Chapter II

THE FINITE POSITIVE CONNECTIVES

In this chapter we consider the compound propositions generated from the elementary propositions by three binary connectives which we call implication, conjunction and alternation. After a few preliminary definitions in §1, we shall consider these connectives informally in §§2 and 3. The considerations will then be formalized, leading to the set up of the episystems LA, LC in §4. The later sections will be concerned with theorems concerning these systems. The elimination theorem, which is the same as Gentzen's Hauptsatz, is proved in §7. The relation to Gentzen's natural system is the subject of §8; while the relations to propositional algebra are taken up in §9.

1. **Preliminary Definitions and Notation.** The three connectives will be symbolized as follows:

$$\begin{array}{lll} & \text{Implication} & (---_1 \supset ---_2) \\ (1) & \text{Conjunction} & (---_1 \wedge ---_2) \\ & \text{Alternation} & (---_1 \vee ---_2) \end{array}$$

We presuppose a system $\mathfrak{S}$ as in Chapter I §8, and shall adhere to the notational conventions there made. Then $\mathfrak{E}$ is the class of elementary propositions of $\mathfrak{S}$.

The use of these functors as connectors may seem at first sight to depart from a more or less standard practice of using them as adjunctors, and using the Hilbert functors "$(---_1 \rightarrow ---_2)$", "$(---_1 \& ---_2)$", "$(---_1$ or $---_2)$" as connectors. But later on we shall formalize the theory, and then the former functors will in fact play the role of adjunctors, so that we are agreeing with the practice rather than the reverse. The notation leads to no confusion so long as we are dealing with an unspecified formal system $\mathfrak{S}$. But when the theory is applied to a particular system $\mathfrak{S}$ it may happen that $\mathfrak{S}$ contains analogous functives as part of its formal machinery. In that case special care must be exercised, and since these functors will then be thought of more as connectors, the Hilbert functors are likely to be appropriate. It is important to realize the difference, in such a case, between the two classes of functors: those introduced here are connectors from the point of view of $\mathfrak{S}$ itself and the connectives form compound propositions from elementary ones; while others are constituents out of which the elementary sentences themselves are formed.

In this chapter the extensions we shall consider are all propositional extensions.

2. **Informal Discussion.** We shall now make an intuitive analysis of the notions $A \supset B$, $A \wedge B$, and $A \vee B$. This analysis will be based on the principle that <u>the meaning of a concept is determined by the conditions under which it is introduced into discourse</u>. The system $\mathfrak{S}'$ referred to is any fixed extension (propositional or term) of $\mathfrak{S}$.

If A and B are elementary then we can define the connectives, according to the above principle, as follows:

a1) $A \supset B$ holds for $\mathfrak{S}'$ if there is a derivation of B from A in $\mathfrak{S}'$; in other words, if B is a theorem in the system $\mathfrak{S}'$ (A) formed by adjoining A to the axioms of $\mathfrak{S}'$.

a2) $A \wedge B$ holds for $\mathfrak{S}'$ if A holds for $\mathfrak{S}'$ and B holds for $\mathfrak{S}'$.

a3) $A \vee B$ holds for $\mathfrak{S}'$ if either A holds for $\mathfrak{S}'$ or B holds for $\mathfrak{S}'$.

These rules are to be understood as the justifications, and the only justifications, for introducing the statements indicated ' into discourse.

It will be observed that the definitions a2) and a3) make sense when A and B are not elementary, and a1) makes sense even when B is not elementary, provided that A is elementary. But when A is compound it does not make sense to talk about adjoining A to the axioms of $\mathfrak{S}'$. We therefore have to say what we mean by "B is a theorem in the system $\mathfrak{S}'$ (A)", or in other words when we admit that B follows from A relative to $\mathfrak{S}'$. This we do as follows:

b1) We admit that C follows from $A \supset B$ relative to $\mathfrak{S}'$ if A is in $\mathfrak{S}'$ and C follows from B in $\mathfrak{S}'$.

b2) We admit that C follows from $A \wedge B$ in $\mathfrak{S}'$ if either C follows from A or C follows from B.

b3) We admit that C follows from $A \vee B$ in $\mathfrak{S}'$ if both C follows from A and C follows from B.

With this understanding all the rules a1 - b3 make sense for any members of $\mathfrak{P}$. If we define the order of a compound proposition as the number of times the connectives are applied in its construction,[1] then the admissibility of a proposition of order n depends on the validity of propositions of order $<n$.[2]

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1. More precisely: an A in $\mathfrak{P}$ is of order 0 when it is in $\mathfrak{S}$; it is of order n + 1 if it is of the form $B \supset C$, $B \wedge C$, or $B \vee C$, where B and C are of orders whose sum is n.

2. Note there may be propositions of higher order in $\mathfrak{S}'$. We could define an order for an elementary statement (see below) as the sum of the orders of all distinct propositions in $\mathfrak{X}$, plus that of A, then our rules so far explain an elementary statement of order n in terms of those of lower order.

In the foregoing we thought of $\mathfrak{S}'$ as an extension of $\mathfrak{S}$. Let us suppose it is formed by adjoining a class $\mathfrak{X}$ of propositions to $\mathfrak{S}$. Then A is valid in $\mathfrak{S}'$ if and only if A follows from $\mathfrak{X}$ in $\mathfrak{S}$; and B follows from A in $\mathfrak{S}'$ if and only if it follows from the class formed by adjoining A to $\mathfrak{X}$ in $\mathfrak{S}$. If we symbolize the latter class as "$\mathfrak{X}$, A" and the first relation as

$$\mathfrak{X} \Vdash A \tag{2}$$

then the latter relation is

$$\mathfrak{X}, A \Vdash B.$$

We read (2) "$\mathfrak{X}$ involves[3] A" or "$\mathfrak{X}$ entails A." In this notation our rules a1 - b3 become:

a1) $\dfrac{\mathfrak{X}, A \Vdash B}{\mathfrak{X} \Vdash A \supset B}$ b1) $\dfrac{\mathfrak{X} \Vdash A;\ \mathfrak{X}, B \Vdash C}{\mathfrak{X}, A \supset B \Vdash C}$

a2) $\dfrac{\mathfrak{X} \Vdash A;\ \mathfrak{X} \Vdash B}{\mathfrak{X} \Vdash A \wedge B}$ b2) $\dfrac{\mathfrak{X}, A \Vdash C}{\mathfrak{X}, A \wedge B \Vdash C}$; $\dfrac{\mathfrak{X}, B \Vdash C}{\mathfrak{X}, A \wedge B \Vdash C}$

a3) $\dfrac{\mathfrak{X} \Vdash A}{\mathfrak{X} \Vdash A \vee B}$; $\dfrac{\mathfrak{X} \Vdash B}{\mathfrak{X} \Vdash A \vee B}$ b3) $\dfrac{\mathfrak{X}, A \Vdash C;\ \mathfrak{X}, B \Vdash C}{\mathfrak{X}, A \vee B \Vdash C}$

The following rules also follow from the interpretation given for (2): If

$$A_1, A_2, \ldots, A_m \vdash B,$$

then

a4) $\dfrac{\mathfrak{X} \Vdash A_i \quad i = 1, 2, \ldots, m}{\mathfrak{X} \Vdash B}$ b4) $\dfrac{\mathfrak{X}, B \Vdash C}{\mathfrak{X}, A_1, \ldots, A_m \Vdash C.}$

where, of course, "$\mathfrak{X}, A_1, \ldots, A_m$" means the class obtained by adjoining $A_1, \ldots, A_m$ to $\mathfrak{X}$. In all these rules the premises are written above the line, the conclusions below. This manner of writing has become traditional with the Hilbert School. Note that a3 and b2 each consist of two separate rules.

In order to formalize these rules it is necessary to complete them by making explicit further assumptions which follow from the intuitive meaning. In the first place, we have said $\mathfrak{X}$ is a class. But if we follow the above rules in a purely mechanical fashion, we shall have not a class, but a sequence of propositions, which may contain repetitions, in the place of $\mathfrak{X}$. Such a sequence of propositions will be called a <u>prosequence</u>.

To express that prosequences which correspond to the same class are equivalent we need the rules

c1) If $\mathfrak{X}_2$ is a permutation of $\mathfrak{X}_1$, then

$$\frac{\mathfrak{X}_1 \Vdash A}{\mathfrak{X}_2 \Vdash A}.$$

3. This is Carnap's word (see Carnap [5] §32, p. 151).

c2) If A is in $\mathfrak{X}$

$$\frac{\mathfrak{X},\ A \Vdash B}{\mathfrak{X} \Vdash B}.$$

To express the fact that if a proposition follows from a class $\mathfrak{X}$ it follows from any larger class we need further the rule

c3) $$\frac{\mathfrak{X} \Vdash B}{\mathfrak{X},\ A \Vdash B}.$$

Finally we need to state those formulas (2) which we accept initially, thus:

d1) If A is in $\mathfrak{X}$, $\mathfrak{X} \Vdash A$.

d2) If A is an axiom, $\mathfrak{X} \Vdash A$.

d3) If $A_1, A_2, \ldots, A_m \vdash B$, then

$$\mathfrak{X},\ A_1, \ldots, A_m \Vdash B.$$

The rules so stated are redundant. Thus c3 is unnecessary if d are stated with a general $\mathfrak{X}$, because the extra A can be added to $\mathfrak{X}$ at the beginning and carried all the way through (see Theorem 2 below). Again, d3 expresses intuitively the same principle as a4, and is easily derived from it in Theorem 1. As for b4 it is readily derived (in Theorem 11, Corollary 2) from d3 and the rule

(3) $$\frac{\mathfrak{X},\ A \Vdash B;\quad \mathfrak{X} \Vdash A}{\mathfrak{X} \Vdash B}$$

This is as obvious intuitively as any of the rules; yet its redundancy can be expected a priori from a careful consideration of the principles according to which the rules were set up. In fact each of the rules a1 - 3 and b1 - 3 constitutes an explanation of a complex concept in terms of something simpler.[4] The rules a4 and d2 say that the rules and axioms of $\mathfrak{S}$ are valid in $\mathfrak{S}'$; while d1 gives the most elementary kind of deducibility. The rules c state simply that $\mathfrak{X}$ is to be regarded as a class. Intuitively one would feel that these cases should suffice to explain the meaning of every statement (2). In other words, (3) and b4 ought to be epitheorems, which in fact they are. The rule (3) is the elimination theorem proved in §7 (Theorem 11). It entails all the usual rules necessary to establish the "natural system" in §8.

The formalization of these ideas will concern us in §4. But before we proceed to it we shall discuss a related set of rules with an entirely different interpretation.

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4. In a sense a4 does too, since one can regard the conclusion of a rule of $\mathfrak{S}$ as more complex than its premises. Such a sense can presumably be made precise by the methods of Gödel's arithmetization and recursive functions. That, however. is irrelevant here.

3. **Classical Form of the Rules.** Gentzen proposed the following modification of the preceding formalism. Let the elementary statements be of the form

$$(4) \qquad \mathfrak{X} \Vdash \mathfrak{Y}$$

where $\mathfrak{Y}$, as well as $\mathfrak{X}$, is an arbitrary prosequence. With this type of elementary statement he associates the rules which are, essentially, the same as those above, except that an arbitrary prosequence $\mathfrak{Z}$ is adjoined to the right side in both premises and conclusion. Thus the analogues of the rules al and bl are:

$$\text{a1)}\ \frac{\mathfrak{X}, A \Vdash B, \mathfrak{Z}}{\mathfrak{X} \Vdash A \supset B, \mathfrak{Z}} \qquad \text{b1)}\ \frac{\mathfrak{X} \Vdash A, \mathfrak{Z}; \quad \mathfrak{X}, B \Vdash C, \mathfrak{Z}}{\mathfrak{X}, A \supset B \Vdash C, \mathfrak{Z}}$$

In addition to these there are rules for permutation, contraction, and weakening on the right which are, so to speak, the duals of c1, c2, c3. The rules will not be stated here in detail, since they are given fully in the formalization below.

This formalism has the following interpretation in terms of truth tables. Let the values 1 and 0 be assigned to all the members of $\mathfrak{E}$ in such a manner that all the axioms, if any, have the value 1, and that if

$$A_1, \ldots, A_m \vdash B$$

then B has the value 1 whenever all the $A_1, A_2, \ldots, A_m$ have the value 1. It is immaterial whether 1 and 0 are interpreted as truth and falsity or not. It is also immaterial whether every A in $\mathfrak{E}$ has a unique value or there are some which may have either value at will - in the latter case the stipulated conditions, both above and below, must hold for all choices of these values. The valuations may then be extended in the usual manner to all A in $\mathfrak{P}$ by the truth tables of Table 1. The statement (4) shall then mean that either some constituent in $\mathfrak{X}$ has the value 0 or some constituent in $\mathfrak{Y}$ has the value 1. Then it will follow that all the rules are valid. Thus we can verify the above rule b1 as follows: the conclusion is valid if there is a constituent in $\mathfrak{X}$ which is 0, if C is 1, or if some constituent in $\mathfrak{Z}$ is 1. If none of these cases occurs then by the first premise A is 1 and by the second premise B is 0, hence by Table 1 $A \supset B$ is 0 and the conclusion is valid. The other rules may be verified in like manner.

This interpretation is appropriate for the case where truth in $\mathfrak{S}$ is a definite concept. But even then there are elementary statements of the form (2) which are valid on the second interpretation but not on the first. An example of such is

$$(5) \qquad \Vdash A \vee (A \supset B).$$

This is clearly valid on the second interpretation for any system $\mathfrak{S}$ and any propositions A, B. Now let $\mathfrak{S}$ be the system with three elementary propositions E_1, E_2, E_3, of which E_3 is an axiom, and no rule of derivation or, perhaps, a single one $E_1 \vdash E_2$.

Then consider the proposition $E_2 \vee (E_2 \supset E_1)$. For this to be valid by Rule a3 we must have either E_2 or $E_2 \supset E_1$. But we do not have E_2 since the only theorem of $\mathfrak{S}$ is E_3, and we do not have $E_2 \supset E_1$, because when we adjoin E_2 to $\mathfrak{S}$ the only theorems are E_2 and E_3. Thus the special case $A = E_2$ and $B = E_1$ of (5) is invalid on the first interpretation. There is no question about definiteness for this system.

If (5) is interpreted according to the principles of §2, it requires that every proposition A either be true or imply every other proposition - in other words that $\mathfrak{S}$ be not only decidable but complete. In this special case we shall see in Chapter IV that the rules of the second interpretation follow from those of the first.

For the case where $\mathfrak{E}$ consists of propositional variables only, with $\mathfrak{A}$ and $\mathfrak{R}$ void, then we shall see that the propositions A for which

$$\Vdash A$$

are the positive formulas of the intuitionistic propositional algebra on the first interpretation, and of the classical algebra on the second. Hence this second interpretation is called classical.

4. **The Systems LA ($\mathfrak{S}$) and LC ($\mathfrak{S}$).** We turn now to the formalizations of these ideas. We formulate two systems LA($\mathfrak{S}$) and LC($\mathfrak{S}$) corresponding to the two interpretations. As explained previously, we shall take "proposition" as a formal category, corresponding to the terms of the system; the role of "proposition" as a significant category of the U-language is taken over by "statement." The term "formula" will be used as synonymous with either "proposition" or "statement" according to the context.

PROPOSITIONS. We shall suppose that the underlying system $\mathfrak{S}$ formulates a category $\mathfrak{E}(\mathfrak{S})$, or simply $\mathfrak{E}$, of <u>elementary propositions</u>, it being definite in any given case whether something belongs to this category or not. We then define $\mathfrak{P}(\mathfrak{S})$ inductively by

a) If A is in $\mathfrak{E}$, it is in $\mathfrak{P}$.

b) If A and B are in $\mathfrak{P}$, then $A \supset B$, $A \wedge B$, $A \vee B$ are in $\mathfrak{P}$.

PROSEQUENCES. A prosequence is a sequence of propositions with repetitions allowed. The elements of the sequence will be called its <u>constituents</u>, it being understood that where repetitions of the same propositions occur, each occurrence is a separate constituent. Constituents which are instances of the same proposition will be said to be <u>alike</u>, those which are instances of different propositions will be said to be <u>distinct</u>. A prosequence may be void, or it may contain any finite or infinite number of constituents. (The admission of an infinite number of constituents is optional by Theorem 5; however, it enables one to state such theorems as Theorem 19.)

We shall denote unspecified prosequences by capital German letters in the list

$$\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \mathfrak{U}, \mathfrak{V}, \mathfrak{W}, \mathfrak{M}.$$

The void prosequences will be indicated by "0" or by a blank space. A prosequence with a single constituent will not be distinguished notationally from that constituent. When the symbols for two or more prosequences are written one after the other, separated by commas, the complex expression shall indicate a prosequence whose constituents are those in all the component prosequences taken together. Thus the notation

$$\mathfrak{X}, \mathfrak{Y}, A_1, A_2, \ldots, A_m, \mathfrak{U}, \mathfrak{X}$$

indicates a prosequence whose constituents are the constituents of $\mathfrak{X}$, then those in $\mathfrak{Y}$, then $A_1, \ldots, A_m$, then those in $\mathfrak{U}$, then those in $\mathfrak{X}$ (repeated). (The complications in the notion of infinite prosequences are admitted, but will not be gone into, since infinite prosequences are optional. Cf. Remark 7 below.)

If A is a proposition and $\mathfrak{X}$ a prosequence (or a class) of propositions we define

$$A \varepsilon \mathfrak{X}$$

to mean A is a constituent (or element) in $\mathfrak{X}$. If $\mathfrak{X}$ and $\mathfrak{Y}$ are prosequences we shall define

$$\mathfrak{X} \leq \mathfrak{Y}$$

to mean that every constituent of $\mathfrak{X}$ is also constituent of $\mathfrak{Y}$;

$$\mathfrak{X} \equiv \mathfrak{Y}$$

to mean $\mathfrak{X} \leq \mathfrak{Y}$ and $\mathfrak{Y} \leq \mathfrak{X}$, in which case each of $\mathfrak{X}$ and $\mathfrak{Y}$ is a permutation of the other;

$$\mathfrak{X} \subseteq \mathfrak{Y}$$

to mean that the propositions which occur in $\mathfrak{X}$ also occur in $\mathfrak{Y}$ (without regard to multiplicity); and

$$\mathfrak{X} = \mathfrak{Y}$$

to mean that $\mathfrak{X} \subseteq \mathfrak{Y}$ and $\mathfrak{Y} \subseteq \mathfrak{X}$.

In the system LA($\mathfrak{S}$) it will be required that any prosequence denoted by "$\mathfrak{Y}$", with or without diacritical marks, shall have a single constituent, while, similarly, one denoted by "$\mathfrak{Z}$" shall be void. In the system LC all prosequences are arbitrary.

ELEMENTARY STATEMENTS. These are of the form

$$\mathfrak{X} \Vdash \mathfrak{Y}$$

where $\mathfrak{X}$ and $\mathfrak{Y}$ are prosequences. We shall call $\mathfrak{X}$ the left prosequence and $\mathfrak{Y}$ the right prosequence; we shall also use the phrases left side and right side respectively. (In LA($\mathfrak{S}$) the elementary statements are of the form (2).)

AUXILIARY STATEMENTS. We also suppose that statements of the form

$$(6) \qquad A_1, A_2, \ldots, A_m \vdash B$$

are defined by $\mathfrak{S}$ for $A_1, \ldots, A_m$, B in $\mathfrak{S}$. Here m may have different values ≥ 0 in the different statements. When a statement of form (6) holds for $m = 0$ we say B is an axiom. (In the interpretation (6) for $m > 0$ means that B follows immediately from $A_1, \ldots, A_m$ by a rule of $\mathfrak{S}$. In this case the statements of form (6) are definite when $\mathfrak{S}$ is definite.) The statements (6) will be treated here as morphological.

PRIME STATEMENTS. An elementary statement is a prime statement in the following cases, and these only:

p1) If $\mathfrak{X}$ and $\mathfrak{Y}$ have a constituent in common; i.e., if there is an A such that $A \varepsilon \mathfrak{X}$ and $A \varepsilon \mathfrak{Y}$.

p2) If $\mathfrak{Y}$ has a constituent A which is an axiom.

RULES. These will be given here with names which are somewhat more mnemonic than those used in § 3. Except for Er the rules are stated in pairs which are roughly dual to one another. Each pair will be assigned a symbol.[5] The rules on the left and right in each pair will be distinguished by writing "ℓ" and "r" respectively after the symbol for the pair. Thus Pr is the rule for introduction of $---_1 \supset ---_2$ on the right (the "deduction theorem" of propositional algebra). Er is peculiar since its dual Eℓ (i.e., b4 above) is redundant.

In stating the rules the almost self-explanatory notation of the Hilbert school is used (as in § 2). The theoretical premises are written above the line, the conclusion below. The morphological premises are indicated separately. The redundant rules have been omitted.

5. This symbol, in the rules introduced both here and later, is suggested by an analogy with certain terms of combinatory logic (see [17]). In that theory the symbols so introduced have the following significance (the symbol is explained in [17] unless another reference is given):

C	Primitive permutation combinator	
W	Primitive reduplication combinator	
K	Primitive cancellation combinator	
P	Implication	
Λ	Conjunction	([26], § 5.4)
V	Alternation	([18], p. 397)
Π	Universal quantifier	
N	Negation	([18], p. 397)

E Rule of Elementary Derivation Er

If $A_1, \ldots, A_m \vdash B$

then
$$\frac{\mathfrak{X} \Vdash A_i, \mathfrak{Z} \quad i = 1, 2, \ldots, m}{\mathfrak{X} \Vdash B, \mathfrak{Z}}$$

C Rules of Permutation

If $\mathfrak{X}_1 \equiv \mathfrak{X}_2$ $\qquad$ $\mathfrak{Y}_1 \equiv \mathfrak{Y}_2$

$$\frac{\mathfrak{X}_1 \Vdash \mathfrak{Y}}{\mathfrak{X}_2 \Vdash \mathfrak{Y}} \qquad \frac{\mathfrak{X} \Vdash \mathfrak{Y}_1}{\mathfrak{X} \Vdash \mathfrak{Y}_2}$$

W Rules of Contraction

If $A \varepsilon \mathfrak{X}$ $\qquad$ $A \varepsilon \mathfrak{Z}$

$$\frac{\mathfrak{X}, A, \Vdash \mathfrak{Y}}{\mathfrak{X} \Vdash \mathfrak{Y}} \qquad \frac{\mathfrak{X} \Vdash A, \mathfrak{Z}}{\mathfrak{X} \Vdash \mathfrak{Z}}$$

P Rules of Implication

If $\mathfrak{Z} \leq \mathfrak{Y}$

$$\frac{\mathfrak{X} \Vdash A, \mathfrak{Z} ; \ \mathfrak{X}, B \Vdash \mathfrak{Y}}{\mathfrak{X}, A \supset B \Vdash \mathfrak{Y}} \qquad \frac{\mathfrak{X}, A \Vdash B, \mathfrak{Z}}{\mathfrak{X} \Vdash A \supset B, \mathfrak{Z}}$$

A Rules of Conjunction

$$\frac{\mathfrak{X}, A \Vdash \mathfrak{Y}}{\mathfrak{X}, A \wedge B \Vdash \mathfrak{Y}} ; \frac{\mathfrak{X}, B \Vdash \mathfrak{Y}}{\mathfrak{X}, A \wedge B \Vdash \mathfrak{Y}} \qquad \frac{\mathfrak{X} \Vdash A, \mathfrak{Z} ; \quad \mathfrak{X} \Vdash B, \mathfrak{Z}}{\mathfrak{X} \Vdash A \wedge B, \mathfrak{Z}}$$

V Rules of Alternation

$$\frac{\mathfrak{X}, A \Vdash \mathfrak{Y} ; \ \mathfrak{X}, B \Vdash \mathfrak{Y}}{\mathfrak{X}, A \vee B \Vdash \mathfrak{Y}} \qquad \frac{\mathfrak{X} \Vdash A, \mathfrak{Z}}{\mathfrak{X} \Vdash A \vee B, \mathfrak{Z}} ; \frac{\mathfrak{X} \Vdash B, \mathfrak{Z}}{\mathfrak{X} \Vdash A \vee B, \mathfrak{Z}}$$

Remarks on these rules. 1.) The rules Cr and Wr are inapplicable in LA($\mathfrak{S}$).

2.) It will be convenient to use "O" to denote an unspecified one of the three functives P, Λ, V, and "$A \circ B$" for any one of $A \supset B$, $A \wedge B$, $A \vee B$. Thus a rule $O\ell$ is any one of the rules $P\ell$, $\Lambda\ell$, $V\ell$; etc.

3.) A constituent of $\mathfrak{X}$, $\mathfrak{Y}$, or $\mathfrak{Z}$ in an application of any of the rules will be called a parametric constituent. Such preserve their identity, so to speak, through the application of the rule; so that we can speak significantly of the same constituents in premise or premises and conclusion. Note that with the single exception of the $\mathfrak{Y}$ in $P\ell$ all parametric constituents appear in all premises.

4.) The use of the rules C will be tacit, in that we regard $\mathfrak{X}_1$ and $\mathfrak{X}_2$ as the same prosequence when $\mathfrak{X}_1 \equiv \mathfrak{X}_2$.[6] Since a permutation establishes a 1 - 1 correspondence, what was said about preservation of parametric constituents holds if a rule is combined with an application of Rule C to the conclusion.

5.) The constituent introduced into the conclusion by a rule Oℓ, Or, or Er will be called the *principal constituent*. This is A o B for Oℓ or Or, and the B for Er. The constituents in the premises which are absorbed to form the principal constituents will be called the *components*. These phrases will be extended to Rule W by calling the two indicated instances of A in the premises the components, while the single constituent which replaces them in the conclusion will be called the principal constituent. Thus the components occur in the premises but not in the conclusion, the principal constituent in the conclusion, but not in the premises, and the parametric constituents in both.

6.) A transformation effected by dropping any number of repetitions of a constituent will be called a *contraction*. Evidently we can speak of a similar transformation on a prosequence as a contraction. If prosequences are finite, a contraction of an elementary statement can be effected by a succession of applications of Rule W.

THEOREMS AND DERIVATIONS. We shall use capital Greek letters Γ, Δ, Θ for unspecified statements and classes of statements.

If Θ is a class of elementary statements, a *deduction on basis* Θ is a sequence $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, of elementary statements such that each Γ_k is either a) in Θ, b) a prime statement, or c) derived from some of its predecessors by an application of one of the rules. A *derivation* is a deduction with void basis. A *normal* deduction or derivation is one in which each Γ_k, except the last, is used as premise for inferring one and only one Γ_m with $m > k$. Evidently a deduction can be converted into a normal deduction by making the necessary repetitions and omissions. A normal derivation can be written schematically as a kind of genealogical tree (cf. §6).

An elementary statement which is the end statement of a derivation is an *elementary theorem*.

An $\mathfrak{S}$-*derivation* is one whose statements are of form (4) with void $\mathfrak{X}$, $\mathfrak{Y}$ having a single constituent. A proposition is $\mathfrak{S}$-derivable if and only if it is the sole constituent in the concluding statement of an $\mathfrak{S}$-derivation. Since the only rule applicable in an $\mathfrak{S}$-derivation is Er, an $\mathfrak{S}$-derivable proposition

6. This means that we regard a prosequence as a set for whose elements there is defined a degree of multiplicity. The roots of an algebraic equation form a similar concept in ordinary mathematics.

is a theorem in 𝔖 when our system is interpreted in the sense intended.[7]

Our assumptions are verified in the particular case where 𝔈 consists of the sequence $E_1, E_2, \ldots$ and the relation (6) is void. The system 𝔖 in that case will be called 𝔇. The systems LA(𝔇) and LC(𝔇) then give two forms of propositional algebra in which the E_k are propositional variables - i.e., indeterminates in the sense of Chap. I §8. These are precisely the systems LJ and LK of Gentzen.[8] Evidently the elementary theorems and theorem-schemes of LA(𝔇) and LC(𝔇) will be valid in LA(𝔖) and LC(𝔖) respectively for any 𝔖; for it is an epitheorem of any formal system that the indeterminates in any derivation can be replaced by arbitrary terms without destroying the validity of the argument.

5. Some Preliminary Theorems. Illustrations of the application of these rules will be given in §6. First, however, we shall prove some general theorems. This is expedient, in that we can then illustrate the theorems in the examples; but the reader may prefer to read this section and the next concurrently.[9]

The theorems stated will hold when additional connectives are formalized in the later chapters. In fact they depend principally on general properties of principal, component, and parametric constituents as defined in §4.[10]

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7. Note that we have actually assumed concerning 𝔖 only that it defines the category 𝔈(𝔖) and relation (6). But in the interpretation 𝔖 is a formal system, 𝔈 its elementary propositions, and (6) the relation of direct derivability.

8. Except, of course, that only the finite positive connectives are considered.

9. The theorems 3 and 4 are not essential to the main argument, but serve to shorten the decision process in §6. Theorem 5 is trivial if only finite prosequences are admitted.

10. The main theorems follow for any set of rules satisfying the following conditions: 1) the principal constituent appears in the conclusion only and is unique; 2) the components appear in the premises only; 3) every other constituent is parametric; 4) to each parametric constituent (p.c.) in the premises there is associated a unique p.c. in the conclusion called its correspondent; this correspondent is like the original and appears on the same side; 5) no two p.c's. in the same premise have the same correspondent; 6) every p.c. in the conclusion is correspondent of some p.c. in the premises; 7) a rule remains valid if a p.c. is omitted in the conclusion together with all those of which it is the correspondent; 8) a rule remains valid if a constituent is added to one or more premises together with a like constituent to serve as correspondent in the conclusion; 9) Except in cases W and Er, the principal constituent is of higher order than the components, and the rule is determined uniquely by that constituent. In Theorems 3 and 4 alone use is made of the fact that the p.c. appear in all the premises except on the right in P𝓁. In the discussion below corresponding constituents are regarded as the same.

The theorems are true for either the LA or the LC formulation. In the former case certain possibilities cannot arise; but the arguments are all valid.

One obvious property it seems unnecessary to formulate. Any LA deduction or derivation is also valid in LC. Conversely an LC deduction in which all prime and basic statements have only one constituent on the right is also an LA deduction. In fact, the restriction to only one constituent on the right in LA need only have been made for the prime statements.

THEOREM 1. If $A_1,\ldots,A_m,B$ are in $\mathfrak{G}$ and are such that (6) holds, then for any prosequences $\mathfrak{X},\mathfrak{Z}$

$$(7) \qquad \mathfrak{X}, A_1,\ldots,A_m \Vdash B,\mathfrak{Z}.$$

Proof. The statement $\mathfrak{X},A_1,\ldots,A_m \Vdash A_i,\mathfrak{Z}$ is a prime statement of type pl for each $i = 1,2,\ldots,m$. From these the conclusion follows by Rule Er.

Remark 1. If we admitted infinitely many constituents on the left in (6), that would correspond, on our interpretation, to admitting formal systems which were not definite. If we were interested in that sort of system, we should presumably not object to having infinitely many premises in Er, which would make LA($\mathfrak{S}$) and LC($\mathfrak{S}$) also indefinite. But if we insist on making LA($\mathfrak{S}$) and LC($\mathfrak{S}$) definite even when ($\mathfrak{S}$) is not, then Theorem 1 will not hold. In these lectures I shall adhere to the restriction that (6) have only a finite number of constituents on the left.

THEOREM 2. The following pair of rules are redundant:
K. Rules of weakening

If $\quad \mathfrak{X}_1 \leqq \mathfrak{X}_2 \qquad\qquad \mathfrak{Y}_1 \leqq \mathfrak{Y}_2$

$$\frac{\mathfrak{X}_1 \Vdash \mathfrak{Y}}{\mathfrak{X}_2 \Vdash \mathfrak{Y}} \qquad\qquad \frac{\mathfrak{X} \Vdash \mathfrak{Y}_1}{\mathfrak{X} \Vdash \mathfrak{Y}_2}$$

Proof. Let $\Delta = (\Gamma_1,\Gamma_2,\ldots,\Gamma_n)$ be a derivation of the premise. Let $\mathfrak{U}$ be a prosequence such that,

$$\mathfrak{X}_2 \equiv \mathfrak{X}_1,\mathfrak{U} \qquad \text{or} \qquad \mathfrak{Y}_2 \equiv \mathfrak{Y}_1,\mathfrak{U}$$

as the case may be. Let Γ_k' $(k = 1,2,\ldots,n)$ be the statement obtained by adjoining $\mathfrak{U}$ on the appropriate side of Γ_k. Then Γ_n' is the conclusion of our theorem. I shall show that $\Gamma_1',\ldots,\Gamma_n'$ is a derivation of Γ_n'.

If Γ_k is a prime statement, then Γ_k' is a prime statement also. If Γ_k is a consequence of $\Gamma_{i_1}, \Gamma_{i_2}, \ldots \Gamma_{i_p}$ by a rule, then Γ_k' is a consequence of $\Gamma_{i_1}',\Gamma_{i_2}',\ldots,\Gamma_{i_p}'$ by the same rule, since $\mathfrak{U}$ is merely added to the parametric constituents in both premises and conclusion. Since $\Gamma_1,\ldots,\Gamma_n$ is a derivation, $\Gamma_1',\ldots,\Gamma_n'$ is also.

Remark 2. Gentzen assumes the rules K, but takes as prime statements only those of the form

$$(8) \qquad A \Vdash A.$$

These replace our prime statements p1. If we had Rule K we could replace the statements p2 by

$$(9) \qquad \Vdash A.$$

This is, of course, an equivalent procedure.

Remark 3. The theorem makes it possible to state the rules with multiple premises in the following form which is sometimes more convenient:

If $A_1, \ldots, A_m \Vdash B$ then

$$P\ell' \quad \frac{\mathfrak{X}_1 \Vdash A, \mathfrak{Z};\ \mathfrak{X}_2, B \Vdash \mathfrak{Y}}{\mathfrak{X}_1, \mathfrak{X}_2, A \supset B \Vdash \mathfrak{Y}, \mathfrak{Z}} \qquad Er' \quad \frac{\mathfrak{X}_i \Vdash A_i, \mathfrak{Z}_i \quad i = 1,2,\ldots,m}{\mathfrak{X}_1, \ldots, \mathfrak{X}_m \Vdash B, \mathfrak{Z}_1, \ldots, \mathfrak{Z}_m}$$

$$V\ell' \quad \frac{\mathfrak{X}_1, A \Vdash C, \mathfrak{Z}_1;\ \mathfrak{X}_2, B \Vdash C, \mathfrak{Z}_2}{\mathfrak{X}_1, \mathfrak{X}_2, A \vee B \Vdash C, \mathfrak{Z}_1, \mathfrak{Z}_2} \qquad \Lambda r' \quad \frac{\mathfrak{X}_1 \Vdash A, \mathfrak{Z}_1;\quad \mathfrak{X}_2 \Vdash B, \mathfrak{Z}_2}{\mathfrak{X}_1, \mathfrak{X}_2 \Vdash A \wedge B, \mathfrak{Z}_1, \mathfrak{Z}_2}$$

For by Rule K these may be reduced to those in §4.

COROLLARY 1. *If $\Gamma_1, \ldots, \Gamma_n$ is a derivation, where Γ_k is of the form*

$$\mathfrak{X}_k \Vdash A_k, \mathfrak{Z}_k,$$

we can suppose without loss of generality that

$$\mathfrak{X}_n \leq \mathfrak{X}_k \qquad\qquad \mathfrak{Z}_n \leq \mathfrak{Z}_k.$$

Proof. Adjoin constituents of $\mathfrak{X}_n, \mathfrak{Z}_n$ as additional parametric constituents throughout the whole derivation as in Theorem 2. Then remove repeated elements by contraction at the end.

COROLLARY 2. *If*

$$\mathfrak{X}_1 \subseteq \mathfrak{X}_2 \qquad\qquad \mathfrak{Y}_1 \subseteq \mathfrak{Y}_2$$

then

$$\frac{\mathfrak{X}_1 \Vdash \mathfrak{Y}}{\mathfrak{X}_2 \Vdash \mathfrak{Y}} \qquad\qquad \frac{\mathfrak{X} \Vdash \mathfrak{Y}_1}{\mathfrak{X} \Vdash \mathfrak{Y}_2}$$

Proof. This is a combination of Rules K, C, and W. (In case we admit infinite prosequences, we need also Theorem 5 in order to derive this corollary.)

THEOREM 3. *If the rules $O\ell$ and, in LC, Or and Er, are so modified as to require the principal constituents to appear in all the premises on the same side as in the conclusion; then the rules W are redundant.*

Proof. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be a derivation of

$$\mathfrak{X}, A \Vdash \mathfrak{Y} \qquad \text{or} \qquad (\mathfrak{X} \Vdash A, \mathfrak{Z})$$

where $A \in \mathfrak{X}$, $(A \in \mathfrak{Z})$. Then I shall show by induction for all $k = 1,2,\ldots,n$ that, if Γ_k' is obtained from Γ_k by a contraction, then Γ_k' can be derived from the modified rules without a contraction.

If Γ_k is a prime statement, then Γ_k' is also. Hence the assertion is true if Γ_k is prime.

Suppose Γ_k is derived by a rule, and that the assertion holds for the premises of the rule. If both component constituents of the contraction are parametric, then they both occur in all the premises leading to Γ_k[11] and these premises may all be contracted on these constituents as components; the contracted premises will be valid by the hypothesis of the induction, and the contracted premises will lead by the same rule to Γ_k'. If one contracted constituent is the principal constituent in Γ_k, then the modified rule will lead to Γ_k'.

Remark 4. If the rules are modified as in Theorem 3, then all constituents of the conclusion must be present in the premises; and hence, in any derivation, all constituents must be present in the prime statements. The essential function of the rules is then to eliminate components. Note that the original rules follow from the modified rules by weakening (Theorem 2).

THEOREM 4. *If the rules are modified as in Theorem 3, and if we then admit as additional rules all further modifications (compatible with the restrictions on LA) in which a component is allowed to appear on the same side in the conclusion, even when it does not appear as parametric constituent in the premise or premises in which it occurs as component (but does occur in all the others); then every derivation according to Theorem 3 becomes a derivation according to the further modified rules if every statement in it is contracted until there are no repeated constituents in any prosequence.*

Remark 5. To clarify the meaning of the new rules I shall list them explicitly for $\wedge r$ thus:

1. $$\frac{\mathfrak{X} \Vdash A, A \wedge B, \mathfrak{Z}; \quad \mathfrak{X} \Vdash B, A \wedge B, \mathfrak{Z}}{\mathfrak{X} \Vdash A \wedge B, \mathfrak{Z}}$$

2. $$\frac{\mathfrak{X} \Vdash A, A \wedge B, \mathfrak{Z}; \quad \mathfrak{X} \Vdash B, A \wedge B, A, \mathfrak{Z}}{\mathfrak{X} \Vdash A \wedge B, A, \mathfrak{Z}}$$

---

11. The one possible exception is in $P\ell$ on the right, where one of the contracted constituents may not occur in $\mathfrak{Z}$. But in that case they are in $\mathfrak{Y}$, and the contracted $\mathfrak{Y}$ will still satisfy the condition $\mathfrak{Z} \leq \mathfrak{Y}$. Thus the rule will still be applicable after the contraction. This case cannot arise in LA. In LC we can suppose without loss of generality that $\mathfrak{Z} \equiv \mathfrak{Y}$ (by Theorem 2).

3. $$\frac{\mathfrak{X} \Vdash A, A \wedge B, B, \mathfrak{Z}; \quad \mathfrak{X} \Vdash B, A \wedge B, \mathfrak{Z}}{\mathfrak{X} \Vdash A \wedge B, B, \mathfrak{Z}}$$

4. $$\frac{\mathfrak{X} \Vdash A, A \wedge B, B, \mathfrak{Z}; \quad \mathfrak{X} \Vdash B, A \wedge B, A, \mathfrak{Z}}{\mathfrak{X} \Vdash A \wedge B, A, B, \mathfrak{Z}}$$

Here 4 is redundant. Many of the new rules are redundant in the other cases also.

Proof of Theorem 4. Let $\Gamma_1, \ldots, \Gamma_n$ be a derivation according to Theorem 3, and let Γ_k' be obtained from Γ_k by dropping all repetitions. I shall show by induction on k that $\Gamma_1', \Gamma_2', \ldots, \Gamma_k'$ is a modified derivation as stated.

If Γ_k is a prime statement, then Γ_k' is also. This takes care of case $k = 1$.

Let Γ_k be deduced by a rule from premises $\Gamma_{i_1}, \Gamma_{i_2}, \ldots, \Gamma_{i_p}$, and let the hypothesis of the induction be verified for these premises. We make the necessary contractions in the premises in three stages, and show that at each stage the rule remains valid from the contracted premises to the contracted Γ_k. First let all possible contractions be made which involve only parametric constituents; then the contractions can be made in all premises and in Γ_k simultaneously:[12] the rule remains valid. Next let all contractions be made in which one of the like constituents is the principal constituent; then the others must be parametric, for in rules O*l* and Or a principal constituent is never like a component, and if that eventuality occurred in Er the conclusion would be the same as one of the premises. Hence all contracted constituents are in all premises and Γ_k, and the parametric constituents can be dropped out in all simultaneously. After all such contractions have been made no further contractions of Γ_k are possible; we have already reached Γ_k'. Any further contractions in the premises will invalidate the original rule; but the inference will still be possible by one of the derived rules.

Remark 6. This theorem shows that we can regard a prosequence as a class, but that the rules become complicated when we do so.

THEOREM 5. *If*

$$\mathfrak{X} \Vdash \mathfrak{Y};$$

then there exist $\mathfrak{X}'$, $\mathfrak{Y}'$, each with a finite number of constituents, such that

$$\mathfrak{X}' \leq \mathfrak{X} \qquad \mathfrak{Y}' \leq \mathfrak{Y}$$

and

$$\mathfrak{X}' \Vdash \mathfrak{Y}'.$$

12. Cf. footnote[11] to Theorem 3.

Proof. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be a derivation of (4) and let Γ_k be

$$\Gamma_k \qquad\qquad \mathfrak{X}_k \Vdash \mathfrak{Y}_k .$$

With each Γ_k we shall associate a Γ_k', viz.,

$$\Gamma_k' \qquad\qquad \mathfrak{X}_k' \Vdash \mathfrak{Y}_k'$$

such that $\mathfrak{X}_k', \mathfrak{Y}_k'$ are finite, Γ_k' is true, and

$$\mathfrak{X}_k' \leq \mathfrak{X}_k \qquad\qquad \mathfrak{Y}_k' \leq \mathfrak{Y}_k .$$

The theorem will then follow when k = n.

If Γ_k is a prime statement of type p1, with A as common constituent, then all conditions are fulfilled if we take Γ_k' to be (8). If it is of type p2 with A as axiom, we can take Γ_k' to be (9).

Now let Γ_k be derived by one of the rules from premises $\Gamma_{i_1}, \Gamma_{i_2}, \ldots, \Gamma_{i_p}$, for which the $\Gamma_{i_1}', \ldots, \Gamma_{i_p}'$ have already been defined. If any of the component constituents which occur in Γ_{i_j} are missing in Γ_{i_j}', they can be reinserted by Rule K (Theorem 2), yielding $\Gamma_{i_1}'', \Gamma_{i_2}'', \ldots, \Gamma_{i_p}''$. Then all conditions are fulfilled if we let Γ_k' be the conclusion of the rule, modified, in the case of multiple premises, as in Remark 3 after Theorem 2. This completes the proof by induction.

Remark 7. For the purpose of analysis of compound propositions one would naturally introduce the finiteness restrictions at the beginning. Gentzen did this. The significance of this theorem is that we can admit infinite prosequences if we want to This is made use of in Theorem 19 below.

6. **The Technique of Elementary Derivations. Decidability.** The way in which the rules were arrived at in § 2 shows that certain decidability properties should be expected of them. In fact, each of the rules O derives a more complex statement from simpler ones, and the complexity once introduced can never be got rid of at a later stage. Thus the application of the rules has a constructive aspect. It ought to be possible by examining an elementary statement to determine what rules it could be a consequence of and from what premises; to examine each of these possibilities in turn to see what might lead to them; and so on. As a matter of fact when $\mathfrak{S}$ is $\mathfrak{D}$, the number of possibilities to be tried is finite, and we eventually end with a derivation of the statement, or a proof that it is not derivable, (Theorem 7).

The process just described is not the shortest method of ascertaining whether an elementary statement is true or not. It can be simplified by the use of Gentzen's "natural" rules, whose validation on the present basis requires the elimination theorem (see §8). Nevertheless it is worth while to illustrate the process with a few examples. In these examples we suppose .

A,B, and C are elementary propositions of $\mathfrak{D}$. We shall suppose W has been eliminated as in Theorem 3.

As a first example consider the statement:

$$\Gamma_1 \qquad \Vdash (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)).$$

In LA($\mathfrak{D}$) the only rule which can lead to this conclusion is Pr; in that case the premise is

$$\Gamma_2 \qquad A \supset (B \supset C) \Vdash (A \supset B) \supset (A \supset C).$$

This could come from either Pℓ or Pr. Suppose we explore the latter possibility first.[13]

The premise would be:

$$\Gamma_3 \qquad A \supset (B \supset C),\ A \supset B \Vdash A \supset C.$$

This again could come from Pℓ or Pr; if it is Pr the premise is

$$\Gamma_4 \qquad A \supset (B \supset C),\ A \supset B,\ A \Vdash C.$$

Here we have no alternative but to try Pℓ. Let us try getting rid of the more complicated premise by an unmodified Pℓ first. Our Γ_4 could originate by Pℓ from Γ_5 and Γ_6, viz.

$$\Gamma_5 \qquad A,\ A \supset B \Vdash A,$$

$$\Gamma_6 \qquad A,\ A \supset B,\ B \supset C \Vdash C.$$

Here Γ_5 is a prime statement; hence Γ_1 is reduced to Γ_6. The latter could come from Pℓ and the statements

$$\Gamma_7 \qquad A,\ B \supset C \Vdash A,$$

$$\Gamma_8 \qquad A,\ B \supset C,\ B \Vdash C.$$

Here Γ_7 is again a prime statement, while Γ_8 could originate via Pℓ from

$$\Gamma_9 \qquad A,\ B \Vdash B,$$

$$\Gamma_{10} \qquad A,\ B,\ C \Vdash C,$$

both of which are prime. Thus the statements just written, in reverse order, constitute a derivation of Γ_1.

The derivation can be exhibited schematically as follows (for explanation of the dot notation see the remarks immediately preceding Theorem 6 below):

$$\dfrac{\dfrac{\dfrac{\dfrac{A \supset B, A \Vdash A \qquad \dfrac{A, B \supset C \Vdash A \qquad \dfrac{A, B \Vdash B \qquad A, B, C \Vdash C}{A, B \supset C, . B \Vdash C}\,\mathrm{P}\ell}{A \supset B, A, B \supset C \Vdash C}\,\mathrm{P}\ell}{A \supset . B \supset C, A \supset B, A \Vdash C}\,\mathrm{P}\ell}{A \supset . B \supset C, A \supset B \Vdash A \supset C}\,\mathrm{Pr}}{A \supset . B \supset C \Vdash A \supset B . \supset . A \supset C}\,\mathrm{Pr}}{\Vdash A \supset . B \supset C : \supset : A \supset B . \supset . A \supset C}\,\mathrm{Pr}$$

13. In this case Pℓ would not work. The elimination theorem shows that Γ_3 is equivalent to Γ_2. A similar remark applies in some other cases below.

As a second example consider the distributive law

Γ_1 $\quad\Vdash (A \wedge (B \vee C)) \supset ((A \wedge B) \vee (A \wedge C)).$

In LA($\mathfrak{D}$) this can come only from Pr with premise

Γ_2 $\quad A \wedge (B \vee C) \Vdash (A \wedge B) \vee (A \wedge C).$

Here we could use $\Lambda\ell$ or Vr. However, we cannot use the unmodified rules since the premises are intuitively false and can be shown to fail by the valuation method of §3. With a $\Lambda\ell$ modified as in Theorem 3 we should get Γ_2 from

Γ_3 $\quad A, A \wedge (B \vee C) \Vdash (A \wedge B) \vee (A \wedge C).$

This in turn follows by unmodified $\Lambda\ell$ from

Γ_4 $\quad A, B \vee C \Vdash (A \wedge B) \vee (A \wedge C).$

Premises which would give Γ_4 from $V\ell$ are

Γ_5 $\quad A, B \Vdash (A \wedge B) \vee (A \wedge C),$

Γ_6 $\quad A, C \Vdash (A \wedge B) \vee (A \wedge C).$

These are symmetric in B and C. It is sufficient to consider Γ_5. This follows by Vr from

Γ_7 $\quad A, B \Vdash A \wedge B,$

which again is a consequence by Λr of the prime statements

Γ_8 $\quad A, B \Vdash A,$

Γ_9 $\quad A, B \Vdash B.$

From this argument we can construct a derivation of Γ_1.

The schematic for the final derivation, including the $W\ell$, is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A,B \Vdash A \qquad A,B \Vdash B}{A,B \Vdash A \wedge B}\,\Lambda r}{A,B \Vdash (A \wedge B) \vee (A \wedge C)}\,Vr \qquad \dfrac{\dfrac{A,C \Vdash A \qquad A,C \Vdash C}{A,C \Vdash A \wedge C}\,\Lambda r}{A,C \Vdash (A \wedge B) \vee (A \wedge C)}\,Vr}{A, B \vee C \Vdash (A \wedge B) \vee (A \wedge C)}\,V\ell}{A, A \wedge (B \vee C) \Vdash (A \wedge B) \vee (A \wedge C)}\,\Lambda\ell}{A \wedge (B \vee C), A \wedge (B \vee C) \Vdash (A \wedge B) \vee (A \wedge C)}\,\left.\begin{matrix}\Lambda\ell\\ W\ell\end{matrix}\right\}\begin{matrix}\text{modified}\\ \Lambda\ell\end{matrix}}{A \wedge (B \vee C) \Vdash (A \wedge B) \vee (A \wedge C)}}{\Vdash (A \wedge (B \vee C)) \supset ((A \wedge B) \vee (A \wedge C))}\,Pr$$

In this example I have excluded certain alternatives by an appeal to intuition - or the valuation of §3. In practice one would use such devices to shorten the work. But in the next example I shall carry the process sketched at the beginning of this section to the bitter end, even though it is clear that Γ_3 is false by the criterion of §3.

The example considered is Peirce's law:

Γ_1 $\quad\Vdash ((A \supset B) \supset A) \supset A.$

We consider this in LA($\mathfrak{D}$). It can come only by Pr from

Γ_2 $\quad (A \supset B) \supset A \Vdash A.$

The only possibility is $P\ell$, for which the premises would be

Γ_3 $\quad (A \supset B) \supset A \Vdash A \supset B,$

Γ_4 $\quad (A \supset B) \supset A, A \Vdash A.$

Here Γ_4 is prime. Γ_3 could come from Pr or $P\ell$; if Pr the premise is

Γ_5 $\quad (A \supset B) \supset A, A \Vdash B,$

if $P\ell$ the premises are

Γ_6 $\quad (A \supset B) \supset A \Vdash A \supset B,$

Γ_7 $\quad (A \supset B) \supset A, A \Vdash A \supset B.$

Now Γ_6 is the same as Γ_3, hence a derivation of Γ_3 by $P\ell$ is impossible. The only possibility is Pr. But Γ_5 can come only by $P\ell$ from

Γ_8 $\quad (A \supset B) \supset A, A \Vdash A \supset B,$

Γ_9 $\quad (A \supset B) \supset A, A, A \Vdash B.$

At this point we can invoke Theorem 4. That theorem says that if we use certain modified rules then we can get a derivation in which there are no repeated constituents. Moreover, a modified rule is only used where an inference could be made by the unmodified rule (= modified rule in sense of Theorem 3) by using a premise with a component repeated. Now Γ_5 is derived from Γ_8 and Γ_9 by unmodified $P\ell$; hence it can only be derived according to Theorem 4 from Γ_8 and the contracted Γ_9, which is Γ_5 itself. Thus Γ_5, and hence Γ_1, Γ_2, Γ_3, is non-derivable in LA($\mathfrak{D}$).[14]

Peirce's law is, however, derivable in LC($\mathfrak{D}$). In fact Γ_2 is The schematic derivation for the proof, which can be constructed from the bottom up, is as follows:

$$\dfrac{\dfrac{\dfrac{A \Vdash B, A}{\Vdash A \supset B, A}\,\mathrm{Pr} \qquad A \Vdash A}{A \supset B \,.\supset A \Vdash A}\,P\ell}{A \supset B \,.\supset A \,.\supset A}\,\mathrm{Pr}$$

I will now state some theorems regarding these matters. The first of these merely lists some special cases which can be derived either by the foregoing techniques or those introduced later. In most cases the proof is simpler than in the foregoing examples. Only those marked * are needed for § 8. The other theorems are connected with the question of decidability.

14. Note it could be derived in LA($\mathfrak{G}$), for instance if A were an axiom.

In connection with these and later formulas the customary dot notation will be used.[15] A group of dots alongside a connector will indicate one end of a parenthesized unit, extending from that point away from the connector until a larger number of dots or the end of the formula is encountered.

THEOREM 6. <u>For any A, B, C ε $\mathfrak{P}(\mathfrak{S})$ the following hold in LA($\mathfrak{S}$):</u>

P_0*	$A, A \supset B \Vdash B,$
PK	$\Vdash A \supset . B \supset A,$
PS	$\Vdash A \supset . B \supset C : \supset : A \supset B . \supset . A \supset C,$
Λ_0	$A, B \Vdash A \wedge B,$
ΛK	$\Vdash A \wedge B . \supset . A,$
ΛK'	$\Vdash A \wedge B . \supset . B,$
Λ_1	$\Vdash A \supset B . \supset : A \supset C . \supset : A \supset . B \wedge C,$
Λ_2	$\Vdash A . \supset : B . \supset . A \wedge B,$
VK	$\Vdash A . \supset . A \vee B,$
VK'	$\Vdash B . \supset . A \vee B,$
V_1	$\Vdash A \supset C . \supset : B \supset C . \supset : A \vee B . \supset . C,$

<u>while the following hold in LC($\mathfrak{D}$) but do not hold in LA($\mathfrak{D}$) for A, B elementary</u>:

V_0	$A \vee B \Vdash A, B,$
Pei*	$\Vdash A \supset B . \supset A : \supset A,$
(5)	$\Vdash A . \vee . A \supset B.$

THEOREM 7. <u>The systems LA($\mathfrak{D}$) and LC($\mathfrak{D}$) are decidable systems</u>.

<u>Proof</u>. The process outlined above has only a finite number of possibilities. In fact, from the finiteness theorem (Theorem 5) and from the fact that for a derivation by Theorem 4 every constituent occurring in the derivation must occur as a constituent or part of a constituent in the result, it follows there are only a finite number of unlike constituents which can occur.

15. This notation is said to have been used by Leibniz. It was extensively used by Peano and the Principia Mathematica (see Jørgensen [51] I p. 177), and by many modern writers, but not a great deal by continental Europeans. For explanation of the customary notation see, e.g., [86] p. 9, [57] Appendix I, [71] § 7, pp. 37 ff., [42] and, more briefly, [2] p. 261, [11], p. 225, [29] p.370; [4] § 4c; [73]; [87] p. 44. Church uses a modification [14] p. 4. A useful generalization is given in [27] and in [83].

By Theorem 4 only a finite number of elementary statements are possible. Thus, any systematic process of testing whether these can be arranged in order so as to yield a derivation of the given statement must come to an end.

Gentzen's method of proof is to write down all possible elementary statements, pick out the prime ones, and test for derivability in some order, starting over again each time a new one is found to be derivable. The process in the text is shorter, and was used by Gentzen himself in special cases. In practice one can shorten the process still more by using known valuations, etc.[16]

THEOREM 8. <u>If $A_1, A_2, \ldots, A_m$ are in $\mathfrak{S}$, and</u>

(10) $$\Vdash A_1, A_2, \ldots, A_m ;$$

<u>then at least one A_k is S-derivable</u>.

<u>Proof</u>. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be a derivation of (10). Since the only rules which can have a conclusion of form (10) are Er and Wr, and since these will have premises of the same form, all the Γ_k are of the form (10).

I shall show by induction on k that the theorem holds for every Γ_k.

If Γ_k is a prime statement it must be of type p2; then the theorem is true since some A_i is an axiom.

If Γ_k is derived from Γ_i by Wr, the case is trivial.

If Γ_k is derived from $\Gamma_{i_1}, \Gamma_{i_2}, \ldots, \Gamma_{i_n}$ by Er, then we can suppose without loss of generality that we have

$$B_1, B_2, \ldots, B_n \vdash A_m,$$

where the premises are

$$\Gamma_i \quad \Vdash A_1, A_2, \ldots, A_{m-1}, B_j \qquad j = 1, 2, \ldots, n.$$

By the hypothesis of the induction every one of these has a constituent which is $\mathfrak{S}$-derivable. If the constituent is A_i, the theorem holds for Γ_k. If not then all B_j are $\mathfrak{S}$-derivable; hence, by the definition of $\mathfrak{S}$-derivation, A_m is.

COROLLARY. <u>If</u> <u>$\Vdash A$ holds in LC($\mathfrak{S}$)</u> - <u>hence a fortiori if it holds in LA($\mathfrak{S}$)</u>, - <u>A is $\mathfrak{S}$-derivable</u>.

THEOREM 9. <u>If in LA</u>

$$\Vdash A \vee B$$

<u>then either $\Vdash A$ or $\Vdash B$ is valid</u>.

16. A valuation scheme which is often convenient for showing non-derivability in LA is that of Gödel [39]. Cf. also Jaśkowski [49]. For other valuations see McKinsey and Tarski [62] and [61], especially Examples 1 and 2 in [61] p. 128. (Other papers by these authors are cited in [62].)

Proof. The statement in the hypothesis cannot be the conclusion of any rule except Vr, and the premise of that rule must be either $\Vdash A$ or $\Vdash B$.

It is convenient, for reference in later chapters, to state the obvious property of our rules, which is fundamental to all the foregoing arguments, as a theorem, viz.,-

THEOREM 10. In any derivation the only rules used, besides C, W, and E, are rules corresponding to connectives which actually appear in the final result.

7. **The Elimination Theorem.** We turn now to the proof of the theorem which Gentzen called his "Hauptsatz," He made several interesting applications of this theorem; in particular he showed that it included an important theorem of Herbrand. The theorem allows a common constituent of two elementary statements to be eliminated. Since none of the rules of §4 has any such character, the transition from premises to conclusion cannot be effected by any sequence of applications of our rules. The proof of the theorem is an inductive process, and shows that any proof of the premises can be transformed into a proof of the conclusion.

THEOREM 11. (Elimination Theorem.) If $\mathfrak{Z} \leq \mathfrak{Y}$,

(11) $$\mathfrak{X}, A \Vdash \mathfrak{Y},$$

and

(12) $$\mathfrak{X} \Vdash A, \mathfrak{Z};$$

then

(13) $$\mathfrak{X} \Vdash \mathfrak{Y}.$$

Remark 8. By Theorem 2 there is no loss of generality due to the restriction $\mathfrak{Z} \leq \mathfrak{Y}$. Without this restriction (13) would be

$$\mathfrak{X} \Vdash \mathfrak{Y}, \mathfrak{Z}.$$

Proof. This will be accomplished in three stages as follows: 1) Reduction to the case where (11) is the conclusion of an instance of a rule $\mathfrak{O}l$ for which A is the principal constituent. The case where A is elementary will be disposed of completely in this stage. 2) Further reduction to the case where (12) is the conclusion of an instance of a rule Or for which A is the principal constituent. 3) Disposition of the case where both these simplifications hold, by an induction on the order of A.

Stage 1. We assume that the theorem is true, for the particular A we are interested in, whenever (11) is the conclusion of a rule $\mathfrak{O}l$ for which A is the principal constituent. This assumption is called the hypothesis of the stage.

Let $\Delta = \Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be a normal derivation of (11) according to the original rules of §4. We neglect applications of Rule C as stated in §4. Let Γ_k be

$$\mathfrak{X}_k, \mathfrak{U}_k \Vdash \mathfrak{Y}_k ,$$

where $\mathfrak{U}_k$ is a prosequence, all of whose constituents, if any, are like A, defined as follows:

a) If $k = n$, $\mathfrak{X}_n$ is $\mathfrak{X}$, $\mathfrak{U}_n$ is the A in (11), $\mathfrak{Y}_n$ is $\mathfrak{Y}$.

b) If $k < n$, Γ_k is used as premise in deriving a unique Γ_m, $m > k$, by a rule R_m. Then $\mathfrak{U}_k$ consists of 1) those parametric constituents of $\mathfrak{U}_m$ which are in Γ_k and 2), in case R_m is Wℓ with principal constituent in $\mathfrak{U}_m$, the two like components.

It then follows by induction that all constituents of $\mathfrak{U}_k$ are like A, and that all are parametric for R_m, or are components of a contraction. Further if $\mathfrak{U}_m$ is void, $\mathfrak{U}_k$ is void also. The definition provides for the possibility that the parametric constituents do not occur in all the premises. We shall assume, as we may by virtue of Theorem 2, that

(14) $\qquad \mathfrak{X} \leq \mathfrak{X}_k, \qquad\qquad \mathfrak{Z} \leq \mathfrak{Y}_k.$

With Γ_k so defined, let Γ_k' be

$$\mathfrak{X}_k \Vdash \mathfrak{Y}_k.$$

I shall show, by induction on k, that every Γ_k' is derivable. There are two main cases, (α) and (β):

(α) Γ_k is prime. Here there are three subcases:

(α1) - Some constituent of $\mathfrak{X}_k$ is in $\mathfrak{Y}_k$. Then Γ_k' is also prime.

(α2) - Some $\mathfrak{Y}_k$ is an axiom. Then Γ_k' is also prime.

(α3) - Some $\mathfrak{U}_k$ is in $\mathfrak{Y}_k$. Then $\mathfrak{Y}_k$ is of the form $A, \mathfrak{Z}_k'$. Then by (14) we can apply Rule K to (12), and obtain

$$\mathfrak{X}_k \Vdash A, \mathfrak{Z}_k',$$

which is Γ_k'.

(β) Γ_k is derived from premises $\Gamma_{i_1}, \Gamma_{i_2}, \ldots, \Gamma_{i_p}$, by a rule R_k. Again there are three subcases:

(β1) - All constituents of $\mathfrak{U}_k$ are parametric. Then Γ_k' is derivable by the same rule R_k from $\Gamma_{i_1}', \ldots, \Gamma_{i_p}'$.

(β2) - R_k is Wℓ with principal constituent in $\mathfrak{U}_k$ and premise Γ_i. Then Γ_k' is the same as Γ_i'.

(β3) - R_k is not Wℓ and has principal constituent in $\mathfrak{U}_k$. Then if all parametric constituents in $\mathfrak{U}_k$ are omitted the rule is still valid. The premises are then $\Gamma_{i_1}', \Gamma_{i_2}', \ldots, \Gamma_{i_p}'$, while the conclusion is

$$\mathfrak{X}_k, A \Vdash \mathfrak{Y}_k .$$

But by Kℓ and (12) we have (in virtue of (14))

$$\mathfrak{X}_k \Vdash A, \mathfrak{Z} .$$

Hence by the hypothesis of the stage we have Γ'_k.

This completes Stage 1. Note that the case where A is elementary has been disposed of completely because case (β3) cannot then arise (the principal constituent of all rules Oℓ is always compound).

Stage 2. The argument for this stage is partially dual to that of Stage 1. It is only necessary to comment on the points where there is a departure from duality.

The standard form for Γ_k is

$$\mathfrak{X}_k \Vdash \mathfrak{U}_k,\ \mathfrak{Z}_k,$$

where we now suppose simply

$$(15) \qquad \mathfrak{X} \leq \mathfrak{X}_k.$$

The form for Γ'_k is $\qquad \mathfrak{X}_k \Vdash \mathfrak{Y},\ \mathfrak{Z}_k,$

provided $\mathfrak{U}_k$ is not void; if $\mathfrak{U}_k$ is void Γ'_k is the same as Γ_k.

In deriving the Γ'_k by induction there are now four subcases under Case (α), viz.,

(α1) - Some constituent of $\mathfrak{Z}_k$ is in $\mathfrak{X}_k$. Then Γ'_k is also prime.

(α2) - Some $\mathfrak{Z}_k$ is an axiom; then Γ'_k is again prime.

(α3) - A is an axiom. This was disposed of completely in Stage 1.

(α4) - A is in $\mathfrak{X}_k$. Then we have by (11) and Kℓ

$$\cfrac{\cfrac{\mathfrak{X}_k, A \Vdash \mathfrak{Y}}{\mathfrak{X}_k \Vdash \mathfrak{Y}}\,W\ell}{\mathfrak{X}_k \Vdash \mathfrak{Y}, \mathfrak{Z}_k}\,Kr$$

The conclusion is Γ'_k.

In case (β3) the argument is as follows. If $\mathfrak{U}_k$ is void, Γ'_k is the same as Γ_k. If not, then from $\Gamma'_{i_1}, \Gamma'_{i_2}, \ldots, \Gamma'_{i_p}$ and R_k we have

$$\mathfrak{X}_k \Vdash A, (\mathfrak{Y}), \mathfrak{Z}_k,$$

where the parentheses around "$\mathfrak{Y}$" indicate that $\mathfrak{Y}$ does not occur if there are no parametric constituents. By applying K to (11) we have

$$\mathfrak{X}_k, A \Vdash \mathfrak{Y}, \mathfrak{Z}_k.$$

Hence by the hypothesis of the stage we have Γ'_k.

Note that since the case where A is elementary was disposed of in Stage 1, we can exclude the possibility of Er in subcase (β3). Further the entire argument is valid in LA if all $\mathfrak{Z}$'s are void and all $\mathfrak{Y}$'s singular (but $\mathfrak{Z}_k$ can be singular if $\mathfrak{U}_k$ is void)

Stage 3. As stated at the beginning of the proof we suppose here that

$$A = B \circ C,$$

and that (11) is a consequence of a rule $\mathfrak{O}\ell$ and (12) of the corresponding rule Or, A being the principal constituent in both cases. We suppose the theorem true for B and C and prove it for A. Since the theorem was proved for $A \varepsilon \mathfrak{G}$ in Stage 1, this proves the theorem by induction on the number of connectives in A.

Let the premises for the rule $O\ell$ be Γ_1 (and Γ_2 if there are two), and those for the rule Or be Γ_3 (and Γ_4). Then the various possibilities are as follows:

<u>Case P</u>. $A = B \supset C$. Then the premises are

Γ_1	$\mathfrak{X} \Vdash B, \mathfrak{Z}'$,	$\mathfrak{Z}' \leq \mathfrak{Y}$,
Γ_2	$\mathfrak{X}, C \Vdash \mathfrak{Y}$,	
Γ_3	$\mathfrak{X}, B \Vdash C, \mathfrak{Z}$,	$\mathfrak{Z} \leq \mathfrak{Y}$.

From Γ_2, Γ_3 and the hypothesis of the induction

$$\mathfrak{X}, B \Vdash \mathfrak{Y}.$$

Hence by Γ_1, and the hypothesis of the induction

$$\mathfrak{X} \Vdash \mathfrak{Y}.$$

<u>Case Λ.</u> $A = B \wedge C$. Here we have two possibilities for the single premise for $\Lambda\ell$. If we call them Γ_1 and Γ_2 we have in this case

Γ_1	$\mathfrak{X}, B \Vdash \mathfrak{Y}$,	Γ_2	$\mathfrak{X}, C \Vdash \mathfrak{Y}$,
Γ_3	$\mathfrak{X} \Vdash B, \mathfrak{Z}$,	Γ_4	$\mathfrak{X} \Vdash C, \mathfrak{Z}$.

From Γ_1 and Γ_3 (or Γ_2 and Γ_4 as the case may be) and the hypothesis of the induction

$$\mathfrak{X} \Vdash \mathfrak{Y} \qquad \text{q.e.d.}$$

<u>Case V</u>. $A = B \vee C$. In this case the premises Γ_3 and Γ_4 are alternative while Γ_1 and Γ_2 are simultaneous. The premises are

Γ_1	$\mathfrak{X}, B \Vdash \mathfrak{Y}$,	Γ_2	$\mathfrak{X}, C \Vdash \mathfrak{Y}$,
Γ_3	$\mathfrak{X} \Vdash B, \mathfrak{Z}$,	Γ_4	$\mathfrak{X} \Vdash C, \mathfrak{Z}$.

From whichever pair holds, and the hypothesis of the induction, we have

$$\mathfrak{X} \Vdash \mathfrak{Y} \qquad \text{q.e.d.}$$

COROLLARY 1. <u>If</u>

$$\mathfrak{X}_0, A_1, \ldots, A_m \Vdash \mathfrak{Y},$$

and

$$\mathfrak{X}_i \Vdash A_i, \mathfrak{Z}_i, \qquad i = 1,2,\ldots,m;$$

then

$$\mathfrak{X}_0, \mathfrak{X}_1, \ldots, \mathfrak{X}_m \Vdash \mathfrak{Y}, \mathfrak{Z}_1, \ldots, \mathfrak{Z}_m.$$ [17]

This follows by induction on m and Remark 8.

COROLLARY 2. If

$$A_1, \ldots, A_m \vdash B,$$

and

$$\mathfrak{X}, B \Vdash \mathfrak{Y};$$

then

$$\mathfrak{X}, A_1, \ldots, A_m \Vdash \mathfrak{Y}.$$

Proof. By Theorem 1

$$\mathfrak{X}, A_1, \ldots, A_m \Vdash B.$$

The result follows by Theorem 11, with $\mathfrak{Z}$ void, and Kℓ.

8. **The Natural System T.** Beside the systems LA and LC which we have just been studying, Gentzen introduced what he called the natural systems - in fact he introduced these first. Gentzen's idea was to stay as close to actual reasoning as possible. In such actual reasoning we argue from suppositions; having established a thesis on the basis of certain suppositions, we later state the thesis as an implication and discharge the suppositions.

The elementary statement of a natural system is thus of the form "A is valid on the suppositions $\mathfrak{X}$." This is, of course, the same idea as the "$\mathfrak{X} \Vdash A$" of § 2. But the basis of the formalization is quite different. Whereas in § 2 we analyzed the meanings of the logical connectives by specifying the conditions under which they could be introduced into discourse, the objective of the natural system is to formulate as simply as possible the rules for their practical manipulation. In view of this difference of objective I shall introduce a new notation, viz.

$$(16) \qquad A \varepsilon \mathfrak{T}(\mathfrak{X})$$

for the elementary statement of the natural system. In such a statement we call A the subject and $\mathfrak{X}$ the supposition. This amounts to saying that we shall use "$\mathfrak{T}(\mathfrak{X})$" as meaning the class of propositions valid on the suppositions $\mathfrak{X}$. For practical purposes (16) will often be abbreviated, as in the statement of the rules below.

17. For void $\mathfrak{Z}$ this is the scheme "Syllogisimus" of Hertz [89]. Cf. footnote 7 in the Introduction.

The natural system will be called hereafter simply the system TA($\mathfrak{S}$).[18]

In stating rules of inference in TA($\mathfrak{S}$) the following abbreviation is convenient. We shall say that a rule

$$\frac{A_1, \ldots, A_m}{B}$$

holds relative to $\mathfrak{X}$ if the rule

$$\frac{A_1 \varepsilon \mathfrak{S}(\mathfrak{X}), \ldots, A_m \varepsilon \mathfrak{S}(\mathfrak{X})}{B \varepsilon \mathfrak{S}(\mathfrak{X})}$$

is valid. This is convenient when all its statements have the same supposition; it will be used when $m \geqq 1$. When some of the premises involve an extra constituent, it will be indicated in brackets over the subject; thus the rule

$$\frac{A_1, \ldots, A_m, \quad \overset{[B]}{A_{m+1}}}{C}$$

shall be said to hold relative to $\mathfrak{X}$ if and only if the rule

$$\frac{A_1 \varepsilon \mathfrak{S}(\mathfrak{X}), \ldots, A_m \varepsilon \mathfrak{S}(\mathfrak{X}), \quad A_{m+1} \varepsilon \mathfrak{S}(\mathfrak{X}, B)}{C \varepsilon \mathfrak{S}(\mathfrak{X})}$$

holds.

The rules for TA($\mathfrak{S}$) involve certain preliminary rules, a rule of elimination and a rule of introduction for each connective, together with a rule analogous to Er which we treat as a rule of introduction, as follows:

RULES FOR THE SYSTEM TA($\mathfrak{S}$)

Preliminary Rules.

t1) $A \varepsilon \mathfrak{S}(A)$.

t2) If $A \varepsilon \mathfrak{A}$, then $A \varepsilon \mathfrak{S}(0)$.

t3) If $\mathfrak{X} \subseteq \mathfrak{Y}$, then $\mathfrak{S}(\mathfrak{X}) \subseteq \mathfrak{S}(\mathfrak{Y})$.

Rules of Introduction and Elimination.

The following hold relative to any supposition $\mathfrak{X}$,

E1 If $A_1, A_2, \ldots, A_m \vdash B$,

$$\frac{A_1, A_2, \ldots, A_m}{B}$$

18. Gentzen called it NJ, but this conflicts with our use of "N" for negation.

Pe $\dfrac{A, A \supset B}{B}$ Pi $\dfrac{\begin{matrix}[A]\\ B\end{matrix}}{A \supset B}$

Ae $\dfrac{A \wedge B}{A}$ $\dfrac{A \wedge B}{B}$ Ai $\dfrac{A, B}{A \wedge B}$

Ve $\dfrac{\begin{matrix} & [A] & [B]\\ A \vee B, & C, & C\end{matrix}}{C}$ Vi $\dfrac{A}{A \vee B}$ $\dfrac{B}{A \vee B}$

To illustrate the technique of using the TA system, we consider the first example of §6, viz.

$$A \supset .\ B \supset C :\supset: A \supset B .\supset. A \supset C.$$

In view of the rule Pi, this will follow if from the supposition

1. $A \supset .\ B \supset C$

we derive

$$A \supset B .\supset. A \supset C.$$

The latter will be valid, subject to 1, if from 1 and

2. $A \supset B$

we derive

$$A \supset C.$$

This in turn will follow if we derive C from 1, 2, and

3. A.

This last is immediate; thus

From 1 and 3: $B \supset C$.

From 2 and 3: B.

By Pe C.

The whole proof can be written schematically, as Gentzen does it, thus:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\overset{\checkmark}{3}}{A} \quad \overset{\overset{\checkmark}{2}}{A \supset B}}{B}\,\text{Pe} \qquad \dfrac{\overset{\overset{\checkmark}{3}}{A} \quad \overset{\overset{\checkmark}{1}}{A \supset .\ B \supset C}}{B \supset C}\,\text{Pe}}{C}\,\text{Pe}}{A \supset C}\,\text{Pi-3}}{A \supset B .\supset. A \supset C}\,\text{Pi-2}}{A \supset .\ B \supset C :\supset: A \supset B .\supset. A \supset C}\,\text{Pi-1}$$

1. $A \supset .\ B \supset C$
2. $A \supset B$
3. A

Here each horizontal bar represents an inference by the rule written at the right, the supposition indicated by number being discharged. (The discharge of a supposition is shown by a check mark.) The scheme shows exactly under what suppositions each indicated proposition is asserted.[19] The reader should note that *the scheme can be constructed by starting with the proposition to be proved and working upwards, writing the suppositions down underneath, at least as rapidly as the proposition can be tested with truth tables.*[20] The latter gives no information about validity in LA.

The technique can be shortened by establishing the following theorem.

THEOREM 12. The following three statements are equivalent:

(a) $B \,\varepsilon\, \mathfrak{S}(\mathfrak{X}, A_1, A_2, \ldots, A_m)$,

(b) $A_1 \supset . A_2 \supset . \ldots \supset . A_m \supset B \,\varepsilon\, \mathfrak{S}(\mathfrak{X})$,

(c) *the rule*

$$\frac{A_1, A_2, \ldots, A_m}{B}$$

holds relative to any $\mathfrak{Y}$ such that $\mathfrak{X} \subseteq \mathfrak{Y}$.

Proof. If (a) holds then (b) follows by successive applications of Pi, discharging the $A_1, \ldots, A_m$ in inverse order. Suppose (b) holds and $\mathfrak{X} \subseteq \mathfrak{Y}$; then if $A_i \,\varepsilon\, \mathfrak{Y}$ $(i = 1, 2, \ldots, m)$ we have $B \,\varepsilon\, \mathfrak{S}(\mathfrak{Y})$, by successive application of Pe , showing that (c) holds. Finally suppose (c) holds. Since, by t1 and t3, we have

$$A_i \,\varepsilon\, \mathfrak{S}(\mathfrak{X}, A_1, \ldots, A_m),$$

we can apply (c) when $\mathfrak{Y} = \mathfrak{X}, A_1, \ldots, A_m$. This gives (a) as consequence of (c), q.e.d.

Using this theorem we can discharge several suppositions at once. Thus the above scheme could be written:

$$\dfrac{\dfrac{\dfrac{\overset{3\checkmark}{A} \qquad \overset{2\checkmark}{A \supset B}}{B}\,\text{Pe} \qquad \dfrac{\overset{3\checkmark}{A} \qquad \overset{1\checkmark}{A \supset . B \supset C}}{B \supset C}\,\text{Pe}}{C}}{A \supset . B \supset C : \supset : A \supset B \supset . A \supset C}\,\text{Pi-1, 2,3}$$

1. $A \supset . B \supset C$
2. $A \supset B$
3. A

---

19. Note applications of the preliminary rules are tacit. Each proposition in the scheme is valid under all the suppositions written over it plus any others which may be desired. Thus in the second line of the scheme we have B from 2 and 3 but $B \supset C$ from 1 and 3. In order to apply Pe we need to apply t3 to both. But the omission of these uses of t3 causes no real difficulty.

20. This is frequently but not always the case. In some of the proofs given later I have found them more readily by the technique of § 6, but present them in the more compact schemes of the natural system.

Another example is

$$A \supset \therefore A \wedge B \,.\, \supset C \,.\, \supset : A \supset C \,.\, \vee \,.\, B \supset C$$

for which the scheme is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\checkmark}{1}\atop A \qquad \overset{\checkmark}{3}\atop B}{A \wedge B}\,\Lambda i \qquad \overset{\checkmark}{2}\atop A \wedge B \,.\supset .\, C}{C}\,Pe}{B \supset C}\,Pi-3}{A \supset C \,.\, \vee \,.\, B \supset C}\,Vi}{A \supset \therefore A \wedge B \,.\, \supset C : \supset : A \supset C \,.\, \vee \,.\, B \supset C}\,Pi-1,2$$

1. A
2. $A \wedge B \,.\supset .\, C$
3. B

We now establish two theorems concerning the relation of the T-system to the system LA.

THEOREM 13. _If $\mathfrak{Z}$ is a fixed prosequence and we interpret_

(16) $A \; \varepsilon \; \mathfrak{T}(\mathfrak{X})$

to mean

(17) $\mathfrak{X} \Vdash A, \mathfrak{Z}$ in LC($\mathfrak{G}$);

then the rules of TA($\mathfrak{G}$) are valid.

Proof. On the above interpretation the rules t1 and t2 are special cases of p1 and p2, while t3 is a special case of Theorem 2, Corollary 2. The rules Ei and Oi are identical with rules Er and Or respectively. It remains to consider the rules Oe.

Proof of Pe. By P_0 (Theorem 6) and Theorem 2,

$$\mathfrak{X}, A, A \supset B \Vdash B, \mathfrak{Z}.$$

By weakening the second premise of Pe,

$$\mathfrak{X}, A \Vdash A \supset B, \mathfrak{Z}.$$

Hence, eliminating $A \supset B$ by the elimination theorem,

$$\mathfrak{X}, A \Vdash B, \mathfrak{Z}.$$

From this and the first premise of Pe we can similarly eliminate A by the elimination theorem. The result is the conclusion of Pe.

Proof of Λe. We prove the left-hand half of Λe only; the proof of the right half is similar. By p1

$$\mathfrak{X}, A \Vdash A.$$

$\therefore (\Lambda\ell)$ $$\mathfrak{X}, A \wedge B \Vdash A.$$

Eliminating $A \wedge B$ between this and the premise of $\wedge$e, we have the conclusion of $\wedge$e.

<u>Proof of Ve</u>. From the second and third premises and rule Vℓ we have

$$\mathfrak{X},\ A \vee B \Vdash C,\ \mathfrak{Z}.$$

Eliminating $A \vee B$ with the first premise we have the conclusion.

<u>Remark</u>. The statement P_0 adduced from Theorem 6 is derived by the rule Pℓ. Thus in all the three cases the rule Oe is deduced from the corresponding rule Oℓ.

COROLLARY 1. <u>If in TA($\mathfrak{S}$)</u>

$$A \ \varepsilon\ \mathfrak{T}(\mathfrak{X});$$

<u>then</u>

$$\mathfrak{X} \Vdash A \quad \text{in LA}(\mathfrak{S}).$$

<u>Proof</u>. This is the case where $\mathfrak{Z}$ is null.

THEOREM 14. <u>If in LA($\mathfrak{S}$)</u>

$$\mathfrak{X} \Vdash A;$$

<u>then in TA($\mathfrak{S}$)</u>

$$A \ \varepsilon\ \mathfrak{T}(\mathfrak{X}).$$

<u>Proof</u>. In this case we show that if (2) is interpreted as meaning (16) the rules of the LA system are epitheorems of the TA system.

We note that rules C, W, and K are all valid by t3. The rules p1 and p2 follow from t1, t2 and Rule K (or t3). (Cf. Remarks 2 and 3 in § 5.) The rules Er and Or are, as before, identical with Ei and Oi. It remains only to consider the rules Oℓ.

In proving these we shall use Theorem 12 and the technique above described. In the schemes below M1 and M2 are the premises of the rule to be derived. The supposition 1 in each case is the principal constituent in the rule; the parametric constituents are not mentioned - all these inferences are valid relative to an arbitrary $\mathfrak{X}$ as explained above. The right side of the rule is a single proposition which we denote here by "C". We show C is valid on the supposition 1 which proves the rule valid by Theorem 12.

<u>Proof of Pℓ</u>.

$$\dfrac{\dfrac{\overset{\text{M1}}{A} \qquad \overset{1}{A \supset B}}{B}\,\text{Pe} \qquad \dfrac{\dfrac{\overset{2}{B}}{C}\,\text{M2}}{B \supset C}\,\text{Pi - 2}}{C}\,\text{Pe}$$

<u>Proof of $\wedge\ell$</u>. We prove the first half only; the proof of the second half is similar.

$$\dfrac{\dfrac{\overset{1}{A\wedge B}}{A}\,\wedge e \qquad \dfrac{\dfrac{\overset{2}{A}}{C}\,\text{M1}}{A\supset C}\,\text{Pi-2}}{C}\,\text{Pe}$$

<u>Proof of $\vee\ell$</u>.

$$\dfrac{\overset{1}{A\vee B} \qquad \dfrac{\overset{2}{A}}{C}\,\text{M1} \qquad \dfrac{\overset{3}{B}}{C}\,\text{M2}}{C}\,\text{Ve-2,3}$$

To make sure the technique is not misunderstood I give below the proof of $P\ell$, putting in all applications of t1 - t3.

$$
\begin{array}{lrll}
 & A & \varepsilon\ \mathfrak{X}(\mathfrak{x}) & \text{by M1} \\
 & A & \varepsilon\ \mathfrak{X}(\mathfrak{x}, A\supset B) & \text{by t3} \\
 & A\supset B & \varepsilon\ \mathfrak{X}(\mathfrak{x}, A\supset B) & \text{by t1, t3} \\
(18) & B & \varepsilon\ \mathfrak{X}(\mathfrak{x}, A\supset B) & \text{by Pe} \\
 & C & \varepsilon\ \mathfrak{X}(\mathfrak{x}, B) & \text{by M2} \\
 & B\supset C & \varepsilon\ \mathfrak{X}(\mathfrak{x}) & \text{by Pi} \\
 & B\supset C & \varepsilon\ \mathfrak{X}(\mathfrak{x}, A\supset B) & \text{by t3} \\
 & C & \varepsilon\ \mathfrak{X}(\mathfrak{x}, A\supset B) & \text{by (18) and Pe} \quad \text{q.e.d.}
\end{array}
$$

We consider now <u>the system TC</u>. This is defined as the system formed by adjoining to the rules for TA the rule[21]

$$\underline{\text{Pk}} \qquad \dfrac{\begin{matrix}[A\supset B]\\ A\end{matrix}}{A}$$

<u>THEOREM 15. Under the interpretation of Theorem 13 the rules of TC are valid. Hence if (16) is valid in TC, (17) is valid in LC</u>.

<u>Proof</u>. Since the rules of TA are valid in LC by Theorem 13, it remains only to consider Pk. But in the third example of §6 we showed that

$$\Vdash A\supset B\,.\supset A:\supset A \quad \text{in LC.}$$

Hence by Theorem 13 and our interpretation

$$A\supset B\,.\supset A:\supset A\ \varepsilon\ \mathfrak{X}(0).$$

This is equivalent to Pk by Theorem 12.

21. This is intuitively equivalent to Peirce's law. It is known that Peirce's law is sufficient to deduce all classical properties of implication from the intuitionistic ones (cf. footnote below, under Theorem 22). The rule was also stated by Popper [68].

It is possible to get an interpretation of LC in TC in the following manner. Intuitively the statement

$$(19) \qquad \mathfrak{X} \Vdash A,\ Z_1, \ldots, Z_n$$

means the same as

$$(20) \qquad \mathfrak{X} \Vdash A \vee Z_1 \vee Z_2 \vee \ldots \vee Z_n.$$

Now the propositions $A \vee B$ and $A \supset B\ .\ \supset B$ have the same truth table (in the sense of Table 1). Hence the above is equivalent, intuitively, to

$$\mathfrak{X} \Vdash Z_1 \supset A\ .\ \supset A, \ldots, Z_n \supset A\ .\ \supset A,$$

and hence to

$$(21) \qquad \mathfrak{X}, Z_1 \supset A, \ldots, Z_n \supset A \Vdash A.$$

The statements (19) and (21) are in fact equivalent in LC. Indeed from (19) and the prime statement

$$\mathfrak{X}, A \Vdash A, Z_1, \ldots, Z_{n-1}$$

we can get

$$\mathfrak{X}, Z_n \supset A \Vdash A, Z_1, \ldots, Z_{n-1}$$

by Pℓ, and so we can continue until we have (21). On the other hand if we start with (21) then we have, by Pr,

$$\mathfrak{X}, Z_1 \supset A, \ldots, Z_{n-1} \supset A \Vdash Z_n \supset A\ .\ \supset A.$$

But we have

$$Z_n \supset A\ .\ \supset A \Vdash A, Z_n$$

by an argument like that used to prove Pei in § 6. From the last two statements and the elimination theorem we have

$$\mathfrak{X}, Z_1 \supset A, \ldots, Z_{n-1} \supset A \Vdash A, Z_n.$$

In this way we can continue until we have (19).

This, in combination with Theorem 15, leads to

THEOREM 16. A sufficient condition that

$$(19) \qquad \mathfrak{X} \Vdash A, Z_1, \ldots, Z_n \qquad \text{in LC}$$

is that

$$(22) \qquad A \ \varepsilon\ \mathfrak{S}(X, Z_1 \supset A, \ldots, Z_n \supset A) \quad \text{in TC}.$$

Proof. If (22) holds, then (21) does by Theorem 15. This leads to (19) by the argument above.

The necessity of the condition (22) is shown by the following theorem.

THEOREM 17. If we interpret

$$(19) \qquad \mathfrak{X} \Vdash A, Z_1, \ldots, Z_n$$

<u>in the TC system as</u>

(22) $\quad A \ \varepsilon\ \mathfrak{S}(\mathfrak{X}, Z_1 \supset A, Z_2 \supset A, \ldots, Z_n \supset A);$

<u>then the rules of the LC system are valid. Hence if (19) holds, so also does (22).</u>

<u>Proof</u>. For the rules $\Lambda\ell$ and $V\ell$ the proofs in Theorem 14 are valid in the present case. We consider the other rules of LC one by one as follows:

<u>p1</u>. If A is in $\mathfrak{X}$, then this follows by t1 and t3. If some Z_i is in $\mathfrak{X}$, then (22) follows by t3 from

$$A \ \varepsilon\ \mathfrak{S}(Z_i, Z_i \supset A),$$

which in turn follows by Pe and Theorem 12.

<u>p2</u>. If A is an axiom, (22) follows by t2 and t3. If some Z_i is an axiom, then (22) follows, by Theorem 12, from the following argument:

$$\dfrac{\overset{t_2}{Z_i} \qquad\qquad \overset{1}{Z_i \supset A}}{A}\ \text{Pe}$$

<u>$C\ell$</u> and <u>$W\ell$</u> follow at once by t3.

<u>$P\ell$</u>. Let the first constituent in $\mathfrak{Y}$ be C, the other constituents, $Z_1, \ldots, Z_n$. As noted in footnote 11 we can suppose $\mathfrak{Z} \equiv \mathfrak{Y}$. Then the premises of the rule we wish to establish give us the rules

$$\text{M1} \qquad \dfrac{\mathfrak{X}, C \supset A,\ Z_1 \supset A, \ldots, Z_n \supset A}{A},$$

$$\text{M2} \qquad \dfrac{\mathfrak{X}, B,\ Z_1 \supset C, Z_2 \supset C, \ldots, Z_n \supset C}{C}.$$

We wish to show that

$$\dfrac{\mathfrak{X}, A \supset B, Z_1 \supset C, \ldots, Z_n \supset C}{C}.$$

The proof scheme for this is as follows (the $\mathfrak{X}$ is not indicated; cf. above):

$$\dfrac{\dfrac{\dfrac{\overset{2}{A \supset B} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{4}{Z_1} \quad \overset{1}{Z_1 \supset C}}{C}\,\text{Pe} \qquad \overset{3}{C \supset A}}{A}\,\text{Pe}}{Z_1 \supset A}\,\text{Pi-4} \qquad Z_2 \supset A \ldots Z_n \supset A}{A}\,\text{M1}}{B}\,\text{Pe}}{C}\,\text{M2}}{C}\,\text{Pk-3}$$

Similar proof for
$i = 2, 3, \ldots, n$

<u>Cr</u>. (Note that this rule is not trivial in this case.) If only the Z's are permuted the rule follows by t3. We consider the case where A is permuted with one of the Z's, which we can now suppose, without loss of generality, to be the first. Let this be B and let the other Z's be renumbered $Z_1, Z_2, \ldots, Z_n$. Then what we wish to prove is the following: given the rule

$$\text{M1} \quad \frac{\mathfrak{X},\, B \supset A,\, Z_1 \supset A, \ldots, Z_n \supset A}{A},$$

to derive the rule

$$\frac{\mathfrak{X},\, A \supset B,\, Z_1 \supset B, \ldots, Z_n \supset B}{B}.$$

The proof scheme for this is the following:

$$\cfrac{\overset{2}{A \supset B} \qquad \cfrac{\cfrac{\cfrac{\cfrac{\overset{1}{Z_1 \supset B} \quad \overset{4}{Z_1}}{B}\,\text{Pe} \qquad \overset{3}{B \supset A}}{A}\,\text{Pe}}{Z_1 \supset A}\,\text{Pi-4} \qquad Z_2 \supset A, \ldots, Z_n \supset A}{A}\,\text{M1}}{\cfrac{B}{B}\,\text{Pk-3}}\,\text{Pe}$$

Similar proof for $i = 2, \ldots, n$

<u>Wr</u>. If A is not a component, the rule follows at once by t3. If the components are A and Z_n then the premise of the rule is

$$A \in \mathfrak{S}(X, Z_1 \supset A, \ldots, Z_{n-1} \supset A, A \supset A).$$

By Theorem 12 this is equivalent to

$$A \supset A \,.\supset A \in \mathfrak{S}(X, Z_1 \supset A, \ldots, Z_{n-1} \supset A).$$

But since $A \supset A$ is in $\mathfrak{S}(0)$, we have by Pe

$$A \in \mathfrak{S}(X, Z_1 \supset A, \ldots, Z_{n-1} \supset A),$$

which is the conclusion of Wr.

<u>Pr</u>. Here the premise, by Theorem 12, is the rule

$$\text{M1} \quad \frac{\mathfrak{X},\, A,\, Z_1 \supset B, \ldots, Z_n \supset B}{B};$$

the desired conclusion is

$$\frac{\mathfrak{X},\, Z_1 \supset .\, A \supset B, \ldots, Z_n \supset .\, A \supset B}{A \supset B}.$$

The proof scheme is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\cancel{3}}{Z_1} \quad \overset{1}{Z_1 \supset . A \supset B}}{A \supset B}\,\mathrm{Pe} \qquad \overset{\cancel{2}}{A}}{B}\,\mathrm{Pe}}{Z_1 \supset B}\,\mathrm{Pi}-3 \qquad \cdots \qquad Z_n \supset B}{\dfrac{B}{A \supset B}\,\mathrm{Pi}-2}\,\mathrm{M1}.$$

Similar proof for $i = 2,3,\ldots,n$

$\underline{\wedge r}$. Here the premises are

$$\mathrm{M1}\ \frac{\mathfrak{X}, Z_1 \supset A, \ldots, Z_n \supset A}{A}, \qquad \mathrm{M2}\ \frac{\mathfrak{X}, Z_1 \supset B, \ldots, Z_n \supset B}{B}.$$

The desired conclusion is

$$\frac{\mathfrak{X}, Z_1 \supset . A \wedge B, \ldots, Z_n \supset . A \wedge B}{A \wedge B}.$$

The proof scheme is

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\cancel{2}}{Z_1} \quad \overset{1}{Z_1 \supset . A \wedge B}}{A \wedge B}\,\mathrm{Pe}}{A}\,\wedge e}{Z_1 \supset A}\,\mathrm{Pi}-2 \qquad Z_2 \supset A, \ldots, Z_n \supset A}{A}\,\mathrm{M1}.$$

Similar proof for $i = 2,3,\ldots,n$

Thus we derive the scheme

$$\frac{Z_1 \supset A \wedge B, \ldots, Z_n \supset A \wedge B}{A}$$

using M1. By a similar proof, using M2, we show B follows from the same premises. If we combine these schemes by $\wedge i$, we have the desired conclusion.

$\underline{\vee r}$. We prove the first half only; the proof of the second is similar. The premise is

$$\mathrm{M1}\ \frac{\mathfrak{X}, Z_1 \supset A, \ldots, Z_n \supset A}{A}.$$

The desired conclusion is

$$\frac{\mathfrak{X}, Z_1 \supset . A \vee B, \ldots, Z_n \supset . A \vee B}{A \vee B}.$$

The proof scheme is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{Z_1 \supset . A \vee B} \qquad \overset{3}{Z_1}}{A \vee B}\,\text{Pe} \qquad \overset{2}{A \vee B . \supset A}}{A}\,\text{Pe}}{Z_1 \supset A, \ldots, Z_n \supset A}\,\text{Pi-3}}{A}\,\text{M1}}{A \vee B}\,\text{Vi}}{A \vee B}\,\text{Pk-2}$$

Similar proof for $i = 2,3,\ldots,n$.

Remark 1. Note that Pk was required in Cr, $P\ell$, and Vr.

Remark 2. An alternative scheme would be to regard (20), rather than (21), as translation of (19). Then we could justify all the rules of LC except Pr without going outside of TA; Pr could be justified most easily by (5). The present procedure has the advantage that no rule for V or Λ is involved unnecessarily; this leads to Theorem 22 (below).

Remark 3. The proofs in the TC system do not have the same constructive character that proofs in TA have.

We may sum up the preceding investigation in a theorem as follows.

THEOREM 18. *A necessary and sufficient condition that*

$$\mathfrak{X} \Vdash A, Z_1, \ldots, Z_n$$

in either LA or LC, is that in the corresponding T system

$$A \,\varepsilon\, \mathfrak{T}(\mathfrak{X}, Z_1 \supset A, \ldots, Z_n \supset A);$$

it being understood that in the case LA-TA $n = 0$.

The following theorem shows that $\mathfrak{T}(\mathfrak{X})$ has the properties of a Folgerungs - relation in a sense defined by Tarski.[22]

THEOREM 19. *The class $\mathfrak{T}(\mathfrak{X})$ has the properties*

(a) $\mathfrak{A} + \mathfrak{X} \subseteq \mathfrak{T}(\mathfrak{X})$,[23]

(b) *If* $\mathfrak{X} \subseteq \mathfrak{Y}$, *then* $\mathfrak{T}(\mathfrak{X}) \subseteq \mathfrak{T}(\mathfrak{Y})$,

(c) $\mathfrak{T}(\mathfrak{T}(\mathfrak{X})) \subseteq \mathfrak{T}(\mathfrak{X})$,

(d) *If* $A \,\varepsilon\, \mathfrak{T}(\mathfrak{X})$, *then there exists a finite set* $X_1, \ldots, X_m$, *such that* $A \,\varepsilon\, \mathfrak{T}(X_1, \ldots, X_m)$.

Proof. Property (a) follows from t1, t2, t3. Property (b) is precisely t3. Property (d) follows, in the light of the equivalence between L system and the T system, from Theorem 5. As for property (c) if $A \,\varepsilon\, \mathfrak{T}(\mathfrak{T}(\mathfrak{X}))$ then by (d) there exist

22. Tarski [78, 79], cf. [80, 75].

23. Here "+" indicates class addition.

$B_1, \ldots, B_n \in \mathfrak{T}(\mathfrak{X})$ such that $A \in \mathfrak{T}(\mathfrak{X}, B_1, \ldots, B_n)$, while $B_i \in \mathfrak{T}(\mathfrak{X})$. Hence by Theorem 12, $A \in \mathfrak{T}(\mathfrak{X})$.

9. Equivalent Propositional Algebras. Systems HA and HC. A propositional algebra, in the ordinary sense, may be looked at from our present point of view as a subclass $\mathfrak{H}$ of $\mathfrak{P}(\mathfrak{D})$ defined as follows:

(a) A certain class of propositions are postulated as belonging to $\mathfrak{H}$. These will be called the *prime propositions* of $\mathfrak{H}$.

(b) The rule of inference

Ph $$\frac{A \qquad A \supset B}{B}$$

is valid in the sense that whenever A and B are in $\mathfrak{P}(\mathfrak{D})$, and the propositions on the top line belong to $\mathfrak{H}$, that on the bottom line does also.

Various schemes of prime propositions may determine the same class $\mathfrak{H}$. We do not distinguish different methods of specifying prime propositions as giving different algebras but rather as different formulations of the same algebra. Thus "prime proposition" is relative to the formulation.

We consider here algebras HA and HC which are equivalent to the systems LA($\mathfrak{D}$) and LC($\mathfrak{D}$), respectively, in the sense that H is the class of propositions A such that $\Vdash$A holds in the corresponding L-system. We consider the problem of characterizing these algebras. We shall see that HA and HC consist of those propositions of the intuitionistic and classical propositional algebras respectively which can be expressed with our connectives.

THEOREM 20. *A set of prime propositions for the algebra HA consists of those obtained from the schemes PK, PS, ΛK, ΛK', Λ_1, VK, VK', V_1 of Theorem 6 by taking for A, B, C any propositions of $\mathfrak{P}(\mathfrak{D})$.*[24]

Proof. Let $\mathfrak{H}$ be propositions generated by Ph from the prime propositions stated. Then, in view of Theorems 13 and 14, it is to be shown that $\mathfrak{H}$ is the same as $\mathfrak{T}(\mathfrak{D})$.

That all the propositions in the above schemes are in $\mathfrak{T}(\mathfrak{D})$ was shown in Theorem 6. That the rule of inference is valid for $\mathfrak{T}(\mathfrak{D})$ was shown in Theorem 12. Hence $\mathfrak{H} \subseteq \mathfrak{T}$.

It remains to show that $\mathfrak{T} \subseteq \mathfrak{H}$. To do this, let $\mathfrak{H}(\mathfrak{X})$ be the class generated by the rule of inference if we adjoin $\mathfrak{X}$ to the prime propositions stated. I shall then show that the rules of TA system are verified if we interpret $\mathfrak{T}(\mathfrak{X})$ as $\mathfrak{H}(\mathfrak{X})$. This will be done for the various rules as follows:

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24. It follows that the algebra HA is the positive part of the propositional algebra of Heyting. See the introduction, footnote 3. This is the positive logic of Hilbert-Bernays [47].

<u>t1</u>. Clear by definition.

<u>t2</u>. Vacuous, since $\mathfrak{S}$ here is $\mathfrak{O}$.

<u>t3</u>. Clear since a derivation in $\mathfrak{H}(\mathfrak{X})$ is also a derivation in $\mathfrak{H}(\mathfrak{Y})$.

<u>Pe</u>. This is Ph.

<u>Λe</u>. Follows by schemes ΛK and ΛK'.

<u>Ve</u>. We reduce this to Pi. If Pi holds then the premises of Ve are $A \supset C$, $B \supset C$, and $A \vee B$; from these we have C by V_1.

<u>Pi</u>. This is the famous deduction theorem.[25] To prove it we first establish the scheme

$$A \supset A. \tag{23}$$

In fact, if we put A for C in PS we have

$$A \supset . B \supset A :\supset: A \supset B . \supset . A \supset A.$$

Hence by PK and the rule of inference

$$A \supset B . \supset . A \supset A.$$

Here B is arbitrary. Taking $B \supset A$ for B and applying PK we have (23).

This established, let $B \varepsilon \mathfrak{H}(\mathfrak{X}, A)$. Then there is a sequence of propositions $B_1, B_2, \ldots, B_n$ such that B_n is B, and every B_k is either A, or is in $\mathfrak{X}$, or is a prime proposition, or is obtainable from B_i and B_j[25a] by Ph. I show by induction that for every k, $A \supset B_k$ is in $\mathfrak{H}(\mathfrak{X})$. If B_k is A this follows by (23). If B_k is in $\mathfrak{X}$, or is a prime proposition, it follows by PK. Finally, if B_k is obtained by Ph from B_i and B_j, then B_j must be $B_i \supset B_k$ (or vice versa). From PS we have

$$A \supset . B_i \supset B_k :\supset: A \supset B_i . \supset . A \supset B_k$$

By the hypothesis of the induction and Ph (twice) we have $A \supset B_k \varepsilon \mathfrak{H}(\mathfrak{X})$.

<u>Λi</u>. Suppose $A \varepsilon \mathfrak{H}(\mathfrak{X})$, $B \varepsilon \mathfrak{H}(\mathfrak{X})$. Then by PK, for any C, $C \supset A$ and $C \supset B$ are in $\mathfrak{H}(\mathfrak{X})$. Hence, by Λ_1, $C \supset . A \wedge B$ is in $\mathfrak{H}(\mathfrak{X})$. Here C is arbitrary; hence take $C = A \supset A$ and apply (23).

<u>Remark</u>. The proof would be easier if we replaced Λ_1 by Λ_2.

<u>Vi</u>. Follows by VK and VK'.

THEOREM 21. <u>A set of prime propositions for the algebra HC consists of those for HA together with either Pei or (5)</u>.

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25. Cf. Hilbert-Bernays [47] p. 155 and Church [14] pp. 9-11. Church has kindly furnished, in a letter, the following information concerning the theorem. The earliest appearance of the theorem, with proof, is in Herbrand [42]. The property Pi is stated as a postulate, Axiom 8*, by Tarski in [79].

25a. Where $i < k$ and $j < k$.

Proof. Let $\mathfrak{h}(\mathfrak{x})$ be the class of propositions generated by adjoining Pei to HA. Then, since HA $\subseteq$ HC, Pei ε HC (Theorem 6), and Ph is valid in HC, we have $\mathfrak{h}(0) \subseteq$ HC. Hence in TC, $\mathfrak{h}(\mathfrak{x}) \subseteq \mathfrak{t}(\mathfrak{x})$. The converse follows, as in the proof of Theorem 20, since all rules of TC except Pk are justified by HA, and Pk follows at once from Pei.

As for (5) it follows from Pei, as follows. Let

$$C \equiv A \vee . A \supset B .$$

Then we have the proof scheme

$$\cfrac{\cfrac{\cfrac{\cfrac{\overset{1}{C \supset A} \qquad \cfrac{\overset{2}{A \supset B}}{C}\,\mathrm{Vi}}{A}\,\mathrm{Pe}}{A}\,\mathrm{Pk}-2}{C}\,\mathrm{Vi}}{C}\,\mathrm{Pk}-1$$

On the other hand, one can derive Pei from (5) in TA, thus

$$\cfrac{\cfrac{\overset{1}{A \vee . A \supset B} \qquad \cfrac{\overset{2}{A \supset B . \supset A} \qquad \overset{3}{A \supset B}}{A}\,\mathrm{Pe} \qquad \overset{4}{A}}{A}\,\mathrm{Ve}-3,4}{A \supset B . \supset A . \supset A}\,\mathrm{Pi}-2$$

THEOREM 22. In any derivation in either HA or HC, the only prime propositions used, in addition to those involving implication only, are from among those relating to connectives which actually appear in the final result.[26]

Proof. This follows from Theorem 10 and from the fact that in establishing the validity of any rule of the L-system, when interpreted from the standpoint of the H-system, we have actually used only prime formulas conforming to the statement in the theorem.

COROLLARY. The propositions of HA which contain implication only are derivable from PK and PS; those of HC from these and Pei.[27]

10. **Concluding Remarks.** We have studied two systems LA($\mathfrak{S}$) and LC($\mathfrak{S}$). The first of these arose by making very precise

26. On this theorem Cf. [22] and Wajsberg [84]. At least a part of the result was mentioned in a letter from Bernays in 1932. Cf. [47] p. 70f.

27. The first of these results was mentioned in Bernay's letter of 1932 and ascribed to Łukasiewicz. It is related to the fact that all combinators can be defined in terms of S and K. The second is a theorem of Tarski-Bernays. Cf. [47] pp. 70f. and [59] Satz 29.

definitions of what we mean in introducing the logical connectives. It is thus semantical and constructive in character. It can therefore be said to represent truthfully the properties of formal deducibility.

On the other hand the system LC arose by a formal analogy with LA. It was probably suggested to Gentzen, as later to Carnap, by a certain lack of duality in LA. This theory was developed in parallel with LA, and most theorems of LA could be extended to it. Thus, although this system has no justification semantically, yet we are able to prove Theorem 8 for it. It is thus just as consistent with $\mathfrak{S}$ as is LA.

Related to the system LA is the system TA of §8. This was called the natural system because it reflects the properties we assume almost instinctively in using our connectives. But to deduce these properties from the definition was a long and tedious process. It made use essentially of the fact that the rules of the underlying system have the form (6). Now there are systems for which the rules do not have that form. Systems with mathematical induction are an example. For such systems the proof of the elimination theorem, and hence the derivation of the system TA, does not hold; and indeed if we were to prove these by any such methods, we should run into conflict with the theorem of Gödel [38].

The intuitionistic algebra HA was invented by the intuitionists because it was in agreement with their constructionist tendencies. But Gödel [40] showed that their arithmetic is no more constructive than the classical. The preceding discussion throws light on the reason for this. Indeed <u>the really nonconstructive moment in the theory of arithmetic is the tacit assumption of the rule of inference, and hence of the elimination theorem</u>.

It will be seen from this that the attachment of the word "natural" to the system TA is, in one sense, a misnomer. Indeed the word "natural" has two senses: 1) as related to the essential nature of the object, and 2) as reflecting what we do instinctively. In the former of these senses a Napierian logarithm is natural, but hardly in the latter. In the former of these senses <u>the natural system of formal deducibility is the system LA</u>.