

**Institute of Mathematical Statistics**  
**LECTURE NOTES—MONOGRAPH SERIES**

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# **Invariant Measures on Group and Their Use in Statistics**

**Robert A. Wijsman**

*University of Illinois at Urbana-Champaign*



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Hayward, California**

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