Chapter V

MODALITIES

Although the notions of modality in logic are as old as
Aristotle, the first attempt to incorporate them in a modern
system of symbolic logic is that of C. I. Lewis.[1] He, like many
others following him, was concerned with an analysis of logical
implication in terms of various combinations of necessity and
negation. From the standpoint of formal deducibility, however,
this is putting the cart before the horse. Indeed the implica-
tion which we defined in the previous chapters was, under cer-
tain assumptions, a strict implication; and we have seen that
if one wishes to adopt a more or less classical standpoint with
respect to negation, the natural system of strict implication
is LD. But the notions of necessity and possibility are inter-
esting on their own account. In this chapter we shall study
definitions of formal concepts analogous to them by the same
methods which have been used in the previous chapters.
    The notion of necessity will be discussed first; then the
notions of possibility and various associated notions. The
treatment is necessarily incomplete; but it is hoped it will be
suggestive to those who make a special study of modal systems.
    We shall use the notations

$$\#A \qquad\qquad \Diamond A$$

to indicate the necessity and the possibility of A respectively.
They may be read as "necessarily A," and "possibly A."

    1. **Analysis of Necessity.** To begin with let it be empha-
sized that we are not attempting to define something in the in-
ner nature of propositions which characterizes some propositions
as necessary and others as contingent. On the contrary we are
defining, in terms of the notion of formal system, an objective
concept, to which we can, with some justice, attach the name
"necessity."
    We can get such a concept by considering, so to speak, a
formal system within a formal system. That is, given a formal
system $\mathfrak{S}$, we may distinguish those theorems which are derivable
from only a part of its rules. The part in question will then
constitute a system $\mathfrak{S}^\#$ within $\mathfrak{S}$. I shall call $\mathfrak{S}^\#$ the <u>inner system</u>
while $\mathfrak{S}$ itself is the <u>outer</u> system. Then we can interpret $\#A$ to
mean that A is derivable in the inner system.

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1. See references in the Introduction.

For example, let $\mathfrak{S}$ be the system considered in IV, §2, with
Rules 1 and 3 and the axioms $0 \leqq 0$, $0 \leqq 1$. Here I have written
the primitive relator as " $\leqq$ " rather than " = " because it now
has the meaning usually associated with " $\leqq$ ". As inner system
we take that based on the first axiom and Rule 1. Then the re-
lation

$$\# \, (r \leqq s)$$

is precisely the relation of equality. (If we were to admit
both rules and only the second axiom in the inner system, this
relation would be the same as $r < s$.)

As noted in the preceding chapter we do not exclude the ap-
plication, in principle, to the case where $\mathfrak{S}$ is a system of or-
dinary discourse, while the inner system is some restricted part
of it. Thus we may take $\mathfrak{S}$ to be a formulation for physics, with
axioms which are established by experiment, while $\mathfrak{S}^{\#}$ is a system
of logic and mathematics. Or we may take the outer system to be
a formulation of biology, the inner system one of physics. Again
we may consider a whole family of outer systems, each constitut-
ing what Carnap [6] would call a "state description," or a fam-
ily derived by permuting the primitive terms as suggested by
McKinsey [60].

In the present theory we shall suppose that we have a single
outer and inner system, and that both are completely formalized.
But the outer system is any system which includes the inner sys-
tem, and may be the inner system itself; so that theorems stat-
ed about the outer system can be specialized to apply to the
inner system.[2]

2. **The L-System for Necessity.** We consider now the formal-
ization of these rules.

We suppose the inner and outer systems both formulated as one
of the systems LA, LC, LM, LJ, LD, LK, with or without quanti-
fiers.[3] The kind of system so involved will be called the un-
derlying system and designated LX. We consider the changes to
be made to adjoin the necessity connective. The new system
formed will be called LXY.

On the level of morphology we should make the following
changes. We extend the definition of $\mathfrak{P}(\mathfrak{S})$ by adding the rule.

If $A \, \varepsilon \, \mathfrak{P}(\mathfrak{S})$, then $\#A \, \varepsilon \, \mathfrak{P}(\mathfrak{S})$,

it being understood that the inner and outer systems both have

2. An example of use of such an inner system in practice is in [19] 2.74.
Here if we call the inner system that formed by the rules 2.71-2.73, the
premise of 2.74 is that $\varphi = \Psi$ be derivable from certain premises relative to
the inner system.

3. In regard to systems with quantifiers the same remarks as in footnote[1]
of Chapter III apply here.

the same $\mathfrak{P}(\mathfrak{S})$. The definition of a prosequence holds without change. We shall use the notation $\#\mathfrak{X}$ to denote a prosequence all of whose constituents have the form $\#A$.[4]

We consider two kinds of <u>elementary statements</u>, viz.,

(1) $\mathfrak{X} \Vdash \mathfrak{Y}$,

(2) $\mathfrak{X} \Vdash \mathfrak{Y}$.

Here (1) refers to the inner system, (2) to the outer. The rules of $LX(\mathfrak{S}\#)$ shall be valid for (1), those of $LX(\mathfrak{S})$ for (2). (These rules differ only in the rules p2 and Er, which are those valid in the inner and outer systems respectively.)

According to the methods of the previous chapters, we should arrive at theoretical rules by asking under what circumstances we can introduce a constituent $\#A$ on the left and right of an elementary statement. As for introduction on the left it is clear that $\#A$ is to be stronger than A; therefore we can allow A to be replaced by $\#A$. On the right, however, we say $\#A$ is a consequence of the necessity of certain premises when A is itself a consequence of the necessity of those premises in the inner system.

The above rule on the right needs some explaining. It might be thought we should say that $\#A$ is a consequence of certain premises when A is a consequence of those premises in the inner system. But since A is always a consequence of itself, we should end up with $\#A$ equivalent to A. Now of course if A is actually true in the inner system, so is $\#A$ (according to our intentions). But we do not want $\#A$ to follow simply from the assumption of A. So we state that if A is valid in the inner system on the basis of certain assumptions $\mathfrak{X}$, then $\#A$ is valid (in either system) on the basis of the assumptions $\#\mathfrak{X}$. (Of course it is valid on the assumptions $\#\mathfrak{X}$ in the inner system by the first rule, if it is valid on the assumptions $\mathfrak{X}$, so that the statement in the preceding paragraph is sufficient.)

One further restriction comes in the extension to cases LC and LK where there may be more than one constituent on the right The analogy which we have followed so far, where we have simply added a $\mathfrak{Z}$ on the right, would lead to paradoxical results if extended to Yr. For we should get

(3) $\#(A \vee B) \mathbin{\supset} . (\#A) \vee B$,

which is not in accord with our intuitive idea of necessity. It is therefore necessary to restrict Yr to the case of a single constituent on the right in all systems.

This leads to the following statement of rules:

RULES FOR NECESSITY:

Yl $\dfrac{\mathfrak{X}, A \Vdash \mathfrak{Y}}{\mathfrak{X}, \#A \Vdash \mathfrak{Y}}$ Yr $\dfrac{\#\mathfrak{X} \Vdash A}{\#\mathfrak{X} \Vdash \#A}$

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4. It is to be obtained from $\mathfrak{X}$ by prefixing $\#$ to all propositions of $\mathfrak{X}$ not already of the form $\#A$.

Here " $\Vdash$ " can refer to either the inner or the outer system; be-
cause the inner system can be regarded as a special case of the
outer system.

These rules, however, do not admit the methods of proof of
II §§ 5-7. So far as Theorem II 2 is concerned, it is clear that
we have to admit as primitive the rules Kr and Kℓ. But even
this does not save the elimination theorem. Indeed the proof of
that theorem in II § 7 depended essentially on the fact that
parametric constituents could be added to premises and conclu-
sion of a rule without invalidating the inference. This is not
true for the rule Yr. Thus the proof of Theorem II 11 does not
carry over to the present case. Whether the elimination theorem
can be established by other methods is not known.[5]

On this account we shall not go further with the system LXY.
However, the following facts concerning it may be noted inciden-
tally. The elimination theorem can be proved if A is non-modal
and LX is such that only prosequences with a single constituent
can appear on the right. It can then be shown that

$$\mathfrak{x} \Vdash \mathfrak{y}$$

is equivalent, for suitable $\mathfrak{n}$, to

(4) $\mathfrak{x}, \mathfrak{n} \Vvdash \mathfrak{y}$

where every constituent of $\mathfrak{n}$ is either an axiom of the outer
system, or is of the form

$$A_1 \supset . A_2 \supset . \ldots \supset . A_m \supset B$$

where

(5) $A_1, A_2, \ldots, A_m \vdash B.$

In case there exists a non-modal proposition M such that 1)

(6) $\Vdash M$

2) if A is an axiom of the outer system, then

(7) $M \Vvdash A$

and 3) whenever (5) holds in the outer system, then

(8) $M, A_1, \ldots, A_m \Vvdash B;$

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5. At least it was not at the time these lectures were delivered. Since
that time I have proved the elimination theorem for LAY, IMY, LJY. It is
probable that the proof can be extended to LDY and IKY, and possibly also to
LCY; but the details are not yet clear. Note that a proof for IKY which re-
tained the decidability theorems would give a decision process for the Lewis
calculus S4; the relation of this to previously known decision procedures is
also not yet clear. The presence of variables seems not to cause any diffi-
culties comparable to those caused by Nx. These matters are left for another
publication.

we say that <u>M expresses the axioms and rules of the outer system in the inner</u>. In that case (4) can be replaced by

$$(9) \qquad\qquad \mathfrak{x}, M \Vdash \mathfrak{Y}.$$

Further, with the addition of the rules Kr and Kℓ there is no trouble about Theorems II 2, II 3, II 4, II 5, II 7, II 8, II 10 in so far as they apply to LX.

3. The T and H Systems for Necessity. Since we do not have the elimination theorem for LXY, we cannot derive the rules of a corresponding T system from those of LXY. Instead we shall postulate them directly. It is clear from the semantical discussion in §§1-2 what postulates we should accept. It can be shown - rather more easily than in the previous chapters - that the rules adopted are equivalent to those of LXY if a rule of elimination is postulated for the latter. However this will not be gone into here.

We suppose that a system TX has been defined (where X is A, C, M, J, D, K or one of these with addition of variables). We extend the definition of $\mathfrak{P}(\mathfrak{S})$ as in §2, and we consider two kinds of elementary statements

$$(10) \qquad\qquad A \; \varepsilon \; \mathfrak{S}\#(\mathfrak{x})$$

$$(11) \qquad\qquad A \; \varepsilon \; \mathfrak{S}(\mathfrak{x})$$

We postulate that (10) shall obey the rules of TX relative to the inner system, (11) the same rules relative to the outer system. (Thus (10) and (11) differ with regard to the rules t2 and E1 only.) We then adjoin the following:

T-RULES FOR NECESSITY.

$$\begin{array}{cccc} & & & (\#) \\ \text{Ye} & \dfrac{\#A}{A} & \qquad \text{Y1} & \dfrac{A}{\#A} \end{array}$$

where the "$(\#)$" over the "A" in Y1 indicates $\#A$ is derived from $\#\mathfrak{x}$ by the rules of the inner system. The rule Y1, written out in full is

$$\frac{A \; \varepsilon \; \mathfrak{S}^{\#}(\#\mathfrak{x})}{\#A \; \varepsilon \; \mathfrak{S}(\#\mathfrak{x})}$$

On the possibility of specializing (11) to (10) we make the same stipulations as in §2.

DEFINITION 1. We define <u>strict implication</u>, $A \rightarrow B$, thus:

$$A \rightarrow B \; . \; \equiv \; . \; \#(A \supset B)$$

We then have the following:

THEOREM 1. <u>The following are in $\mathfrak{S}^{\#}(\mathfrak{S})$ whatever the system $\mathfrak{S}$:</u>

$$(a) \qquad\qquad \# A \supset A,$$

(b) $\# A \supset \# \# A,$

(c) $A \rightarrow B . \supset . \# A \rightarrow \# B.$

Further we have the property

(d) If $A \varepsilon \mathfrak{S}^{\#}(0),$ then $\# A \varepsilon \mathfrak{S}^{\#}(0).$

Proof. Property (a) follows at once by Ye and P1. Property
(d) follows at once by Y1. Properties (b) and (c) follow by the
following schemes:

$$
\begin{array}{c}
\overset{\large\checkmark}{1} \\[2pt]
\hline
\# A \\
\hline
\# \# A \\
\hline
\end{array}
\quad
\begin{array}{l}
\\
\text{Y1} \\
\text{P1 - 1}
\end{array}
$$

(b)

$\# A \supset \# \# A.$

(c)

$$
\begin{array}{cc}
\overset{\large\checkmark}{1} & \overset{\large\checkmark}{2} \\
\dfrac{A \rightarrow B}{A \supset B}\ \text{Ye} & \dfrac{\# A}{A}\ \text{Ye} \\
& \ \text{Pe}
\end{array}
$$

$$
\dfrac{B}{\# B}\ \text{Y1}
$$

$$
\dfrac{}{\# A \supset \# B}\ \text{P1 - 2}
$$

$$
\dfrac{}{\# A \rightarrow \# B}\ \text{Y1}
$$

$$
\text{P1 - 1}
$$

$A \rightarrow B . \supset . \# A \rightarrow \# B$

We define the system HXY as the set of propositions A such
that

$$A \varepsilon \mathfrak{S}^{\#}(0)$$

for $\mathfrak{S}^{\#} = \mathfrak{D}$. This system is an algebra $\mathfrak{H}$ closed under the rule
Ph (since Ph is a special case of Pe) and the rule Yh:

$$\text{Yh} \quad \frac{A \varepsilon \mathfrak{H}}{\# A \varepsilon \mathfrak{H}}.$$

We consider first the case where only Ph is admitted, so as not
to have to revise previous proofs to take care of the new rule.

THEOREM 2. With respect to the rule Ph, a set of prime
propositions for HXY consists of all those of the form $\# A$ where
A is a prime proposition of HX, together with those in the
schemes

(a) $\# A . \supset . A,$

(a') $\# A . \rightarrow . A,$

(b') $\# A . \rightarrow . \# \# A,$

(c') $A \rightarrow B . \rightarrow : \# A . \rightarrow . \# B.$

Proof. Let $\mathfrak{H}(\mathfrak{X})$ be the propositions generated from $\mathfrak{X}$, to-
gether with the prime propositions indicated in the theorem, by

the rule Ph. Then since, by Theorem 1, every such prime propo-
sition is in $\mathfrak{x}^{\#}(0) \subseteq \mathfrak{x}^{\#}(\mathfrak{x})$,[6] and since $\mathfrak{x}^{\#}(\mathfrak{x})$ is closed with
respect to Ph,

$$\mathfrak{H}(\mathfrak{x}) \subseteq \mathfrak{x}^{\#}(\mathfrak{x}).$$

If we show the converse, then the theorem follows by taking $\mathfrak{x}$
void.

    To prove the converse we show that $\mathfrak{H}(\mathfrak{x})$ obeys the rules for
$\mathfrak{x}^{\#}(\mathfrak{x})$. So far as the non-modal rules are concerned this follows
from the relevant cases in the previous chapters. To take care
of the modal rules we observe first that the rule

$$\text{Ph'} \qquad \frac{A, A \rightarrow B}{B}$$

is valid by Ph and (a). Then the modal rules are verified as
follows:

    <u>Ye</u>. This follows at once from (a) and Ph.

    <u>Yi</u>. Let $A \varepsilon \mathfrak{H}(\mathfrak{x})$ and let $B_1, B_2, \ldots, B_n$ be a derivation of A
in $\mathfrak{H}(\mathfrak{x})$ - i.e., B_n is A and every B_k is either a prime proposi-
tion, a member of $\mathfrak{x}$, or is derived from predecessors B_i, B_j by
Ph. If the premises of Rule Yi are fulfilled, every B_k which is
a member of $\mathfrak{x}$ is of the form $\#C_k$. If B_k is prime, then either
B_k is of the form $\#C_k$ or is an instance of (a). We now show by
induction on k that $\#B_k$ is in $\mathfrak{H}(\mathfrak{x})$ for every k. If B_k is of the
form $\#C_k$ this follows by (b') and Ph'. If B_k is an instance of
(a), then $\#B_k$ is an instance of (a'). If B_k follows from B_i and
B_j by Ph, then, without loss of generality, B_j is $B_i \supset B_k$. By
the hypothesis of the induction we have $\#B_i$ and $B_i \rightarrow B_k$. From
the second of these and (c'), Ph' we have $\#B_i \rightarrow \#B_k$. Hence by
Ph' we have $\#B_k$,   q.e.d.

    THEOREM 3.   <u>With respect to the rules Ph and Yh a set of</u>
<u>prime propositions for HXY consists of those of HX together with</u>
<u>the schemes</u>

    (a) $\#A \supset A,$

    (b) $\#A . \supset . \#\#A,$

    (c) $A \rightarrow B . \supset : \#A . \rightarrow . \#B.$

    <u>Proof</u>. Let $\mathfrak{H}_2$, be the system generated in Theorem 2, $\mathfrak{H}_3$ that
generated in this theorem. Then, since the prime propositions of
$\mathfrak{H}_2$ can be derived in $\mathfrak{H}_3$ by Yh, we have $\mathfrak{H}_2 \subseteq \mathfrak{H}_3$. The converse fol-
lows since Yh, which is a special case of Yi, is a derived rule
of $\mathfrak{H}_2$, and the prime propositions of $\mathfrak{H}_3$ are valid in $\mathfrak{H}_2$.
    The schemes (a) (b) (c) of Theorem 3 are due to Gödel [41].
He stated also the following interesting properties, which it is

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6. One uses (d) to derive (a'), (b'), (c').

not difficult to verify: The system HKY is equivalent to the
system S4 of Lewis;[7] further the propositions of HJ are all ver-
ified in HKY if one makes the following translations:[8]

HJ	HKY
$\neg$ A	$\# \neg$ A
A $\supset$ B	$\#$ A $\supset$ $_\#$B
A $\wedge$ B	A $\wedge$ B
A $\vee$ B	$\#$A $\vee$ $\#$B.

   The above considerations completely determine HKY. But HKY
is the algebra we get when $\mathfrak{S}$ is $\mathfrak{D}$. It is possible that in con-
nection with particular systems $\mathfrak{S}$ one would get stronger systems
of strict implication. On this point Cf. McKinsey [60]. Carnap
[10] arrives at the system S5[9].

   4. **Discussion of Possibility.** In conclusion I shall add a
few sketchy remarks about possibility.
   It is customary to think of possibility and necessity as
inter-definable by means of negation. But possibility may also
be regarded as an independent notion to be defined according to
the preceding methods in its own right. Indeed that is the only
course open to us if we have a system without negation. If we
were to do that we would proceed somewhat as follows.
   Consider a family of formal systems.[10] Let $\mathfrak{S}_1$ be a system of
this family. We may regard A as possible in $\mathfrak{S}_1$ if it is true
in some stronger $\mathfrak{S}_n$ of the family. Then we can say A is possi-
ble on the suppositions $\mathfrak{X}$ if A is derivable from $\mathfrak{X}$ in some
stronger $\mathfrak{S}_n$; and that if $\Diamond$ B holds in the suppositions $\mathfrak{X}$, A,
then it holds on the suppositions $\mathfrak{X}$, $\Diamond$A. If we suppose the
rules and axioms of $\mathfrak{S}_n$ can be expressed in $\mathfrak{S}_1$ by M_n, and if we
follow the same analogy we have been following for multiple
right sides, our rules become

$$\frac{\mathfrak{X}, A \Vdash \Diamond B, \mathfrak{Z}}{\mathfrak{X}, \Diamond A \Vdash \Diamond B, \mathfrak{Z}} \qquad\qquad \frac{\mathfrak{X}, M_n \Vdash A, \mathfrak{Z}}{\mathfrak{X} \Vdash \Diamond A, \mathfrak{Z}}$$

With these we can proceed as in the foregoing theory.[11] Evident
ly there will be no trouble with the elimination theorem. There

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   7. [57] p. 499.
   8. For other similar properties see [62].
   9. [A rule-theoretic formulation which leads to S5 has been communicated
to me by R. Feys, (Spring, 1949).]
   10. On ways of thinking of such a family cf. the remarks in §1.
   11. The proofs of the statements in the following remarks have not yet
been carried through. Consequently they are only tentative. In partic-
ular, no consideration has yet been given to the cases where $\mathfrak{Z}$ is non-
void.

would be some revision in the case of the decidability theorems, on account of the elimination of M_n. The rules of the T system would be

$$\frac{\Diamond A \qquad \overset{[A]}{\Diamond B}}{\Diamond B} \qquad\qquad \frac{A}{\Diamond A}$$

and the prime propositional schemes for an H-system:

(a) $A \supset \Diamond A$

(b) $\Diamond \Diamond A \supset \Diamond A$

(c) $\Diamond (A \supset B) . \supset . \Diamond A \supset \Diamond B.$

The following remarks are now relevant:

1) If we have classical negation we arrive again at the Lewis system S4. Further if we have also necessity, (a), (b), (c) are all satisfied if we define

$$\Diamond A \equiv \neg \# \neg A$$

But (b) fails, at least apparently, if we do not have classical negation.

2) If we have an M such that

$$M \mathrel{\Vdash} M_n$$

then all properties hold if we define

$$\Diamond A \equiv M \supset A.$$

The existence of such an M may seem problematical in the general case, since any such M might be absurd. But it can be realized in certain cases - e.g., we have three systems - an <u>inner system</u> 6_0, a <u>real system</u> 6_1, and an <u>outer system</u> 6_2, and M be defined as M_2.[12] The situation is parallel to a definition of $\neg A$ as $A \supset F$.

3) If we define

$$\Diamond A \equiv \neg \neg A$$

then (a), (b), (c) are satisfied even in HM.

4) It is clear that the essential element in the Glivenko theorem is that double negation satisfies (a) and (c). (Property (b) is true but incidental.) Property (c) - which both necessity and possibility have in common, is the really essential one. In fact we can argue generally in TA that if

$$A \ \varepsilon \ \mathfrak{L}(\mathfrak{X})$$

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12. In such a case we have an interesting distinction between $\Diamond A$ and $\# \Diamond A$. Thus take the system of IV § 2. Let the inner system contain the rules 1, 2, and 3 and the axiom 0 = 0. Let the real system be obtained by adjoining the axiom 0 = 3 and let M be 0 = 2. Then 1 = 3 is necessarily possible, 1 = 2 only contingently so.

then

$$\Diamond A \; \varepsilon \; \mathfrak{L}(\Diamond \mathfrak{X})$$

This argument is the gist of the deduction theorem (P1 in Theorem II 19), the Glivenko theorem, the proof of Π1 in Theorem III, and of Y1 in Theorem 2.

5) There is an interesting possibility if we have a system with two kinds of negation. Suppose we had refutability and absurdity at the same time. If we use " — " and "⫟ " respectively as the connectors in these two cases, then if the formulation is acceptable from the standpoint of interpretation, the ◊A defined by

$$\Diamond A = - \hspace{-0.5em}\rightarrow A$$

will mean that A can be adjoined to the system without making it absurd. This is an example of a definition of possibility which does not have the above properties. But it requires that we have a non-classical negation.