Preface

The birational classification theory of higher-dimensional algebraic varieties is one of the most important subjects in algebraic geometry. We need to construct, study, and calculate various birational invariants for the classification. In the opinion of the author, these invariants related to the canonical divisor K_X of a variety X are most essential for the purpose of recovering the information vital to determine the variety X.

In dimension one, for example, the (geometric) genus $p_g(X) = \dim |K_X| + 1$ characterizes the nature of a non-singular projective curve X. In arbitrary dimension, the theory of Kodaira dimension by Iitaka [**43**], [**44**] tells us that the plurigenera $P_m(X) = \dim |mK_X| + 1$ for $m > 0$ and the Kodaira dimension $\kappa(X)$ determined by $P_m(X)$ also characterize the nature of a variety X.

The minimal model theory for higher-dimensional varieties also requires the information on the canonical divisor at the heart of its construction. For example, the following assertion gives the first step to construct minimal models: 'if K_X is not nef, then there exist an extremal ray and its contraction morphism.'

The author thinks that the canonical divisor can be compared to the 'navel' of a variety in the sense that the navel is an origin of a human body.

We list the following important conjectures arising from the study of canonical divisors:

- The canonical ring is a finitely generated algebra over the base field;
- Iitaka's conjecture $C_{n,m}$: $\kappa(X) \geq \kappa(X/Y) + \kappa(Y)$ for an algebraic fiber space $X \to Y$;
- Existence and termination of flips in the minimal model program;
- The abundance conjecture: K_X is semi-ample for a minimal model X;
- Deformation invariance of plurigenera, etc.

The study of canonical divisors has two sides: One is the study of properties valid not only for the canonical divisor but also for all the divisors in general. The other is finding some theorems, valid specifically for the divisors close to the canonical divisor, from Hodge theory, from the analytic methods for complex analytic varieties, or from the Frobenius maps of schemes of characteristic $p > 0$.

The study of the divisors in general is an old subject in algebraic geometry. The notion of linear systems originally comes from the classical projective geometry. A linear system on a variety parametrizes effective divisors linearly equivalent to a fixed divisor. The rational map into a projective space defined by the linear system is important for analyzing the structure of the variety. Iitaka's theory of D-dimension begins with the study of the complete linear systems $|mD|$ for all natural numbers m. Compared to the properties depending on the linear equivalence class, the numerical properties, i.e., the properties depending on the numerical equivalence class, however, are still not well understood. In particular, it is usually hard to show the expected boundedness properties of certain important numerical invariants. The existence of the Zariski-decomposition is directly related to such boundedness. The decomposition is a numerical analogue of the decomposition $|D| = |M| + F$ as the sum of the movable part $|M|$ and the fixed part F of a linear complete system $|D|$. Zariski [**151**] constructed and studied the decomposition of effective divisors in the case of surfaces. In higher dimensions, we need to blow up the variety for getting the decomposition in general. But it is not clear that we can get the decomposition after a finite number of such blowups. The Zariski-decomposition is a useful tool for the birational geometry if it exists. For example, Kawamata [**57**] showed that the existence for the canonical divisor implies the finite generation of the canonical ring of a variety of general type.

Looking at the other side, we expect some positivity result for the canonical divisor. For example, as the abundance conjecture predicts, we expect that many members in pluricanonical systems $|mK_X|$ exist in such a way that the m-genus $P_m(X)$ behaves like $m^{\nu(X)}$ as a function of m, where $\nu(X)$ is the numerical Kodaira dimension of X.

The Kodaira vanishing theorem [**67**] on projective varieties is derived from the E_1-degeneration of Hodge spectral sequence. The vanishing theorem and its generalization by Kawamata [**51**] and Viehweg [**146**] are powerful tools to analyze the divisors; for example, the Riemann-Roch formula and the vanishing enable us to compute the dimension of the space of global sections, in particular the dimension of a linear system, satisfying some numerical positivity. In some cases, they guarantee the existence of members in the linear system, enough to reconstruct the variety from their data.

The theory of variation of Hodge structure is important for the study of fiber spaces. The weak positivity of the direct images of relative pluricanonical sheaves shown by Viehweg [**147**] is based on the positivity of Hodge filtration studied by Griffiths [**32**]. The weak positivity is essential for the proof of $C_{n,m}$ in some cases.

The Calabi conjecture proved by Yau [**150**] and the Kobayashi–Hitchin correspondence between stable and Einstein-Hermitian vector bundles proved by Donaldson [**12**] and by Uhlenbeck and Yau [**142**] give us strong results, which are usually not derived by algebraic methods. They are related to the abundance conjecture above (cf. [**83**]).

The E_1-degeneration of the Hodge spectral sequence for varieties of characteristic zero was proved in [**11**] by an argument using the Frobenius maps . The existence of rational curves for a non-singular projective variety X with K_X not nef was proved by Mori [**85**], [**86**], in which a deformation theory combined with the Frobenius maps plays an important role. There seems to be no other known method proving the existence of rational curves even now.

This article is a revised version of the RIMS-preprint [**104**] of the same title, supplemented by further two preprints [**105**] and [**106**]. Here, we shall discuss mainly the following four topics: Zariski-decomposition, numerical D-dimension, addition theorems, and invariance of plurigenera.

The author had tried to show the existence of Zariski-decomposition of arbitrary pseudo-effective $\mathbb{R}$-divisors since the mid '80s ([**100**], [**101**]), and found counterexamples for big divisors in 1994 ([**103**]). He also considered numerical D-dimension in the mid '80s ([**100**]). The abundance conjecture has a meaning even for non-minimal varieties. Some addition theorems analogous to Iitaka's conjecture $C_{n,m}$ are proved in some logarithmic situation. They induce an abundance theorem for the case of numerical Kodaira dimension zero. The argument for invariance of plurigenera in this article is a kind of algebraic modification and improvement of Siu's argument in [**130**], and is not a direct continuation of the old papers [**96**], [**98**] of the author.

Many results treated in the present book are published for the first time, though they have been circulating in the preprint forms for quite some time. Some readers will find some of the results here quite new and will find some notation different from the usual convention.

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