## Chapter IV

## NEGATION

The negation connective was postponed in the above discussion because of difficulties in its definition. There is no single definition for the negation of even an elementary proposition; we shall find that we have to introduce different concepts having partially the nature of negation. On the other hand the formal complexities are rather less than those in Chapter III. In fact we shall revert to the simpler formulations of Chapter II, since variables can be adjoined to any of our episystems by the methods of Chapter III.[1]

The plan of this chapter is similar to that of the previous ones. We start by giving an intuitive discussion, then formalize it, and then derive theorems about the resulting episystems. The situation is complicated by the various different ways of defining negation.

1. **Preliminary Analysis.** We consider first various definitions for the negation of an elementary proposition.

Since the criterion for the truth of an elementary proposition is the existence of a proof, the most direct meaning for falsity would be the non-existence of a proof. This notion of negation will be called <u>invalidity</u>. To establish it constructively for a single elementary proposition A, it is necessary to find some definite property which can be proved by recursion to hold for every elementary theorem, and then to show that A does not have the property.[2] In a decidable system every elementary proposition is either true or invalid. A system is consistent, in the ordinary sense, if a single elementary proposition can be shown to be invalid. Thus in many systems, like abstract set theory, we are unable to establish invalidity in a single case.

One fatal objection to taking invalidity as a connective, parallel to the others considered so far, is that the notion is not extensible. I.e., if A is invalid in a system $\mathfrak{S}$, it may become valid when $\mathfrak{S}$ is extended. In this respect it is totally unlike the connectives $P, \Lambda, V, \Pi$, and $\Sigma$. It is therefore necessary to consider other notions of a negative nature.

One such notion is that of implying every proposition. This notion will be called <u>absurdity</u>.[3] An absurd proposition, then,

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1. This has not been carried through in detail for the systems considered below. But it is not anticipated that there will be any difficulty about it.

2. Cf. the proofs of invalidity in II §6.

3. This word is used by the intuitionists for negation. Since this form

is one such that $A \supset B$ is true for every proposition B - it would be sufficient to say for every elementary proposition in $\mathfrak{S}^*$. Since a system in which every elementary proposition is true is trivial, an absurd proposition is intuitively unacceptable. Thus absurdity has the nature of a negation. A system $\mathfrak{S}$ in which every elementary proposition is either true or absurd is complete in the standard sense. If an elementary proposition is both true and absurd, then every proposition is absurd; in that case the system is ordinarily called inconsistent, but is perhaps better called inconsistent in the sense of absurdity, or simply absurd.

Carnap[4] has made an interesting suggestion which leads to another form of negation. We shall adopt his name for this concept: refutability. Suppose the primitive frame for $\mathfrak{S}$ specifies not only rules for true elementary propositions, but also false, i.e., refutable ones. This definition will take the following form: a certain definite class $\mathfrak{F}$ of propositions will be declared directly refutable, just as the axioms are directly true; then every proposition which implies a directly refutable proposition will be said to be refutable. If every elementary proposition is either true or refutable the system is complete in the sense of refutability; if some proposition is both true and refutable it is inconsistent in the sense of refutability, or simply refutable. (In that case it will turn out that every proposition is refutable.)

The notion of invalidity can be resurrected as a criterion for negation if it be regarded, not as a definition of negation, but as a sufficient condition for it. Thus if we have a constructive decision process which sometimes gives an answer, we can say a proposition is false if the process gives a negative answer for validity in the basic system (or some other fixed system). The proposition can then be regarded as false in any extension. Such a definition leads to a form of refutability in which the propositions false by the criterion are directly refutable. We shall not adopt a special name for this kind of negation, since "refutability" covers it. The name "invalidity" we shall reserve for the non-extensible kind of negation considered at the beginning.

These definitions can easily be extended so as to apply to compound propositions. In such a case absurdity can be included in refutability, provided we can formulate a single absurd proposition which we take as directly refutable. (In some cases we may even be able to find an elementary proposition for this purpose; see the example below.)

form of negation leads to an intuitionistic calculus, it is appropriate to use their word for it. Carnap [6] prefers "comprehensiveness," but this does not indicate that the notion is a kind of negation, and entails inconsistency in the usual sense.

4. [6] p. 163.

None of these kinds of negation will have all the properties of classical negation. The latter we shall call classical falsity. It has two peculiarities: first, that every proposition is either true or false, and second, that every false proposition is absurd.[5] Evidently a formal refutability will have these properties only under rather special circumstances. But of course classical falsity is important from the point of view of ordinary discourse. (Cf. §8.)

2. **An Example from Number Theory.** Before going further we shall consider a rudimentary arithmetic system which, although trivial in itself, yet illustrates some of these notions. The primitive frame for this system is as follows:

Primitive terms - a single one : 0.

Primitive operations - a unary one: $(---)'$.

Primitive predicates - a binary one: $(---_1) = (---_2)$.

Axiom: $0 = 0$.

Rule 1. If $r = s$, then $r' = s'$.

In dealing with this system we shall use the ordinary notation of arithmetic, including the symbol for addition, although it is not defined in the system.

The system is decidable. Its elementary theorems consist of the equations $r = s$ where r and s are the same number. Any equation in which r and s are different terms is invalid. No elementary proposition is absurd, but the following compound propositions are:

$$(x)(y) \;.\; x = y.$$

$$(x) : x = 0 \;.\wedge.\; 0 = x.$$

If now we add the specification that $3 = 5$ and $1 = 7$ are directly refutable, then the following are refutable:

$$0 = 2,\ 1 = 3,\ 2 = 4,\ 3 = 5,\ 0 = 6,\ 1 = 7.$$

All other equations $r = s$ with unequal r and s are invalid, but neither refutable nor absurd.

Suppose now we adjoin the following rules one after the other:

Rule 2. If $r = s$, then $s = r$.

Rule 3. If $r = s$ and $s = t$, then $r = t$.

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5. For this reason classical falsehood will sometimes be referred to as complete absurdity.

Rule 4. If $r' = s'$, then $r = s$.

Rule 5. If $0 = r'$, then $0 = r$.

Then after each rule is adjoined the following become refutable or absurd:

After Rule 2: $2 = 0$, $3 = 1$, $4 = 2$, $5 = 3$, $6 = 0$, $7 = 1$ are refutable.

After Rule 3: $0 = 1$, $1 = 2$, $2 = 3$, $3 = 4$, and converses, also $0 = 3$, $1 = 4$, and converses, are refutable; $0 = 1$, $1 = 0$ are absurd.

After Rule 4: $r = r'$, $r' = r$ are absurd (all $r$); $r = r''$, $r'' = r$, $r = r'''$, $r''' = r$, $r = r + 6$, $r + 6 = r$ are refutable (all $r$).

After Rule 5 all invalid propositions become absurd. The system is complete in the sense of absurdity, and decidable. Refutability has now the properties of classical falsity (at least so far as elementary propositions are concerned).

3. **Formulation of the Rules.** According to the preliminary discussion there are three main kinds of negation for which we formulate systems as follows:[6]

LM Minimal system - for refutability in general.

LJ Intuitionist system - for absurdity in general.

LK Classical system - for classical falsity or complete absurdity.

In addition there is a fourth system, which we shall call the system LD, which bears somewhat the same relation to LK that LM does to LJ; that is the specialization of refutability for complete systems. This system - although its algebra was mentioned by Johansson - has been very little studied. When quantifiers are adjoined, by the method of Chapter III, the resulting systems will be called LM*, LJ*, LK*, and LD* respectively.

We shall begin by formulating the notation and morphology, which is common to all the systems, then proceed to the theoretical rules. It will be necessary to precede the latter with some further discussion, because the argument in §1 did not go far enough to determine the rules completely. The formal theoretical rules will be stated at the end.

NOTATION. The negation of A will be denoted[7]

$$\neg A.$$

6. The minimal calculus was first considered by Johansson in [52]; for the Heyting system see the references in the introduction. The abbreviations "LJ" and "LK" were used by Gentzen [35]; "LM" by Johansson.

7. The symbol "$\neg$" was introduced by Heyting [44]. It was also used by Gentzen.

The letter "$\mathfrak{F}$" will denote the class of directly refutable propositions. This is a definite class supposed to have been specified in advance. The symbols "$F_1$," "$F_2$," "$F_i$," "$F_j$,"... will henceforth designate elements of $\mathfrak{F}$. The letter "F", however, will have a special meaning in § 5.

MORPHOLOGICAL RULES. If quantifiers are not involved we need simply the rule

(a) If $A \varepsilon \mathfrak{P}$, then $\neg A \varepsilon \mathfrak{P}$,

while if quantifiers are present we add to the rules of Chapter III § 3 the following:

(a*) If $A \varepsilon \mathfrak{P}(u)$, then $\neg A \varepsilon \mathfrak{P}(u)$.

(b) The variables which occur in $\neg A$ are the same as those which occur in A.

(c) The variables bound in $\neg A$ are the same as those bound in A.

(d) $(Sb \,{}^{s}_{u}) \neg A \equiv \neg (Sb \,{}^{s}_{u}) A$.

We turn now to the theoretical rules. We shall consider the rules for LM first.

The following rules agree with the preliminary discussion of § 1.

(1) $$\frac{\mathfrak{X} \Vdash A}{\mathfrak{X}, \neg A \Vdash F_i} \qquad\qquad \frac{\mathfrak{X}, A \Vdash F_i}{\mathfrak{X} \Vdash \neg A}$$

The right-hand rule clearly agrees with our intentions when $F_i$ is any element of $\mathfrak{F}$. The left-hand rule, taken intuitively, states that a system which contains both A and $\neg$ A entails $F_i$; this will agree with our intentions if $F_i$ is some element of $\mathfrak{F}$, but not necessarily any element.

Now suppose we express the fact that $\mathfrak{X}$ entails some proposition of $\mathfrak{F}$ by an elementary statement of the form

(2) $$\mathfrak{X} \Vdash$$

where the right prosequence is void.[8] Then the rules (1) take the form

(3) $$\frac{\mathfrak{X} \Vdash A}{\mathfrak{X}, \neg A \Vdash} \qquad\qquad \frac{\mathfrak{X}, A \Vdash}{\mathfrak{X} \Vdash \neg A}$$

We need also, in order to express the full force of the right half of (1),

(4) $$\frac{\mathfrak{X} \Vdash F_i}{\mathfrak{X} \Vdash}$$

8. Note that none of the previous rules introduces an elementary statement of this form.

Let us call a prosequence satisfying (2) a refutable prosequence. Then, if refutability is specialized to absurdity, a refutable prosequence should entail every proposition. This leads to the rule

(5) $$\frac{\mathfrak{X} \Vdash}{\mathfrak{X} \Vdash A}$$

This rule is a special case of Kr; it will be called hencefort. Kj. It is adopted as a primitive rule for LJ.

In accordance with the formal analogy we have been following in the previous chapters we form rules for LK by adjoining an arbitrary prosequence $\mathfrak{Z}$ to premise and conclusion. The meaning of the system LK is to be found later.

Our fourth system, LD, is that obtained by adjoining to LM a law of excluded middle. This can be expressed by the rule Nx below. That is a rule of quite different character from the others, in that it allows the elimination of a constituent $\neg A$ which is of higher order than the A which is left.[9] However, I have not been able to find a better formulation for LD. It will be found that many of our theorems still hold for LD.

After these preliminaries, the statement of the rules is as follows:

## RULES FOR NEGATION

The prosequence $\mathfrak{Z}$ is void in LM, LJ, LD; arbitrary in LK. In case variables are present, the same appropriate range is added to all premises and conclusions.

N Negation

Nℓ $$\frac{\mathfrak{X} \Vdash A, \mathfrak{Z}}{\mathfrak{X}, \neg A \Vdash \mathfrak{Z}}$$ Nr $$\frac{\mathfrak{X}, A \Vdash \mathfrak{Z}}{\mathfrak{X} \Vdash \neg A, \mathfrak{Z}}$$

Nx $$\frac{\mathfrak{X}, \neg A \Vdash A}{\mathfrak{X} \Vdash A}$$ for LD only.

K Weakening

Kj $$\frac{\mathfrak{X} \Vdash}{\mathfrak{X} \Vdash A}$$ for LJ only.

F Direct refutability

Fr $$\frac{\mathfrak{X} \Vdash F_1, \mathfrak{Z}}{\mathfrak{X} \Vdash \mathfrak{Z}}$$

In the rule Nx the indicated A on the right of the conclusion is the principal constituent, while the indicated A and $\neg A$ in the premise are component constituents.

9. In other words one of the components is of higher order than the principal constituent.

**4. Fundamental Theorems.** We now inquire what changes are necessary in order to extend the theorems of II §§ 5-7 to the case where negation is present.

Theorems II 2, II 3, II 4, and II 5 carry over without essential change. Likewise the first two stages of the proof of the elimination theorem go through without change. Except for LD, where the rule Nx requires special treatment, the proof of the elimination theorem reduces to the consideration of the following additional case under Stage 3:

<u>Case N</u>. $A \equiv \neg B$. Then the hypotheses of the theorem are

$$\mathfrak{X}, \neg B \Vdash \mathfrak{Z}_1 \qquad\qquad \mathfrak{X} \Vdash \neg B, \mathfrak{Z}_2.$$

These hypotheses arise by N$\ell$ and Nr respectively from the premises

$$\mathfrak{X} \Vdash B, \mathfrak{Z}_1 \qquad\qquad \mathfrak{X}, B \Vdash \mathfrak{Z}_2.$$

From these, by the hypothesis of the induction, we have

$$\mathfrak{X} \Vdash \mathfrak{Z}_1, \mathfrak{Z}_2,$$

which is the conclusion of the theorem.[10]

To take care of Nx in LD it seems best to modify the proof of Stage 2. Let the hypothesis of the stage be that the elimina tion theorem is valid in the case where the second hypothesis of the theorem comes directly from a rule Or, but not from Nx. Let $\Gamma_k$ be

$$\mathfrak{X}_k, \mathfrak{U}_k \Vdash \mathfrak{V}_k, \mathfrak{W}_k,$$

where $\mathfrak{U}_k$, $\mathfrak{V}_k$, and $\mathfrak{W}_k$ are defined as follows:

a) $\mathfrak{U}_n$ and $\mathfrak{W}_n$ are void, $\mathfrak{V}_n$ is the single instance of A.

b) Let $\Gamma_k$ be used in $\Delta$ as premise for deriving $\Gamma_m$ by a rule $R_m$, and let $\mathfrak{U}_m, \mathfrak{V}_m, \mathfrak{W}_m$ be already defined. Then $\mathfrak{U}_k, \mathfrak{V}_k, \mathfrak{W}_k$ shall contain those and only those constituents assigned to them by the following specifications:

b1) Every parametric constituent of $\mathfrak{U}_m$ or $\mathfrak{V}_m$ is in $\mathfrak{U}_k$ or $\mathfrak{V}_k$ respectively;

b2) If $R_m$ is W$\ell$ and the principal constituent is in $\mathfrak{U}_m$, then the two like components are in $\mathfrak{U}_k$;

b3) If $R_m$ is N$\ell$ and the principal constituent is in $\mathfrak{U}_m$, then the component is in $\mathfrak{V}_k$. (Note $\mathfrak{V}_m$ and $\mathfrak{W}_m$ are void.)

b4) If $R_m$ is Nx for which $\mathfrak{V}_m$ is principal constituent, then $\mathfrak{V}_m \equiv \mathfrak{V}_k$ and the component on the left (which is $\neg A$) is in $\mathfrak{U}_k$.

This defines $\mathfrak{U}_k$, $\mathfrak{V}_k$ for all k. Then $\mathfrak{X}_k$ and $\mathfrak{W}_k$ are the rest necessary to make up $\Gamma_k$. It will be seen that $\mathfrak{V}_k$ is either void

10. Cf. Remark after statement of Theorem II 11.

or consists of a single A, the constituents of $\mathfrak{U}_k$ are all like $\neg$ A, and $\mathfrak{M}_k$ is void unless $\mathfrak{B}_k$ is void. As before we can suppose without loss of generality that

$$\mathfrak{X} \subseteq \mathfrak{X}_k.$$

Define $\Gamma_k'$ as the statement obtained from $\Gamma_k$ by replacing every constituent of $\mathfrak{B}_k$ by Y and every constituent of $\mathfrak{U}_k$ by $\neg$ Y Then the inductive proof that every $\Gamma_k'$ is derivable proceeds as follows:

($\alpha$) Suppose $\Gamma_k$ is prime, then there are the following subcases:

($\alpha_1$) Some $\mathfrak{B}_k$ is an axiom. This is impossible since A is compound.

($\alpha_2$) Some $\mathfrak{M}_k$ is an axiom. Then $\Gamma_k'$ is also prime.

($\alpha_3$) Some $\mathfrak{M}_k$ is in $\mathfrak{X}_k$. Then $\Gamma_k'$ is also prime.

($\alpha_4$) Some $\mathfrak{B}_k$ is in $\mathfrak{X}_k$. Then by weakening of the first hypothesis

$$\mathfrak{X}_k, A \Vdash Y.$$

$$(W\ell) \quad \mathfrak{X}_k \Vdash Y.$$

whence $\Gamma_k'$ follows by weakening.

($\alpha_5$) Some $\mathfrak{B}_k$ is in $\mathfrak{U}_k$. This is impossible since the constituents of $\mathfrak{U}_k$ and $\mathfrak{B}_k$ are not alike.

($\alpha_6$) Some $\mathfrak{M}_k$ is in $\mathfrak{U}_k$. In this case $\Gamma_k'$ is obtained by weakening from

$$\mathfrak{X}_k, \neg Y \Vdash \neg A.$$

The last is obtained thus:

$$\dfrac{\dfrac{\mathfrak{X}_k, A \Vdash Y}{\mathfrak{X}_k, A, \neg Y \Vdash}\, N\ell}{\mathfrak{X}_k, \neg Y \Vdash \neg A.}\, Nr$$

where the first statement is obtained as in ($\alpha_4$).

($\beta$) Let $\Gamma_k$ be derived from $\Gamma_i'$, (and $\Gamma_j'$) by $R_k$. Then there are the following subcases:

($\beta_1$) If the constituents of $\mathfrak{U}_k$, $\mathfrak{B}_k$ are all parametric, then $\Gamma_k$ can be obtained from $\Gamma_i'(\Gamma_j')$ by $R_k$.

($\beta_2$) If $\Gamma_k$ is obtained from $\Gamma_i$ by $W\ell$, $N\ell$, or Nx, then $\Gamma_k'$ can be obtained from $\Gamma_i'$ by the same rule.

($\beta_3$) If $\mathfrak{B}_k$ is principal constituent of $R_k$ and $R_k$ is Or, then we can apply the hypothesis of the stage in connection with a weakened first hypothesis as before.

This completes the proof of the elimination theorem.

We note in passing that we can, without loss of generality, restrict Nx to the case where A is not of the form $\neg$ B. This does not disturb the elimination theorem, and from that theorem we can reestablish the general Nx by the following argument.[11]

$$\dfrac{\dfrac{\dfrac{\mathfrak{X},B \Vdash B}{\mathfrak{X},B,\neg B \Vdash}\,N\ell \qquad \dfrac{\dfrac{\mathfrak{X},\neg\neg B \Vdash \neg B}{\mathfrak{X},B,\neg\neg B \Vdash \neg B}\,K\ell \qquad \dfrac{\dfrac{\mathfrak{X},B \Vdash B}{\mathfrak{X},B,\neg B \Vdash}\,N\ell}{\mathfrak{X},B \Vdash \neg\neg B}\,Nr}{\mathfrak{X},B \Vdash \neg B}\,\text{Elim. th.}}{\dfrac{\mathfrak{X},B \Vdash}{\mathfrak{X} \Vdash \neg B}\,Nr}\,\text{Elim. th.}$$

We now consider the theorems of II §6.

Theorem II 7 runs into the difficulty that Nx and Fr allow constituents to be eliminated. But we have seen that we can, without loss of generality, restrict Nx so that the number of possibilities is still finite. As for Fr we shall take it as part of the definition of $\mathfrak{D}$ that $\mathfrak{F}(\mathfrak{D})$ is void. Then Theorem II 7 is valid in all the systems without quantifiers.

In Theorem II 8 we have to note that II (10) may now come from Nx, Kj, or Fr. The rule Nx can arise only in LD, in which case the proof of Theorem II 8 fails.[12] The case Fr cannot give rise to a conclusion of form II(10) in LM, LJ, or LD (because the $\mathfrak{Z}$ is void there), while Kj can occur only in LJ. Thus in LM the proof in II is valid. As for Kj in LJ the premise would have to have both prosequences void; such a statement can come only from Fr in case some $F_i$ is derivable; in that case the system is absurd and Theorem II 8 fails (unless the system was already absurd without using negation). If the system is not absurd, Theorem II 8 is true. In LK the system is again absurd if some $F_i$ is derivable. If not I shall consider only the case where all $F_i$ are elementary. Then the premise of Fr is of the same form as II (10) and the same induction as before shows that Theorem II 8 is valid.

Theorem II 9 is valid in LM. In LJ II 9 is obvious if the system is absurd; otherwise the proof in II is valid. In LD and LK, of course, II 9 is not true (Theorem 2 (p) below).

Theorem II 10 is disturbed by the fact that Fr and Nx can eliminate constituents. On account of Nx, we cannot assert Theorem II 10 for LD.[13] In the other cases Theorem II 10 is valid if the $F_i$ are all elementary, or, at most, contain implication only.

Summing up this discussion, we have:

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11. This shows that the case of Nx where A is $\neg$ B is valid in LM.

12. But for LD analogues of Theorems 14 and 15 (below) can be proved for arbitrary $\mathfrak{S}$; from these Theorem II 8 follows.

13. See, however, Theorems 14 and 15 below.

THEOREM 1. *When negation is introduced the theorems II 2, II 3, II 4, II 5, II 7, and the elimination theorem are valid in all four systems; Theorem II 8 is valid in LM, in LJ if not absurd, and in LK if not absurd and all $F_i$ are elementary[14]; Theorem II 9 is valid in LM and LJ; Theorem II 10 is valid in LM, LJ, LK if all $F_i$ are elementary, or contain implication only.*

Finally, in analogy with Theorem II 6 we have:

THEOREM 2. *The following are valid in LM:*

(a) If $\mathfrak{X} \Vdash$, then $\mathfrak{X} \Vdash \neg B$,

(b) $A, \neg A \Vdash$,

(c) $A \supset \neg A \Vdash \neg A$,

(d) $A \Vdash \neg\neg A$,

(e) $A \Vdash \neg A \supset \neg B$,

(f) $A \supset B \Vdash \neg B \supset \neg A$,

(g) $A \supset \neg B \Vdash B \supset \neg A$,

(h) $A \supset B \Vdash A \supset \neg B . \supset \neg A$,

(i) $A \supset \neg A \Vdash B \supset \neg A$.

*The following are valid in LJ:*

(k) If $\mathfrak{X} \Vdash$, then $\mathfrak{X} \Vdash B$,

(l) $A, \neg A \Vdash B$,

(m) $\neg A \Vdash A \supset B$.

*The following are valid in LD:*

(n) $A \supset B, \neg A \supset B \Vdash B$,

(o) $\neg A \supset A \Vdash A$,

(p) $\Vdash A \vee \neg A$.

*The following is valid in LK:*

(q) $\neg\neg A \Vdash A$.

5. **The Natural Systems T.** Let F be a primitive proposition, which we think of as being a minimum refutable proposition - the logical sum of $\mathfrak{F}$, if we like. Then we adopt as rules for TM:

$$\underline{\text{Ne}}: \quad \frac{A \quad \neg A}{F} \qquad\qquad \underline{\text{Ni}}: \quad \frac{\begin{matrix}[A]\\ F\end{matrix}}{\neg A}$$

14. As for LD cf. footnote 12.

These rules were stated by Gentzen. Note that they amount to defining negation by

$$(8) \qquad \neg A \equiv A \supset F.$$

(Ne gives the inference from left to right, Ni that from right to left.) As rules for TJ we take in addition Gentzen's third rule, viz.,

$$\underline{\text{Nj}}: \frac{F}{A}$$

while for TD we take in addition to Ne, Ni

$$\underline{\text{Nd}}: \quad \frac{\begin{matrix}[\neg A] \\ A\end{matrix}}{A}$$

We also shall consider

$$\underline{\text{Nk}}: \quad \frac{\neg\neg A}{A}$$

In all systems we have the rules

$$\underline{\text{Fi}}: \quad \frac{\text{Fi}}{F}$$

The proposition F introduced by these rules, cannot be used just like any proposition. We have to adopt the following:

RESTRICTION AS TO F. *F shall not be used as supposition, nor as component of a compound proposition*. This implies in particular that it shall not be used as premise of a rule Oi.

THEOREM 3. *If, for some fixed $\mathfrak{Z}$,*[15] *we interpret*

$$(9) \qquad A \;\varepsilon\; \mathfrak{X}(\mathfrak{x}) \qquad\qquad F \;\varepsilon\; \mathfrak{X}(\mathfrak{x})$$

*respectively as*

$$(10) \qquad \mathfrak{x} \Vdash A, \mathfrak{Z} \qquad\qquad \mathfrak{x} \Vdash \mathfrak{Z};$$

*then the rules of TM, TJ, TD, and TK are verified in the corresponding system L*.

*Proof*. If F is treated like any other proposition then the rules not involving negation are valid by Theorems II 13, II 15, and III 10. We have only to consider the complications due to the possible presence of the second form of (9). But this second form cannot be introduced by t2; and it cannot be introduced by t1 or Pk, by the restrictions on F. If it occurs in premise or conclusion of an inferential rule of T, then that rule must be t3, Ve or Σe; and by direct inspection of the proofs of these cases it will be seen that the corresponding L-deduction is valid.

15. Note that this $\mathfrak{Z}$ is necessarily void in all the cases except K.

The rules involving negation may be verified as follows:

<u>Ne</u> By hypothesis of Ne

$$A \,\varepsilon\, \mathfrak{S}(\mathfrak{X}), \qquad \neg A \,\varepsilon\, \mathfrak{S}(\mathfrak{X}).$$

$$\therefore \qquad \mathfrak{X} \Vdash A, \mathfrak{Z}, \qquad \mathfrak{X} \Vdash \neg A, \mathfrak{Z}.$$

$$\therefore \quad (\mathrm{N}\ell) \quad \mathfrak{X}, \neg A \Vdash \mathfrak{Z}, \qquad \mathfrak{X} \Vdash \neg A, \mathfrak{Z}.$$

Hence by the elimination theorem

$$\mathfrak{X} \Vdash \mathfrak{Z}. \qquad \text{q.e.d.}$$

<u>Ni</u> By hypothesis of Ni

$$F \,\varepsilon\, \mathfrak{S}(\mathfrak{X}, A)$$

$$\therefore \qquad \mathfrak{X}, A \Vdash \mathfrak{Z}$$

$$\therefore \quad (\mathrm{Nr}) \quad \mathfrak{X} \Vdash \neg A, \mathfrak{Z} \qquad \text{q.e.d.}$$

<u>Nj</u> This follows at once by Kj.

<u>Nd</u> Follows at once by Nx.

<u>Nk</u> We show

$$\neg \neg A \Vdash A,$$

whence the conclusion follows by the elimination theorem.
The proof is simply

$$\cfrac{\cfrac{A \Vdash A}{\Vdash A, \neg A}\,\mathrm{Nr}}{\neg \neg A \Vdash A}\,\mathrm{N}\ell$$

<u>Fi</u> Follows at once by Fr.

<u>Remark</u>. The theorem would not be true without the restrictions on F. For the inference

$$\frac{A \,\varepsilon\, \mathfrak{S}(\mathfrak{X}) \qquad F \,\varepsilon\, \mathfrak{S}(\mathfrak{X})}{A \wedge F \,\varepsilon\, \mathfrak{S}(\mathfrak{X})}$$

is valid; but the corresponding L-inference

$$\frac{\mathfrak{X} \Vdash A \qquad \mathfrak{X} \Vdash}{\mathfrak{X} \Vdash A \wedge F}$$

is not.

THEOREM 4. <u>If we interpret</u>

(11) $\mathfrak{X} \Vdash A \qquad \mathfrak{X} \Vdash$

*respectively as*

$$A \in \mathfrak{X}(\mathfrak{x}) \qquad\qquad F \in \mathfrak{X}(\mathfrak{x});$$

*then the rules of LM, LJ, LD are valid respectively in TM, TJ, TD.*

*Proof*. The positive rules were taken care of in II, III except for the complications due to the void right side. The latter cannot occur in connection with any of the rules on the right, since each of these rules requires the presence of a right constituent in all premises and in the conclusion. As for the rules on the left, all except P$\ell$ have the same right side in all premises and conclusion, while in P$\ell$ the first premise cannot have a void right side and the second has the same right side as the conclusion. Further a prime statement cannot have a void right side. Hence all inferences are valid if the void right side is replaced by F. Then the void right side is no longer exceptional.

This takes care of the positive rules. The negative rules are then verified as follows:

*N$\ell$* by Ne, thus

$$\dfrac{\overset{(\text{Hp.})}{A} \qquad \overset{1}{\neg A}}{F}\ \text{Ne}$$

*Nr* by Ni.

*Nx* follows directly from Nd.

*Kj* follows by Nj.

*Fr* follows by Fi.

THEOREM 5. *The rule Nk is equivalent to the rules Nj and Nd together, and implies the rule Pk of Chapter II.*

*Proof*. Derivation of Nk:

$$\cfrac{\cfrac{\cfrac{\overset{\checkmark}{\overset{2}{\neg A}} \qquad \overset{1}{\neg \neg A}}{F}\ \text{Ne}}{A}\ \text{Nj}}{A}\ \text{Nd - 2}$$

Derivation of Nj:

$$\cfrac{\cfrac{F}{\neg\neg A}\ \text{Ni}}{A}\ \text{Nk}$$

Derivation of Nd. By Theorem II 12 the rule is equivalent to

$$\dfrac{\neg A \supset A}{A}$$

which is derived thus:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{\neg A \supset A} \qquad \overset{\checkmark}{\overset{2}{\neg A}}}{A}\,\text{Pe}}{F}\,\text{Ne}}{\neg\neg A}\,\text{Ni - 2}}{A}\,\text{Nk}$$

Derivation of Pk:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\checkmark}{\overset{3}{A}} \qquad \overset{\checkmark}{\overset{2}{\neg A}}}{F}\,\text{Ne}}{B}\,\text{Nj}}{A \supset B}\,\text{Pi - 3} \qquad \overset{1}{A \supset B .\supset A}}{A}\,\text{Pe}}{A}\,\text{Nd - 2}$$

We may note incidentally that Nd follows from Pk on substitution of F for B.

DEFINITION 1. The prosequence $\neg\mathfrak{Y}$ shall be the prosequence consisting of the negatives of all the members of $\mathfrak{Y}$.

THEOREM 6. <u>If we interpret</u>

(12) $$\mathfrak{X} \Vdash \mathfrak{Y}$$

<u>as meaning</u>

(13) $$F \varepsilon \mathfrak{S}(\mathfrak{X}, \neg\mathfrak{Y});$$

<u>then all the rules of the LK system are verified in TK</u>.

<u>Proof</u>. We show first that if $\mathfrak{Y}$ is non-void, say

$$\mathfrak{Y} = A, Z_1, Z_2, \ldots, Z_n,$$

then (13) is equivalent to

(14) $$A \varepsilon \mathfrak{S}(\mathfrak{X}, Z_1 \supset A, Z_2 \supset A, \ldots, Z_n \supset A).$$

In fact, by Theorem II 12, (13) and (14) are equivalent respectively to the rules

$$\text{M1}\ \dfrac{\mathfrak{X}, \neg A, \neg Z_1, \neg Z_2, \ldots, \neg Z_n}{F}$$

$$\text{M2}\ \dfrac{\mathfrak{X}, Z_1 \supset A, Z_2 \supset A, \ldots, Z_n \supset A}{A}.$$

We deduce M2 from M1 by the scheme

$$\dfrac{\dfrac{(\mathfrak{X}) \qquad \dfrac{\dfrac{\dfrac{\overset{1}{Z_1 \supset A} \qquad \overset{\overset{\checkmark}{3}}{Z_1}}{A}\,\mathrm{Pe} \qquad \overset{\overset{\checkmark}{2}}{\neg A}}{F}\,\mathrm{Ne}}{\neg Z_1}\,\mathrm{Ni}\text{-}3, \qquad \ldots, \neg Z_n}{\dfrac{F}{\dfrac{\neg\neg A}{A}\,\mathrm{Nk}}\,\mathrm{Ni}\text{-}2}\,\mathrm{M1}$$

(Similar proof for $i = 2,3,\ldots,n$).

The derivation of M1 from M2 is

$$\dfrac{\dfrac{(\mathfrak{X}) \qquad \dfrac{\dfrac{\dfrac{\overset{1}{\neg Z_1} \qquad \overset{\overset{\checkmark}{3}}{Z_1}}{F}\,\mathrm{Ne}}{A}\,\mathrm{Nj}}{Z_1 \supset A}\,\mathrm{Pi}\text{-}3, \ \ldots, \quad Z_n \supset A}{A}\,\mathrm{M2} \qquad \neg A}{F}\,\mathrm{Ne}$$

(Similar proof for $i = 2,3,\ldots,n$).

This established, it follows by Theorem II 17 that the positive rules are verified except for the complication that a $\mathfrak{Y}$ may be void. But if a void $\mathfrak{Y}$ is involved in any way, the situation is the same as in Theorem 4.

It remains simply to verify the rules for negation. The translations into form (13) of these rules are:

$N\ell$ $$\frac{F \,\varepsilon\, \mathfrak{S}(\mathfrak{X}, \neg A, \neg Z_1, \ldots, \neg Z_n)}{F \,\varepsilon\, \mathfrak{S}(\mathfrak{X}, \neg A, \neg Z_1, \ldots, \neg Z_n)},$$

Nr $$\frac{F \,\varepsilon\, \mathfrak{S}(\mathfrak{X}, A, \neg Z_1, \ldots, \neg Z_n)}{F \,\varepsilon\, \mathfrak{S}(\mathfrak{X}, \neg\neg A, \neg Z_1, \ldots, \neg Z_n)},$$

Fr $$\frac{F \,\varepsilon\, \mathfrak{S}(\mathfrak{X}, \neg F_1, \neg Z_1, \ldots, \neg Z_n)}{F \,\varepsilon\, \mathfrak{S}(\mathfrak{X}, \neg Z_1, \ldots, \neg Z_n)},$$

Kj $$\frac{F \,\varepsilon\, \mathfrak{S}(\mathfrak{X}, \neg Z_1, \ldots, \neg Z_n)}{F \,\varepsilon\, \mathfrak{S}(\mathfrak{X}, \neg A, \neg Z_1, \ldots, \neg Z_n)}.$$

Here $N\ell$ is obvious while Kj follows by t3. The schemes for deriving the others follow:

Nr. $$\dfrac{\dfrac{\dfrac{\overset{1\ \ \overset{\checkmark}{4}\qquad\qquad 2}{\mathfrak{X}, A, \neg Z_1, \ldots, \neg Z_n}}{F}\,\mathrm{Hp.}}{\neg A}\,\mathrm{Ni}\text{-}4 \qquad \overset{3}{\neg\neg A}}{F}$$

$$
\mathrm{Fr}\quad
\dfrac{
\dfrac{\dfrac{\overset{\checkmark}{\overset{2}{\mathrm{Fi}}}}{\mathrm{F}}\,\mathrm{Fi}}{\neg\,\mathrm{Fi}}\,\mathrm{Ni}-2
\qquad
\overset{1}{\neg Z_1,\ldots,\neg Z_n,\mathfrak{X}}
}{\mathrm{F}}
$$

THEOREM 7. *A necessary and sufficient condition that*

$$\mathfrak{X} \Vdash A \qquad\qquad \mathfrak{X} \Vdash$$

*hold in any one of the systems LM, LJ, LD, LK, is that in the corresponding T system we have respectively*

$$A \in \mathfrak{T}(\mathfrak{X}) \qquad\qquad F \in \mathfrak{T}(\mathfrak{X}).$$

*Proof*. The sufficiency follows by Theorem 3. The necessity for the cases LM, LJ, LD follows by Theorem 4; and for LK by Theorem 6 completed by the following argument in the left-hand case:

$$F \in \mathfrak{T}(\mathfrak{X}, \neg A).$$

$\therefore \qquad \neg\neg A \in \mathfrak{T}(\mathfrak{X}) \qquad$ by Ni.

$\therefore \qquad A \in \mathfrak{T}(\mathfrak{X}) \qquad$ by Nk.

The following theorem shows that we can dispense with F as primitive notion.

THEOREM 8. *If we define F by*

$$F \equiv \neg G,$$

*where G is any theorem, then Ne, Ni and Nj follow by the rules of TM from the schemes*

(a) $\dfrac{A, \neg A}{\neg B}$ (b) $\dfrac{\dfrac{[A]}{\neg A}}{\neg A}$

*and* (c) $\dfrac{A \quad \neg A}{B}$

*respectively*.

*Proof*. Ne follows from (a) by taking B to be G. Then we can derive Ni thus:

$$
\dfrac{\dfrac{\dfrac{\overset{1}{A \supset \neg G} \qquad \overset{\checkmark}{\overset{2}{A}}}{\neg G}\,\mathrm{Pe} \qquad G}{\neg A}\,(a)}{\neg A}\,(b)\text{-}2
$$

Hence $A \supset F . \supset . \neg A$ which gives Ni by Pe.

Next taking A to be G in (c) we have Nj.

6. **The Propositional Algebras.** As in Chapter II, § 9 we define the algebras HM, HJ, HD, and HK as the algebras in which the propositions generated are the A's for which $\Vdash A$ holds in LM($\mathfrak{D}$), LJ($\mathfrak{D}$), LD($\mathfrak{D}$), and LK($\mathfrak{D}$) respectively.

In the first four theorems I shall state a set of prime propositions for these algebras. In each of these cases it is clear from what has preceded that the propositions belonging to the schemes stated are valid in the algebra, so that the set $\mathfrak{H}$ of propositions generated from the prime propositions stated is included in the algebra. It is only necessary to show that the $\mathfrak{H}(\mathfrak{x})$ generated by adding $\mathfrak{x}$ to the prime propositions obeys the rules for $\mathfrak{X}(\mathfrak{x})$. In most cases this is immediate.

THEOREM 9. <u>A set of prime propositions for the algebra HM consists of those of HA together with</u>

(a) $$A \supset . \neg A \supset \neg B,$$

(b) $$A \supset \neg A . \supset . \neg A.$$

<u>Proof</u>. These rules validate the rules (a) and (b) in Theorem 8.

<u>Remark</u>. Various simpler systems for this algebra are known. So far as I know the first to consider the algebra as a separate entity was Johansson [52], who used the Heyting axiom ((h) of Theorem 2) as single prime formula-scheme. The scheme (g) of Theorem 2 is also sufficient as single axiom scheme.[16] Hilbert-Bernays used Theorem 2(f), (d), and (p) for HK, the first two of these generate HM. The schemes given here come directly from Gentzen's rules.

On the other hand the scheme (b) alone is not sufficient. Thus consider the Boolean algebra formed on 0, 1, $\alpha$, $\beta$. Let $A \supset B$ have its normal interpretation in such an algebra, and let $\neg A$ be non-normal, viz.,

| A \ B | $A \supset B$: 0 | $\alpha$ | $\beta$ | 1 | $\neg A$ |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| $\alpha$ | $\beta$ | 1 | $\beta$ | 1 | 1 |
| $\beta$ | $\alpha$ | $\alpha$ | 1 | 1 | 1 |
| 1 | 0 | $\alpha$ | $\beta$ | 1 | 0 |

16. Łukasiewicz [58]; cf. Bernays [3], p. 44. Of course one applies Pr to the schemes cited from Theorem 2.

The interpretation for $A \vee B$ and $A \wedge B$ are also the normal ones, viz.,

| $A \vee B$ | 0 | $\alpha$ | $\beta$ | 1 | $A \wedge B$ | 0 | $\alpha$ | $\beta$ | 1 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | $\alpha$ | $\beta$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $\alpha$ | $\alpha$ | $\alpha$ | 1 | 1 | $\alpha$ | 0 | $\alpha$ | 0 | $\alpha$ |
| $\beta$ | $\beta$ | 1 | $\beta$ | 1 | $\beta$ | 0 | 0 | $\beta$ | $\beta$ |
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | $\alpha$ | $\beta$ | 1 |

Then $A \supset \neg A . \supset \neg A$ always has value 1, but $A \supset \neg\neg A$ has the value $\beta$ for $A = \alpha$, while all the positive postulates always have the value 1.[17] Also if $A$ and $A \supset B$ have value 1, so does $B$.

THEOREM 10. *A set of prime propositions for the algebra HJ consists of those for HA together with the following schemes:*[18]

(a) $$A \supset \neg A . \supset \neg A.$$

(b) $$A \supset . \neg A \supset B.$$

*Proof*. These rules validate the rules (b) and (c) of Theorem 8. Note that (a) of Theorem 8 is a consequence of (c) (the present (b)).

THEOREM 11. *A set of prime propositions for the algebra HD consists of those of HM together with the scheme:*[19]

(a) $$\neg A \supset A . \supset . A.$$

*Proof*. This validates the scheme Nd.

THEOREM 12. *The algebra HK consists of all propositions having the value 1 in the classical evaluation. This is the same as all those in TK ($\mathfrak{D}$). A set of prime propositions for HK consists of those of HJ and HD together, or of those of HM together with the scheme*

$$\neg\neg A \supset A.$$

*Proof*. Let $\mathfrak{H}_1$ be the propositions of HK, $\mathfrak{H}_2$ those having the value 1 on the classical evaluation, $\mathfrak{H}_3$ those in the algebra above generated,[20] and $\mathfrak{H}_4$ those in TK($\mathfrak{d}$). Then by the same

17. This contradicts a statement by Wajsberg [84] which he says is based on a letter from Scholz.

18. These schemes are given in Wajsberg [84] and credited to the Münster school.

19. This scheme appears in [59], Satz 6.

20. On the equivalence of the two bases for $\mathfrak{H}_3$, cf. Theorem 5. For shorter bases cf. [47] pp. 64-71, [59] pp. 35-37, [76], [81] pp. 147 ff.

argument as in the three previous theorems $\S_3 \subseteq \S_4$. By Theorem 3, $\S_4 \subseteq \S_1$. By the argument of II, §3, $\S_1 \subseteq \S_2$. The relation $\S_2 \subseteq \S_3$ is established by reduction to conjunctive normal form. Since this is a standard technique, it will not be given here.[21]

We note that the systems LM, LJ, LK have the character noted in II Theorem 10 - viz., that a rule relating to a connective is not used unless the connective actually appears. Since we derived all the positive rules in Chapters II and III we have:

THEOREM 13. *The propositions not involving negation which are valid in HM and HJ are those valid in HA; those not involving negation valid in HK are the same as those valid in HC.*

This extends Theorem II 22 to include negation. The only reservation is that we have to add Pei to the prime schemes in order to get HC.

To get a similar result for LD we need some further theorems. The following generalizes a well-known result of Glivenko [37].

THEOREM 14. *A necessary and sufficient condition that A be derivable in HD (HK) is that $\neg\neg$ A be derivable in HM(HJ).*

*Proof*. The necessity of the condition rests upon the rule

$$(15) \qquad \frac{\neg\neg A \qquad \neg\neg(A \supset B)}{\neg\neg B}$$

derivable in TM by the following scheme:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overset{\checkmark}{\overset{5}{A}} \qquad \overset{\checkmark}{\overset{4}{A \supset B}}}{B}\ \text{Pe} \qquad \overset{\checkmark}{\overset{3}{\neg B}}}{F}\ \text{Ne}}{\neg A}\ \text{Ni - 5} \qquad \overset{2}{\neg\neg A}}{F}\ \text{Ne}}{\neg(A \supset B)}\ \text{Ni - 4} \qquad \overset{1}{\neg\neg(A \supset B)}}{F}}{\neg\neg B}\ \text{Ni - 3}$$

Then it follows that a derivation by the rule Ph is carried over by the correspondence $A \rightarrow \neg\neg A$ into a deduction by the rule (15), which can then be converted into a deduction by Ph. To show the deduction can be made into a derivation it is then only necessary to show that the prime propositions in the original system go over into derivable propositions.

21. See for example Hilbert-Ackerman [46], Bernays [3] §4. The theorem is due to Post [69]. The roots of the theorem go clear back to Boole. It is a standard theorem of Boolean algebra that if the coefficients in the "development" of a function are all 1, then the function is identically 1. Since these coefficients are the values in a truth table, the theorem amounts to showing that HK can be interpreted as a Boolean algebra. For other methods of proof see Church [14] and Quine [70]. The latter gives further references.

This is evidently true for the propositions originally present in HM(HJ) by the rule

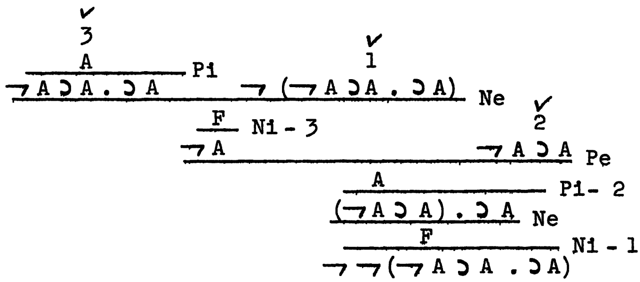
$$\frac{A}{\neg\ \neg A},$$

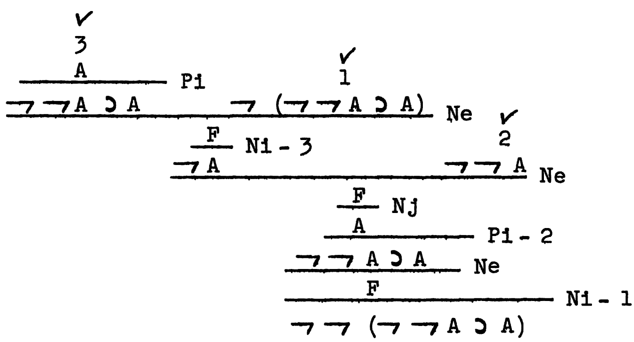
which follows immediately from Ne and Ni. Hence the necessity of the condition depends on

$$\neg\ \neg\ (\neg A \supset A\ .\supset A) \quad \text{in HM.}$$

$$\neg\neg\ (\neg\ \neg A \supset A) \quad \text{in HJ.}$$

These are established thus:[22]





We turn now to the sufficiency of the condition. If we have $\Vdash \neg\ \neg$ B in LM, then this must come by Nr from $\neg$ B $\Vdash$, and this in turn by N$\ell$ either from $\Vdash$ B, or $\neg$ B $\Vdash$ B. In the former case B is already in HM, in the latter case $\neg$ B $\supset$ B is in HM and so B is in HD. If $\neg\ \neg$ B is in HJ, of course we pass directly to B in HK.

THEOREM 15. <u>If B is a proposition not involving negation and B is valid in HD, then B is valid in HA</u>.

22. I found these and several other schemes easier to discover in the IM system, but the presentation in the TM system is more compact.

Proof. By Theorem 14 $\neg\neg$ B is valid in HM. As in the proof of Theorem 14 either $\Vdash$ B or $\neg$ B $\Vdash$ B is valid in LM. In the former case, since B does not involve negation, it is valid in LA by Theorem II 10 and Theorem 1. In the latter case if we go through the proof of $\neg$ B $\Vdash$ B backwards we can never have a void right side, and hence the $\neg$ B on the left cannot have been introduced by $N\ell$. Hence the $\neg$ B must be parametric clear back to the beginning of the proof. The proof is then valid if it is omitted. This reduces the second case to the first.

7. **Concluding Remarks.** 1. In the foregoing we have, in effect, started out by making semantical definitions of concepts we use in the epitheoretic study of formal systems. The rigorous following out of these initial definitions has led us to the systems LA, LM, and LJ, and their associated T and H forms, according to the type of negation involved. These then are the acceptable systems from the constructive standpoint.

2. Although the main focus of our attention is on the constructive systems, at the same time we have carried along the systems LC, LD, and LK, assigning to them properties suggested by an analogy with the others. It is to be determined a posteriori what uses, if any, these systems may have. It has turned out that LK represents the standpoint of classical falsity, wherein implication is no longer a relation of deducibility but a truth function. The system LC, although not containing negation, nevertheless contains those and only those positive properties of implication which follow from its definition as a truth function. The system LD, although designed to include a law of excluded middle, nevertheless partakes of the deducibility character of the constructive systems, in that the positive properties of implication are the same as for LA. But it is noteworthy how little effect the idealistic assumptions inherent in these systems have upon the major theorems.

3. In regard to the multiple prosequences which characterize LK, that was undoubtedly suggested by a lack of symmetry of the situation in LA. We could of course get along without it; i.e., we could characterize LK by LM with Nx and Kj together, and LC by LA with the rule

$$\frac{\mathfrak{X}, A \supset B \Vdash A}{\mathfrak{X} \Vdash A}$$

This would make LK seem undecidable. Although LK is decidable by truth tables, yet Gentzen showed important consequences could be derived from his formulation. Thus the representation by multiple prosequences is a real contribution. It would be desirable to have something similar for LD.

4. By generalizing the ideas of formal system we can consider ordinary discourse as constituting such a system. Of

course, it is not a definite system, and our systems L will not be definite either. But our definitions make sense if $\mathfrak{S}$ is simply a body of intuitively acceptable propositions. In ordinary discourse, however, at least from certain points of view, we think of having such complete knowledge that the law of excluded middle is acceptable. Therefore LD or LK is acceptable; LK if we accept

$$\neg A \vee B \,.\supset.\, A \supset B$$

(i.e., if we have "material implication"), LD if we do not. Thus LD is the natural system of strict implication. It is not sufficient, however, for applicability of LD that every elementary proposition be decidable. Thus in the example of § 2, with Rules 1-4 only, take

$$F \equiv :(\exists x).\ o = x'$$

$$A \equiv : 3 = 5 \,.\supset.\, 1 = 2.$$

Then A, although demonstrably invalid, is not refutable. This is because refutability in the sense of the above F, is not the same as invalidity in the original system. If we should formulate refutability so as to be equivalent to such invalidity, the system LD would be applicable.