Preface

This book was originally written in Japanese for undergraduate students in the Department of Mathematics of Saitama University. In fact, the first hand-written draft was prepared for a series of lectures on the viscosity solution theory for undergraduate students in Ehime University and Hokkaido University.

The aim here is to present a brief introduction to the theory of viscosity solutions for students who have knowledge on Advanced Calculus (*i.e.* differentiation and integration on functions of several-variables) and hopefully, a little on Lebesgue Integration and Functional Analysis. Since this is written for undergraduate students who are not necessarily excellent, I try to give "easy" proofs throughout this book. Thus, if you do not feel any difficulty to read User's guide [6], you should try to read that one.

I also try not only to show the viscosity solution theory but also to mention some related "classical" results.

Our plan of this book is as follows: We begin with our motivation in section 1. Section 2 introduces the definition of viscosity solutions and their properties. In section 3, we first show "classical" comparison principles and then, extend them to viscosity solutions of first- and second-order PDEs, separately. We establish two kinds of existence results via Perron's method and representation formulas for Bellman and Isaacs equations in section 4.

We discuss boundary value problems for viscosity solutions in sections 5. Section 6 is a short introduction to the L^p-viscosity solution theory, on which we have an excellent book [4].

In Appendix, which is the hardest part, we give proofs of fundamental propositions.

In order to learn more on viscosity solutions, I give a list of "books":

A popular survey paper [6] by Crandall-Ishii-Lions on the theory of viscosity solutions of second-order, degenerate elliptic PDEs is still a good choice for undergraduate students to learn first. However, to my experience, it seems a bit hard for average undergraduate students to understand.

Bardi-Capuzzo Dolcetta's book [1] contains lots of information on viscosity solutions for first-order PDEs (Hamilton-Jacobi equations) while Fleming-Soner's [10] complements topics on second-order (degenerate) elliptic PDEs with applications in stochastic control problems.

Barles' book [2] is also nice to learn his original techniques and French language simultaneously !

It has been informed that Ishii would write a book [15] in Japanese on viscosity solutions in the near future, which must be more advanced than this.

For an important application via the viscosity solution theory, we refer to Giga's [12] on curvature flow equations. Also, I recommend the reader to consult Lecture Notes [3] (Bardi-Crandall-Evans-Soner-Souganidis) not only for various applications but also for a "friendly" introduction by Crandall, who first introduced the notion of viscosity solutions with P.-L. Lions in early 80s.

If the reader is interested in section 6, I recommend him/her to attack Caffarelli-Cabré's [4].

As a general PDE theory, although there are so many books on PDEs, I only refer to my favorite ones; Gilbarg-Trudinger's [13] and Evans' [8]. Also as a textbook for undergraduate students, Han-Lin's short lecture notes [14] is a good choice.

Since this is a text-book, we do not refer the reader to original papers unless those are not mentioned in the books in our references.

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