An Ω–relation is an expression $X = Y$ where X and Y are Ω–words. Then if S is an Ω–semigroup, $X = Y$ is either true or false in S. Now let R be a finite set of Ω–relations and let K be an Ω–relation. Then $R \Rightarrow K$ means that K is true in every Ω–semigroup in which all of the relations in R are true. The word problem for Ω–semigroups is to find an algorithm by which, given R and K, we can decide if $R \Rightarrow K$.

We shall show that the word problem for Ω–semigroups is unsolvable. (This was proved independently by Post and Markov.) Let W' be the symmetric process constructed above. Let R consist of the relations $X = Y$ such that $X \rightarrow Y$ is in W' (and hence $Y \rightarrow X$ is in W'). We shall show that $X \Rightarrow_{W'} Y$ iff $R \Rightarrow X = Y$. Hence the word problem for Ω–semigroups is unsolvable even for this particular R.

Clearly $X \Rightarrow_{W'} Y$ implies $R \Rightarrow X = Y$. To prove the implication in the other direction, we construct an Ω–semigroup. First note that the relation $X \Rightarrow_{W'} Y$ between X and Y is an equivalence relation on the class of Ω–words; this follows from the fact that W' is symmetric. Let X^* be the equivalence class of X. Let S be the set of all these equivalence classes; and define a binary operation $\cdot$ on S by $X^* \cdot Y^* = (XY)^*$ (where XY is X followed by Y). A little thought shows that $(XY)^*$ depends only on the equivalence classes X^* and Y^*; so our definition makes sense. It is easy to see that S is then a semigroup; the unit element is the equivalence class of the empty word.

We make S into an Ω–semigroup by letting the symbol a represent a^*; the word X then represents X^*. If $X = Y$ is in R, then X and Y are equivalent; so $X^* = Y^*$; so $X = Y$ is true in S. It follows that if $R \Rightarrow X = Y$, then $X = Y$ is true in S and hence $X \Rightarrow_{W'} Y$. This completes our proof.

11. Undecidable Theories

We shall see how some problems of the following type can be shown to be

unsolvable: find an algorithm by which we can decide if a given sentence is derivable from a system of axioms. The approach given here is due to Tarski. Although we include all necessary definitions, the reader will probably need some familiarity with first–order theories and their models to see what is going on.

A language is a finite set L of symbols, each of which is designated as either a k–ary relation symbol or a k–ary function symbol for some k. A 0–ary function symbol is called a constant. A stucture M for L then consists of the following: (a) a non–empty class $|M|$, called the universe of M; (b) for each k–ary relation symbol R in L, a subset R_M of $|M|^k$; (c) for each k–ary function symbol F in L, a mapping F_M of $|M|^k$ into $|M|$. Members of $|M|$ are called individuals of M. Note that if c is a constant, then c_M is a mapping of $|M|^0$ onto $|M|$ and hence can be identified with an individual of M.

Let L be a language. We introduce an infinite sequence of symbols called variables. We use x, y, and z for variables. We introduce some expressions, called terms, by the following rules: (a) a variable is a term; (b) if F is a k–ary function symbol and $t_1,...,t_k$ are terms, then $F(t_1,...,t_k)$ is a term. We use s and t for terms.

An atomic formula is either an expression of the form $s = t$, or an expression of the form $R(t_1,...,t_k)$, where R is a k–ary relation symbol. The formulas are obtained by the rules: (a) an atomic formula is a formula; (b) if ϕ and ψ are formulas, then $\neg\phi$, $\phi \vee \psi$, $\phi \,\&\, \psi$, $\phi \rightarrow \psi$, and $\phi \longleftrightarrow \psi$ are formulas; (c) if ϕ is a formula, then $\exists x\phi$ and $\forall x\phi$ are formulas.

An occurrence of x in a formula ϕ is bound if it occurs in a part of ϕ of the form $\exists x\psi$ or $\forall x\psi$; otherwise it is free. A sentence is a formula with no free variables.

Let M be a structure for L. If t is a term and each variable in t represents a particular individual of M, then t represents a particular individual of M. If ϕ is a formula and each free variable of ϕ represents a particular individual of M,

the ϕ is either true or false in M. In particular, if ϕ is a sentence, the ϕ is either true or false in M.

A theory consists of a language L and a set of sentences in L; these sentences are called the axioms of the theory. If T is a theory, a model of T is a structure M for the language of T such that every axiom of T is true in M. A theorem of T is a sentence of the language of T which is true in every model of T. The decision problem for a theory T is the following: find an algorithm for deciding if a given sentence of the language of T is a theorem of T. If this problem is unsolvable, we say that T is undecidable.

A theory T' is a finite extension of a theory T if T' is obtained from T by adding a finite number of new axioms.

11.1. PROPOSITION. If T' is a finite extension of T and T' is undecidable, then T is undecidable.

Proof. Let ψ be the conjunction of the axioms added to T to get T'. For every ϕ, ϕ is a theorem of T' iff $\psi \to \phi$ is a theorem of T. Hence a solution to the decision problem for T would give a solution to the decision problem for T'. □

A structure M is strongly undecidable if every theory having M as a model is undecidable. We shall construct such a structure.

Let L be the language whose only member is the binary function symbol $\cdot$. Then every semigroup is a structure for L. Now let Ω be an alphabet, and let L_Ω be obtained from L by adding all the members of Ω as new constants. Then every Ω–semigroup is a structure for L_Ω. We identify an Ω–word $a_1 \ldots a_n$ with the term $a_1 \cdot \ldots \cdot a_n$ of L_Ω; so every Ω–relation is an atomic formula of L_Ω. (Strictly speaking, we should tell how to insert parentheses in the term $a_1 \cdot \ldots \cdot a_n$; but the way in which this is done is immaterial because of the associative law.)

Now let R and S be as at the end of §10. Then R is a finite set of Ω–relations such that there is no algorithm for deciding, given X and Y, whether

$R \Rightarrow X = Y$; and S is an Ω–semigroup such that for all X and Y, $X = Y$ holds in S iff $R \Rightarrow X = Y$. We consider S as a structure for L_Ω. We shall show that S is strongly undecidable.

Let T have S as a model. By 11.1, it is sufficient to prove that some finite extension T' of T is undecidable. We obtain T' from T by adding as new axioms the two semigroup axioms,

$$\forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z))$$

and

$$\exists y \forall x(y \cdot x = x \,\&\, x \cdot y = x),$$

and the members of R. Then S is a model of T'.

We show that $X = Y$ is a theorem of T' iff $R \Rightarrow X = Y$; this will clearly show that T' is undecidable. If $X = Y$ is a theorem of T', then it is true in S; so $R \Rightarrow X = Y$. Now let $R \Rightarrow X = Y$; we must show that $X = Y$ is true in every model S' of T'. But S' is clearly an Ω–semigroup in which every member of R is true; so $X = Y$ is true in S'.

If we consider S as a structure for L, we have a new structure, and we cannot immediately conclude that it is strongly undecidable. However, we shall show that this is the case.

If every symbol of the language of M is a symbol of the language of M' and $F_M = F_{M'}$ for every symbol F of the language of M, we say that M' is an expansion of M. If, in addition, every symbol of M' which is not a symbol of M is a constant, we say that M' is an inessential expansion of M.

11.2. PROPOSITION. If M' is an inessential expansion of M and M' is strongly undecidable, then M is strongly undecidable.

Proof. Let M be a model of T. Obtain T' from T by adding the constants of M' not constants of M as new symbols (but adding no new axioms). Clearly M' is a model of T'; so T' is undecidable. It therefore suffices to show that a solution of the decision problem for T would give a solution of the decision problem for T'.

Let ϕ be a sentence of T'. Then there is a formula ψ of T such that ϕ is obtained from ψ by replacing the free occurrences of $x_1,...,x_k$ by $c_1,...,c_k$, where $c_1,...,c_k$ are new constants. To prove the desired result, it suffices to show that ϕ is a theorem of T' iff $\forall x_1...\forall x_k\psi$ is a theorem of T. If $\forall x_1...\forall x_k\psi$ is a theorem of T, it is a theorem of T'; so ϕ is a theorem of T'. Suppose ϕ is a theorem of T'. We must show that if N is a model of T, then ψ is true in N when $x_1,...,x_k$ represent individuals of N. Expand N to N' by letting $(c_i)_{N'}$ be the individual represented by x_i. Then N' is a model of T'; so ϕ is true in N'; so ψ is true in N when $x_1,...,x_k$ represet $(c_1)_{N'},...,(c_k)_{N'}$. □

It follows that S considered as a structure for L is strongly undecidable. This gives two interesting undecidable theories. The first has the language L and the two semigroup axioms as its axioms; it is called the theory of semigroups. The second has the language of L and no axioms.

We now introduce our main method for obtaining new strongly undecidable structures from old ones. Let ϕ be a formula of M, and let $x_1,...,x_k$ include all the variables free in ϕ. Let R be the k–ary relation in $|M|$ defined by: $R(a_1,...,a_k)$ holds iff ϕ is true in M when $x_1,...,x_k$ represent $a_1,...,a_k$ respectively. We say that R is the relation defined in M by ϕ using $x_1,...,x_k$. Note that if R is defined in M by ϕ using $x_1,...,x_k$, then for any $y_1,...,y_k$, R is defined in M by some ϕ' using $y_1,...,y_k$. A relation is definable in M if it is defined in M by some formula using some sequence of variables.

Let M and M' be structures such that $|M| \subseteq |M'|$. Then M is definable in M' if $|M|$ is definable in M'; R_M is definable in M' for every relation symbol R of M; and the graph of F_M is definable in M' for every function symbol F of M. We shall show that if this is the case and M is strongly undecidable, then M' is strongly undecidable.

A formula of M is special if every atomic formula occuring in ϕ is either of the form $x = y$ or $R(x_1,...,x_k)$ or of the form $F(x_1,...,x_k) = y$. For each formula ϕ

of M, we shall construct a special formula ϕ_s of M such that ϕ is true in M iff ϕ_s is true in M for every assignment of meanings to the free variables. If will clearly suffice to do this for atomic ϕ. We use induction on the number n of occurrences of function symbols in ϕ. If $n = 0$, ϕ_s is ϕ. Otherwise, there is a formula ψ having n–1 occurrences of function symbols such that ϕ results from ψ by replacing the free occurrences of x by $F(x_1,...,x_k)$. We then take ϕ_s to be $\exists x(x = F(x_1,...,x_k)\ \&\ \psi_s)$.

Now suppose that M is definable in M'. For each formula ϕ of M we shall construct a formula ϕ^* of M' such ϕ^* is true in M' iff ϕ is true in M when the free variables are assigned meanings in $|M|$. We can suppose that ϕ is special; otherwise we replace ϕ by ϕ_s. Let ψ be an atomic formula in ϕ not of the form $x = y$. If ψ is of the form $R(x_1,...,x_k)$, then R is definable by some χ in M' using $x_1,...,x_k$. Replace ϕ by χ. A similar procedure takes care of the case that ψ is $F(x_1,...,x_k) = y$. Finally, replace each part $\exists x\psi$ or $\forall x\psi$ of ϕ by $\exists x(\chi\ \&\ \psi)$ or $\forall x(\chi \rightarrow \psi)$ where χ is a formula such that $|M|$ is defined by χ in M' using x. The resulting formula is ϕ^*. (The reader unfamiliar with logic is advised to check all the details.)

We are going to define a finte set Q of sentences true in M' with the following property: if N' is a structure for the language of M' in which all the sentences in Q are true, then there is a structure N for the language of M such that N is definable in N' using the same formulas used to define M in M'. It will follows that ϕ^* is true in N' iff ϕ is true in N. Let $|M|$ be defined by χ in M' using x. If F is a k–ary function symbol in the language of M, let the graph of F_M be defined by ψ_F in M' using $x_1,...,x_k,y$. The sentences of Q must insure that the set defined by χ in N' is non–empty and that the relation defined by ψ_F in N' is the graph of a function. Let $\chi(t)$ be obtained from χ by replacing the free occurrences of x by t; and let ψ'_F be obtained from ψ_F by replacing the free occurrences of y by y'. Then Q contains the sentence $\exists x\chi$, and, for each F, the

two sentences

$$\forall x_1 \ldots \forall x_k(\chi(x_1) \,\&\, \ldots \,\&\, \chi(x_k) \to \exists y(\chi(y) \,\&\, \psi_F))$$

and $\forall x_1 \ldots \forall x_k \forall y \forall y'(\chi(x_1) \,\&\, \ldots \,\&\, \chi(x_k) \,\&\, \chi(y) \,\&\, \chi(y') \,\&\, \psi_F \,\&\, \psi'_F \to y = y')$.

11.3. PROPOSITION. If M is definable in M' and M is strongly undecidable, then M' is strongly undecidable.

Proof. Let M' be a model of T'; we must show that T' is undecidable. Since Q is finite and every sentence in Q is true in M', we may suppose by 11.1 that the sentences of Q are axioms of T'. Let T be the theory with the language of M whose axioms are all ϕ such that ϕ^* is a theorem of T'. For any such ϕ, ϕ^* is true in M'; so ϕ is true in M. Thus M is a model of T; so T is undecidable.

It is thus sufficient to show that a solution to the decision problem for T' would give a solution of the decision problem for T. We show this by showing that ϕ is a theorem of T iff ϕ^* is a theorem of T'. If ϕ^* is a theorem of T', then ϕ is an axiom of T and hence a theorem of T. Suppose that ϕ is a theorem of T; we must show that ϕ^* is true in every model N' of T'. Let N be definable in N' by the same formulas used to define M in M'. It is enough to show that ϕ is true in N; and for this, it is enough to show that N is a model of T. If ψ is an axiom of T, then ψ^* is a theorem of T'; so ψ^* is true in N'; so ψ is true in N. □

We now construct a theory PO. The language of PO consists of a binary relation symbol $<$. Then axioms of PO are

$$\forall x \neg(x < x)$$

and

$$\forall x \forall y \forall z(x < y \,\&\, y < z \to x < z).$$

A model of PO is called a partially ordered set. (We have chosen to present partially ordered sets in terms of the $<$ relation; it would make no essential difference if we used the $\leq$ relation instead.)

We shall construct a strongly undecidable partially ordered set. Recall that we have constructed a strongly undecidable structure M whose language

consists of a binary function symbol F. Hence by 11.2 and 11.3, it will suffice to construct a model M' of PO such that M is definable in an inessential extension of M'.

Let $M_1 = |M| \cup \{1,2,3\}$, where 1,2,3 are objects not in $|M|$. Let M_2 be the set of ordered pairs (x,i) where $x \in |M|$ and $i \in \{1,2,3\}$. Let M_3 be the set of ordered triples (x,y,z) such that $x,y,z \in |M|$ and $F_M(x,y) = z$. Let $|M'| = M_1 \cup M_2 \cup M_3$. We define $<_{M'}$ as follows. If $x \in M_1$, $w <_{M'} x$ is false for all w. If $\langle x,i\rangle \in M_2$, then $w <_{M'} \langle x,i\rangle$ holds for $w = x$ and $w = i$. If $\langle x,y,z\rangle \in M_3$, then $w <_{M'} \langle x,y,z\rangle$ if w is one of $\langle x,1\rangle$, $\langle y,2\rangle$, $\langle z,3\rangle$, x, y, z, 1, 2, or 3. Clearly M' is a partially ordered set.

For $x,y,z \in |M'|$ we have

$$x \in |M| \leftrightarrow \forall y \neg (y < x) \;\&\; x \neq 1 \;\&\; x \neq 2 \;\&\; x \neq 3,$$

$$F(x,y) = z \leftrightarrow x,y,z \in |M| \;\&\; \exists u \exists x_1 \exists y_1 \exists z_1 (x < x_1 \;\&\; y < y_1 \;\&\; z < z_1 \;\&\; 1 < x_1 \;\&\; 2 < y_1 \;\&\; 3 < z_1 \;\&\; x_1 < u \;\&\; y_1 < u \;\&\; z_1 < u).$$

(In proving the second equivalence from right to left, one should first note that we must have $x_1,y_1,z_1 \in M_2$ and $u \in M_3$.) It follows easily that M is definable in M'', where M'' is an inessential expansion of M' formed by adding three new constants to represent 1, 2, and 3.

It follows that PO is undecidable. It also follows that a theory whose language consists of one binary relation symbol and which has no axioms is undecidable.

Many other strongly undecidable structures can be constructed by these methods. However, the proof that M is definable in M' often requires a very detailed analysis of M and M'.

12. Relative Recursion

Let Φ be a set of *total* functions. We generalize the notion of computable to allow us to use the values of the functions in Φ at any arguments we wish in the course of the computation. Following Turing, we picture the computation as