0,1];[1,0,0]) blown up. Now take $x_0 = ([1,0];[1,0,0])$.

The reason for condition (ii) is to allow us to construct a wrong way map $g_j^{\#} : H_k(X_j, \mathbb{Z}) \to H_{k+2m_j}(\Gamma_j, \mathbb{Z})$. If we try to define $g_j^{\#}(\text{cycle}) = \text{closure } g_j^{-1}(\text{cycle})$, then the point x_0 in the above example will certainly cause a problem. We must therefore be able to stratify X_j so that the set of bad points in each stratum is a subvariety of the stratum and then consider only cycles on X_j that are transverse to the strata. Thus a nice complex of transverse cycles is obtained on X_j that admits a wrong way chain map into the chains of Γ_j. In the example above we may stratify the components of $X^{\mathbb{C}^*}$ with one stratum each. Nice 0-cycles and 1-cycles in X_1 will avoid x_0. When a wrong way homomorphism $g^{\#}$ exists, the plus homomorphism is defined as the composition

$$(1.8) \qquad H_k(X_j, \mathbb{Z}) \xrightarrow{g^{\#}} H_{k+2m_j}(\Gamma_j, \mathbb{Z}) \to H_{k+2m_j}(X, \mathbb{Z})$$

where the latter map is induced by the projection $\Gamma_j \to X$. We then have

THEOREM 4 [CG]. If the action $\mathbb{C}^* \times X \to X$ is good as $\lambda \to 0$, then the plus isomorphisms (1.8) are valid for all k. Moreover, for almost every k cycle z on X_j, the class of $\mu_k(z)$ is represented by the k cycle $\overline{p_j^{-1}(z)}$ on X.

Examples of actions that are good as $\lambda \to 0$:

(i) if X is smooth, then any $\overline{X_j^+}$ in X that is a union of plus cells in X with the induced $\mathbb{C}^*$ action;

(ii) any X in which each X_j^+ is smooth.

It is hoped that a more general setting in which the plus isomorphisms are valid will be found. At the present, all the examples we know of singular varieties with a plus isomorphism have a good action. Hopefully, it will eventually be shown that the plus isomorphisms are valid whenever the plus cells are locally trivial affine space bundles.

Example 7. Let Y be, as in Example 6, the cone with vertex $x \in \mathbb{CP}^3 - \mathbb{CP}^2$ over a smooth curve X in $\mathbb{CP}^2$. Then the action defined in Example 6 is good as $\lambda \to 0$ but not good as $\lambda \to \infty$. The plus isomorphism takes the form

$$H_0(\{x\}) \cong H_0(Y)\ , \quad H_i(X) \cong H_{i+2}(Y)\ , \quad 0 \le i \le 2\ .$$

These are the well known Thom isomorphisms [MS]. There is no minus isomorphism however if the genus of X is greater than zero.

Example 8. The Schubert cycle $X = \Omega(2,4)$ in $G_2(\mathbb{C}^4)$ with action induced by $\lambda \cdot (z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,\lambda^{-1}z_4)$

has two fixed point components: the source X_1 is a $\mathbb{CP}^1$ and the sink X_2 is a $\mathbb{CP}^2$ containing the singular point. The plus decomposition of X is locally trivial, and since X_1^+ is $X - X_2$ and $X_2^+ \cong \mathbb{CP}^2$, both plus cells are smooth. Therefore this action is good as $\lambda \to 0$. The minus decomposition fails to be locally trivial. In fact one can easily verify that if V denotes the singular point $\mathbb{C}^2$ of X, then $V^- \cong \mathbb{C}^2$ while $W^- \cong \mathbb{C}$ for any other 2 plane $W \in X_2$. The plus and minus homomorphisms take the form

$$H_{k-4}(X_1) \oplus H_k(X_2) \stackrel{\approx}{\to} H_k(X) \leftarrow H_k(X_1) \oplus H_{k-2}(X_2)$$

The minus homomorphism is neither injective nor surjective.

It would be interesting to know if there exist examples of actions that are singularity preserving as $\lambda \to 0$ that are not good as $\lambda \to 0$. We mention a partial result from [CG].

THEOREM 5. Let X have an action that is singularity preserving as $\lambda \to 0$. Suppose that for any stratum A of X and for any component X_j of $X^{\mathbb{C}^*}$, either $A \cap X_j = \emptyset$ or $\overline{(A \cap X_j)^+} = \overline{A} \cap \overline{X_j^+}$. Then the action is good as $\lambda \to 0$

We close this chapter with two questions.

1. In the case of a good action, how does the mixed

Hodge structure on X relate to the mixed Hodge structure on $X^{\mathbb{C}^*}$?

2. If X has a not necessarily good action with isolated fixed points, do the odd homology groups of X vanish?

5. HOLOMORPHIC VECTOR FIELDS AND THE COHOMOLOGY RING

It is a basic fact that the cohomology ring of a smooth projective variety X admitting a holomorphic vector field V with isolated zeroes $Z \neq \emptyset$ is determined on Z. To be precise let Z denote the variety with structure sheaf $\mathcal{O}_Z = \mathcal{O}_X/i(V)\Omega^1$ where $i(V): \Omega^p \to \Omega^{p-1}$ denotes the contraction of holomorphic p-forms to (p-1)-forms. Then $i(V)$ defines a complex of sheaves

$$0 \to \Omega^n \xrightarrow{i(V)} \Omega^{n-1} \to \dots \to \Omega^1 \to \mathcal{O} \to 0 .$$

which is locally free resolution of $\mathcal{O}_Z$ since V has isolated zeros.

It follows from general facts that there exists a spectral sequence with $E_1^{-p,q} = H^q(X,\Omega^p)$ abutting to $H^0(X,\mathcal{O}_Z)$. The key fact proved in [CL_1] is that if X is compact Kaehler, then this spectral sequence degenerates at E_1 as long as $Z \neq \emptyset$. As a consequence of the finiteness of Z and $i(V)$ being a derivation, we have

THEOREM 6 [CL_2]. If X is a smooth projective variety admitting a holomorphic vector field V with $Z = \mathrm{zero}(V)$ finite but nontrivial, then

(i) $H^p(X,\Omega^q) = 0$ if $p \neq q$ (consequently $H^{2p}(X,\mathbb{C}) = H^p(X,\Omega^p)$ and $H^{2p+1}(X,\mathbb{C}) = 0$), and

(ii) there exists a filtration

$$H^0(X,\mathcal{O}_Z) = F_n \supset F_{n-1} \supset \cdots \supset F_1 \supset F_0 .$$

where $n = \dim X$, such that $F_i F_j \subset F_{i+j}$ and having the property that as graded rings

$$\bigoplus_p F_p/F_{p+1} \cong \bigoplus_p H^{2p}(X,\mathbb{C}) . \tag{2.1}$$

For example, if V has only simple zeros, in other words if Z is nonsingular, then $H^0(X,\mathcal{O}_Z)$ is precisely the ring of complex valued functions on Z. Thus, algebraically, $H^0(X,\mathcal{O}_Z)$ can be quite simple. The difficulty in analyzing the cohomology ring is in describing the filtration F.

Example 9. For each holomorphic action of $\mathbb{C}^*$, one also has the infinitesimal generator, i.e. the holomorphic vector field V obtained by differentiating the action with respect to λ:

$$V_x = \frac{d}{d\lambda}(\lambda\cdot x)\Big|_{\lambda=1} .$$

Clearly, the fixed point set of $\mathbb{C}^*$ coincides with zero set of V. One can easily show that the infinitesimal generator of the $\mathbb{C}^*$ action (1.1) on $\mathbb{CP}^n$ in local affine coordinates $\zeta_1 = z_1/z_0, \ldots, \zeta_n = z_n/z_0$ at the fixed point $[1,0,\ldots,0]$ is the holomorphic vector field

$$V = \sum_{i=1}^n (a_i - a_0)\zeta_i \, \partial/\partial\zeta_i \tag{2.2}$$

on $\mathbb{CP}^n$

Let us continue this example by exhibiting the filtration. The holomorphic vector field (2.2) on $\mathbb{CP}^n$ has isolated zeros if $a_0 < a_1 < \ldots < a_n$. Also, the cohomology ring of $\mathbb{CP}^n$, $\oplus H^{2i}(\mathbb{CP}^n, \mathbb{C})$, has the structure of a polynomial ring, on one generator of degree two, truncated at degree $2n$. That generator is in fact the cohomology class of the closed two form Ω on $\mathbb{CP}^n$. Now since Z is non-singular and finite, $(\mathcal{O}_Z)_\zeta = \mathbb{C}$ for each $\zeta \in Z$ so $H^0(Z,\mathcal{O}_Z)$ is the ring of all complex valued functions on Z. We will let $(\lambda_0,\ldots,\lambda_n)$ denote the function whose value at $[1,0,\ldots,0]$ is λ_0 etc. Then it can be shown that

$$F_0 = \langle (1,\ldots,1) \rangle \simeq H^0(\mathbb{CP}^n, \mathbb{C})$$

$$F_1 = \langle (1,\ldots,1), (a_0,\ldots,a_n) \rangle$$

and $(a_0,\ldots,a_n)$ is sent to Ω under the isomorphism.

(2.1). In general,

$$F_i = \langle (1,\ldots,1),\ (a_0,\ldots,a_n),\ \ldots,\ (a_0^i,\ldots a_n^i) \rangle$$

For example, the linear independence of the $(a_0,\ldots,a_n)^i$ for $0 \le i \le n$ follows from the van der Monde determinant

$$\det \begin{pmatrix} 1 & \cdots & 1 \\ a_0 & \cdots & a_n \\ \vdots & & \\ a_0^n & \cdots & a_n^n \end{pmatrix} = \prod_{i<j} (a_j - a_i)$$

Example 10. Vector fields on G/B. Let h denote the Lie algebra of H. We call a vector $v \in h$ regular if the set of fixed points of the one parameter group $\exp(tv)$ of H acting on G/B by left translation is exactly $(G/B)^H$. Set $V = \frac{d}{dt}\exp(tv)|_{t=0}$ so that $Z = \mathrm{zero}(V) = (G/B)^H$. Clearly, the zeros of V are all simple.

6. BOREL'S THEOREM AND HOLOMORPHIC VECTOR FIELDS

For $w \in W$ and $v \in h$, $w.v$ will denote the action of W on h. W thus acts effectively on h and on h^* in the usual way: $w\cdot f(v) = f(w^{-1}\cdot v)$ for $f \in h^*$. To every character $\alpha \in X(H)$, one associates the holomorphic line bundle $L_\alpha = (G \times \mathbb{C})/B$ on G/B where $(g,z)b = (gb,\alpha(b^{-1})z)$ and where α has been extended to B by the usual convention. Now $d\alpha \in h^*$ and since G is semi-simple, the $d\alpha$, for all $\alpha \in X(H)$, span h^*. Thus there is a well defined linear map $\beta : h^* \to H^2(G/B,\mathbb{C})$ determined by the condition $\beta(d\alpha) = c_1(L_\alpha)$ for any $\alpha \in X(H)$. Let β also denote the algebra homomorphism $\beta : R = \mathrm{Sym}(h^*) \to H^\bullet(G/B,\mathbb{C})$ extending β, where $\mathrm{Sym}(h^*)$ is the symmetric algebra of h^*. W acts on R, so denote by I_W (resp. I_W^+) the ring of invariants of W in R (resp. $f \in I_W$ such that $f(0) = 0$). Borel proved that β is a surjective homomorphism whose kernel is $R\,I_W^+$. Consequently, since $R\,I_W^+$ is a homogeneous ideal, β induces an isomorphism of graded rings

$$\bar{\beta} : R/R\,I_W^+ \xrightarrow{\sim} H^\bullet(G/B,C) \tag{2.3}$$

The purpose of this section is to show how Borel's theorem relates to vector fields. Note that $H^0(G/B,\mathcal{O}_Z) = C^W$ for any vector field on G/B generated by a regular vector in h. We will begin with a more detailed description of $H^0(G/B,\mathcal{O}_Z)$. Define a linear map $\Psi_v : h^* \to H^0(G/B,\mathcal{O}_Z)$ by $\Psi_v(\omega)(w) = -w\cdot\omega(v)$. Then Ψ_v can be extended to an algebra homomorphism $\Psi_v : R \to H^0(G/B,\mathcal{O}_Z)$. For any $v \in h$, let I_v denote the ideal in R generated by all $\phi \in I_W$ such that $\phi(v) = 0$. The ring R/I_v is only graded when I_v is homogeneous, i.e. only when $v = 0$ and $R/I_v = R/RI_W^+$. However R/I_v is always filtered by degree. Namely, if

$p = 0,1,\ldots,$ set $(R/I_v)_p = R_p/R_p \cap I_v$ where $R_p = \{f \in R : \deg f \le p\}$. Notice that $I_v \subset \ker\Psi_v$; for if $\phi \in I_W$ and $\phi(v) = 0$, then for all $w \in W$, $\psi_v(\phi)(w) = (w\cdot\phi)(v) = \phi(v) = 0$. In fact it is shown in [C] that for v in a dense open set in h, Ψ_v induces an isomorphism

$$(2.4) \qquad \bar{\Psi}_v : R/I_v \to H^0(G/B, \mathcal{O}_Z)$$

preserving the filtration, i.e. $\bar{\Psi}_v((R/I_v)_p) = F_p$. Consequently, for each p, the natural morphism $F_1^{\otimes p} \to F_p$ is onto.

The first step in the proof is to identify elements in $H^0(G/B, \mathcal{O}_Z)$ that determine the Chern classes $c_1(L_\alpha)$ for $\alpha \in X(H)$. To accomplish this we recall the theory of V-equivariant Chern classes. A holomorphic line bundle L on X is called V-equivariant if the derivation $V : \mathcal{O}_X \to \mathcal{O}_X$ lifts to a derivation $\tilde{V} : \mathcal{O}_X(L) \to \mathcal{O}_X(L)$; i.e. a $\mathbb{C}$-linear map satisfying $\tilde{V}(fs) = V(f)s + f\tilde{V}(s)$ if $f \in \mathcal{O}_X$, $s \in \mathcal{O}_X(L)$. Since $V(f) = i(V)df$, $\tilde{V}$ defines a global section of $\mathrm{End}(\mathcal{O}_X(L) \otimes_{\mathcal{O}_X} \mathcal{O}_Z) \cong \mathcal{O}_Z$; i.e. $\tilde{V} \in H^0(X, \mathcal{O}_Z)$. It is shown in [$CL_2$] that

(i) $\tilde{V} \in F_1$ and has image $c_1(L)$ under the isomorphism (2.1), and

(ii) every line bundle on X is V-equivariant if $Z \neq \emptyset$.

The calculation of $c_1(L_\alpha)$ is provided by the following lemma

Lemma. Given $\alpha \in X(H)$ there exists a lifting $\tilde{V}_\alpha$ of V to $\mathcal{O}(L_\alpha)$ so that in $H^0(G/B,\mathcal{O}_Z)$, $\tilde{V}_\alpha(wB) = -d\alpha(w^{-1}\cdot v)$ where $v \in h$ is the regular vector corresponding to V.

In other words, $\psi_v(d\alpha)w = -(w\cdot d\alpha)(v) = -d\alpha(w^{-1}\cdot v)$ so, since the $d\alpha$ span h^*, $\psi_v(h^*) \subset F_1$. The remainder of the proof is outlined in [C]. Complete details will appear in $[A_2]$.

To prove Borel's theorem (2.3), note that we have, for each regular $v \in h$, a commutative diagram

$$\begin{array}{ccc} h^* & \xrightarrow{\Psi_v} & F_1/F_0 \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle i_v} & \\ & H^2(G/B,\mathbb{C}) & \end{array}$$

where i_v is an isomorphism, and Ψ_v is surjective. Consequently β is surjective. Moreover, this results in a commutative diagram for each $p \geq 1$

$$\begin{array}{ccc} R_p/R_{p-1} & \longrightarrow & F_p/F_{p-1} \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle i_v} & \\ & H^{2p}(G/B,\mathbb{C}) & \end{array} \qquad (2.5)$$

where Ψ_V is surjective and i_V is an isomorphism. Thus $\beta : R \to H^{\bullet}(G/B,C)$ is surjective. To complete the proof, one must show that $\ker \beta = R\, I_W^+$. But because $\dim R/R\, I_W^+ = \dim H^0(G/B,\mathcal{O}_Z) = |W|$, it suffices to show that $RI_W^+ \subset \ker \beta$, and this is surprisingly easy. In fact, if $f \in R_p \cap RI_W^+$, then $\Psi_V(f) \subset F_{p-1}$ due to the fact that $\Psi_V(I_W^+) \subset F_0$. Hence, by commutativity of (2.5), $\beta(f) = 0$, and Borel's theorem is proved.

7. HOLOMORPHIC VECTOR FIELDS WITH ONE ZERO

So far we have considered only vector fields with simple isolated zeros, i.e. vector fields with the maximal number of zeros. At the other extreme are vector fields with exactly one zero. Suppose V has exactly one zero at $p \in X$ and let $V = \sum a_i \partial/\partial z_i$ in holomorphic local coordinates near p. Then $H^0(X,\mathcal{O}_Z) \cong \mathbb{C}[z_1,\ldots,z_n]/(a_1,\ldots,a_n)$ so the cohomology ring $H^{\bullet}(X,\mathbb{C})$ is the graded ring associated to a certain filtration of $\mathbb{C}[z_1,\ldots,z_n]/(a_1,\ldots,a_n)$. Let's consider a basic example.

Example 11. Let V be the holomorphic vector field on $\mathbb{CP}^n$ generated by $\exp(tM)$ where M is the $(n+1)\times(n+1)$ matrix

$$M = \begin{pmatrix} 0 & \\ \vdots & I_n \\ 0\cdots0 & \end{pmatrix}$$

The unique zero of V is $[1,0,\dots,0]$, and in the affine coordinates $\zeta_1,\dots,\zeta_n$ at $[1,0,\dots,0]$,

$$V = (\zeta_2 - \zeta_1^2)\partial/\partial\zeta_1 + (\zeta_3 - \zeta_1\zeta_2)\partial/\partial\zeta_2 + \dots$$

$$+ (\zeta_n - \zeta_1\zeta_{n-1})\partial/\partial\zeta_{n-1} - \zeta_1\zeta_n\partial/\partial\zeta_n$$

hence $H^0(\mathbb{CP}^n, \mathcal{O}_Z) = \mathbb{C}[\zeta_1]/(\zeta_1^{n+1})$. This is already the cohomology ring of $\mathbb{CP}^n$. In, fact using the theory of equivariant Chern classes, it is shown in [CL_3] that ζ_1 corresponds to $c_1(\mathcal{O}(1))$ under the isomorphism of Theorem 6. The existence of the grading on $H^0(\mathbb{CP}^n, \mathcal{O}_Z)$ follows from the fact that the $\mathbb{C}^*$ action $\lambda\cdot[Z_0,Z_1,\dots,Z_n] = [Z_0,\lambda Z_1,\dots,\lambda^n Z_n]$ on $\mathbb{CP}^n$ has the property $d\lambda\cdot V = \lambda^{-1}V$ which implies that the functions that define the ideal $i(V)\Omega^1$ are homogeneous (with respect to the action) and hence that $H^0(\mathbb{CP}^n, \mathcal{O}_Z)$ is graded. In general we know the following

THEOREM 7 [ACLS]. Let X be a projective manifold having a holomorphic vector field with only isolated zeros but having zeros. Suppose there exists a $\mathbb{C}^*$ action $(\lambda,x) \to \lambda\cdot x$ on X so that $d\lambda\cdot V = \lambda^k V$ for some integer $k \neq 0$. Then $H^0(X,\mathcal{O}_Z)$ is a graded ring in which the filtration by degree coincides with the filtration $F_\bullet$ of Theorem 6. Consequently, $H^0(X,\mathcal{O}_Z)$ and $H^\bullet(X,\mathbb{C})$, the cohomology ring of X with complex coefficients, are isomorphic graded rings.

Applications of this theorem to the algebraic homogeneous spaces G/P will appear in a later paper. In the G/P case a regular unipotent one-parameter subgroup of G will generate a holomorphic vector field with exactly one zero and this subgroup will imbed in an $SL(2,\mathbb{C}) \subset G$ by the Jacobson-Morosov Lemma [Ja]. The maximal torus in this $SL(2,\mathbb{C})$ provides the $\mathbb{C}^*$ action of Theorem 7 where $k = 2$. Thus $H^{\bullet}(G/P,\mathbb{C})$ can be viewed as an analytic ring. Its relations will be reflected in the structure of an infinitesimal neighborhood of the zero. It would be interesting to know if the generalized Schubert cycles on G/B, i.e. the closures of the Bruhat cells, admit an intrinsic characterization in the ring $H^0(G/B,\mathcal{O}_Z)$. The Poincare duals of these classes in $H^{\bullet}(G/B,\mathbb{C})$ are calculated explicitly in [BGG]. We will return to this question in §9.

8. A REMARK ON RATIONALITY

The condition of Theorem 7 that X admit a holomorphic vector field V and a $\mathbb{C}^*$ action so that $d\lambda \cdot V = \lambda^k V$ for some integer $k \neq 0$ is equivalent to requiring that $\lambda \cdot \phi(t) \cdot \lambda^{-1} = \phi(\lambda^k t)$ for all $\lambda \in \mathbb{C}^*$, $t \in \mathbb{C}$, where $\phi : \mathbb{C} \to Aut(X)$ is the one parameter subgroup (of the group $Aut(X)$ of automorphisms of X) generated by V. When the identity component $\mathrm{Aut}_0(X)$ is semi-simple and ϕ is a unipotent one parameter subgroup, i.e. a G_a action [H], the Jacobson-Morosov Lemma [Ja] guarantees the existence of an $SL(2,\mathbb{C}) \subset \mathrm{Aut}_0(X)$ in which $\phi(\mathbb{C}) = \left\{\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} : t \in \mathbb{C}\right\}$. If $(\lambda,x) \to \lambda\cdot x$ denotes the $\mathbb{C}^*$ action on X induced by the maximal torus in $SL(2,\mathbb{C})$, then $\lambda\cdot\phi(t)\cdot\lambda^{-1} = \phi(\lambda^2 t)$. Using this fact, it is possible to prove a result of Deligne [D].

THEOREM 8 Suppose X is a smooth projective variety such that $\mathrm{Aut}(X)$ is semi-simple. Suppose that there exists a holomorphic vector field on X generated by a G_a action whose fixed point set is rational (as a projective subvariety of X). Then X is rational.

Outline of proof. By a theorem of Sommese [S_1], if $\mathrm{Aut}_0(X)$ is semi-simple, then any $\mathbb{C}^* \subset \mathrm{Aut}_0(X)$ has fixed points on X. It follows, by Blanchard's theorem [M, p. 25], that X can be imbedded in some $\mathbb{CP}^N$ so that each $g \in SL(2,\mathbb{C}) \subset \mathrm{Aut}(X)$ is induced by a projective transformation. By Theorem 7.1 of [CS_3], V is tangent to the fibres of the plus cells in X, hence the sink X_r of X is contained in zero(V). Therefore, assuming that X is not contained in any hyperplane of $\mathbb{CP}^N$, $X_r = X \cap L = \mathrm{zero}(V) \cap L$ for some linear subspace L of $\mathbb{CP}^N$. It follows that the sink X_r of X is rational, so X is rational, by the corollary to Theorem 1.

The question of whether the existence of a holomorphic vector field on X having isolated zeros implies X is rational has been considered by Lieberman in $[L_1]$,$[L_2]$, and by Deligne [D]. By the induction argument in $[L_2]$ one can reduce this problem to showing the

Conjecture: A smooth projective variety that admits a holomorphic vector field with exactly one zero is rational.

9. CLOSING REMARKS

Borel's Theorem $R/RI_W^+ \cong H^{\bullet}(G/B,\mathbb{C})$ has another interpretation due to Kostant [Kos]. Namely, R/RI_W^+ can be seen to be the ring $\mathbb{C}[h\cap\eta]$ of functions on the (nonreduced) variety $h \cap \eta$, where η is the nilpotent cone in g. A problem of Kostant is to understand in an intrinsic manner how Schubert calculus works in $\mathbb{C}[h\cap\eta]$. The isomorphism of Theorem 7 may shed some light on this problem since we now have available the fact that $H^{\bullet}(G/B,\mathbb{C})$ is isomorphic to $H^0(G/B,\mathcal{O}_Z)$ for the vector field associated to any regular element in η. In the same spirit as Kostant, one may ask

Question. Suppose V is a holomorphic vector field with one zero having an associated $\mathbb{C}^*$ action so that

$d\lambda \cdot V = \lambda^k V (k \neq 0)$. Find intrinsically the elements in $H^0(X, \mathcal{O}_Z)$ associated to the $\overline{X_j^+}$ by Poincaré duality.

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