# DECIDABILITY QUESTIONS FOR THEORIES OF MODULES

F. POINT

## §0. Introduction.

Which finite rings with identity have a decidable theory of unitary left modules? This question has been raised by S. Burris and R. McKenzie in their paper on decidable varieties with modular congruence lattices. They showed that if a locally finite variety with modular congruence lattice does not decompose as a product of a discriminator variety and an affine variety, then it interprets the theory of all finite graphs. Then, they reduced the problem of classifying the decidable locally finite affine varieties to the problem of classifying the finite rings which have a decidable theory of modules.

First, we will see how this question arises in the context of decidable locally finite varieties. Then, we will restrict our attention to the decidability question for theories of modules. We will establish a connection between the decidability of the theory of modules over a finite-dimensional algebra and the representation type of that algebra.

This leads to the following questions: what are the relationships

- between the theory of $R$-modules and the theory of finitely generated $R$-modules?
- between theories of modules which are Morita equivalent?

## §1. Locally finite varieties.

A *variety* is a class of $L$-structures, where the language $L$ only contains function symbols, defined by some set of equations (or equivalently closed under products, substructures and homomorphisms). A variety is *locally finite* if every finitely generated algebra is finite.

S. Burris and R. McKenzie proved a decomposition theorem for decidable locally finite varieties with modular congruence lattice. They show that it decomposes as the product of a discriminator variety and an affine variety. (See [B,M]).

R. McKenzie and M. Valeriote generalized their decomposition theorem for decidable locally finite varieties. Before stating the result of McKenzie and Valeriote, we make this notion of decomposition precise.

DEFINITION. *$\mathcal{V} = \mathcal{A} \otimes \mathcal{B}$ means that $\mathcal{V}$ is generated by $\mathcal{A}$ and $\mathcal{B}$ and that there exists a term $\tau(x,y)$ such that $\mathcal{A} \models \tau(x,y) = x$ and $\mathcal{B} \models \tau(x,y) = y$. If $M \in \mathcal{V}$, there exist unique up to isomorphism $A \in \mathcal{A}$, $B \in \mathcal{B}$ such that $M \cong A \times B$.*
*If $\mathcal{V} = \mathcal{A} \otimes \mathcal{B}$, then $\mathcal{V}$ is decidable iff $\mathcal{A}$ and $\mathcal{B}$ are decidable.* (See [B,M].)

THEOREM 1. (See [M,V].) *If $\mathcal{V}$ is a locally finite decidable variety, then there are three subvarieties of $\mathcal{V}$, $\mathcal{A}$, $\mathcal{S}$ and $\mathcal{D}$ such that $\mathcal{V} = \mathcal{A} \otimes \mathcal{S} \otimes \mathcal{D}$ where $\mathcal{A}$ is an affine variety, $\mathcal{S}$ is a strongly abelian variety and $\mathcal{D}$ is a discriminator variety.*

This theorem reduces the question of classifying the decidable locally finite varieties to classifying the decidable locally finite discriminator, strongly abelian and affine varieties. M. Valeriote settled the question for decidable locally finite strongly abelian varieties.

*A. Strongly abelian varieties.*

DEFINITION. *An algebra $A$ is strongly abelian if for every term $t$ and tuples $\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}$ from $A$ such that length($\overline{a}$) = length($\overline{c}$) and length($\overline{b}$) = length($\overline{d}$) = length($\overline{e}$), then*

$$(t(\overline{a},\overline{b}) = t(\overline{c},\overline{d}) \Rightarrow t(\overline{a},\overline{e}) = t(\overline{c},\overline{e})).$$

EXAMPLE. Any algebra which language only contains unary functions and constants is strongly abelian.

Valeriote in his thesis characterized the decidable locally finite strongly abelian varieties.

DEFINITION. *Let $W$ be a multi-sorted unary variety.*
*$W$ is linear if for all non constant terms $s(x), t(y)$ where $x$ and $y$ are of the same sort, there exists $w(z)$ such that*

$$\begin{array}{ll} \textit{either} & W \models t(x) = \omega(s(x)) \\ \textit{or} & W \models s(x) = \omega(t(x)). \end{array}$$

THEOREM 2. (See [V].) *A locally finite strongly abelian variety is decidable iff it is bi-interpretable with a multi-sorted linear unary variety.*

*B. Discriminator varieties.*

DEFINITION. *A discriminator variety $\mathcal{V}$ is a variety such that there is a class $\mathcal{K}$ included in $\mathcal{V}$ and a ternary term $t(x,y,z)$ such that $\mathcal{V}$ is generated by $\mathcal{K}$ and for every $A \in \mathcal{K}$,*

$$\begin{aligned} t^A(x,y,z) &= z \quad \text{if } x = y \\ &= x \quad \text{if } x \neq y. \end{aligned}$$

EXAMPLES. Boolean algebras ($\mathcal{K}$ consists of the Boolean algebra with 2 elements), rings satisfying $x^m = x$.

THEOREM 3. (See [W], [B,W].) *Every finitely generated discriminator variety with finite language is decidable.*
*A finitely generated discriminator variety $\mathcal{V}$ is decidable iff $Th(\mathcal{K})$ is decidable where $\mathcal{K}$ is the class of simple algebras in $\mathcal{V}$.*

Burris and Werner used the sheaf representation of a discriminator variety and a generalization of a technique of Ershov for bounded Boolean powers of a finite structure.

Recent progress has been made by Burris, McKenzie and Valeriote concerning the question of which are the decidable locally finite discriminator varieties. The first observation is that the class of simple elements is definable and so decidable. Therefore by theorem 1, it can be decomposed as the product of a discriminator variety, a strongly abelian variety and an affine variety. By a result of Burris on undecidability of iterated discriminator variety, no discriminator term can appear (see [B]). They settled the question for homogeneous discriminator varieties where the class of simple elements is strongly abelian. (See [B,M,V].)

Added in proof: Valeriote and Willard settled the question for $\mathcal{K}$ an affine variety. The corresponding discriminator variety $\mathcal{V}$ is decidable if and only if $\mathcal{K}$ is polynomially equivalent to a variety of left modules over a finite semi-simple ring. (See [V,W].)

*C. Affine varieties.*

An affine variety $\mathcal{A}$ is polynomially equivalent to a variety of unitary left $R$-modules and there exists a term $t(x,y,z)$ such that $t(x,y,z) = x - y + z$ (see [F,M]). If $\mathcal{A}$ is locally finite, then $R$ is finite.

The problem of classifying the decidable locally finite affine varieties effectively reduces to the problem of determining which are the finite rings $R$ for which the variety of unitary left $R$-modules is decidable. (See [B,M].)

We will examine this question in more details, but first let us come back to theorem 1.

McKenzie and Valeriote show that if the locally finite variety $\mathcal{V}$ does not decompose as $\mathcal{A}\otimes\mathcal{S}\otimes\mathcal{D}$ then it interprets the class of all (finite) graphs. (Actually they interpreted $BP^1$ which is a subclass of the class of Boolean pairs: $(A,S) \in BP^1$ if $A$ is atomic and the atoms of $A$ are included in $S$). Which finishes the proof by the result of Lavrov who proved that the set of sentences true in all graphs and the set of sentences which become false in some finite graph are recursively inseparable (see [L]).

Now we are going to make a digression to point out the connection between having few models and being decidable for strongly abelian varieties.

DEFINITION. *Let $\mathcal{K}$ be a class of $L$-structures and let $I(\mathcal{K},\lambda)$ be the number of non isomorphic models in $\mathcal{K}$ of cardinality $\lambda$. Then $\mathcal{K}$ has few models if there exists $\lambda > |L|$ such that $I(\mathcal{K},\lambda) < 2^\lambda$.*

A consequence of the proof of McKenzie and Valeriote is that if $\mathcal{V}$ is a locally finite variety which has few models, then $\mathcal{V}$ decomposes as $\mathcal{A}\otimes\mathcal{S}\otimes\mathcal{D}$. We may eliminate $\mathcal{D}$ since any non trivial discriminator variety contains an algebra whose complete theory is unstable.

For the strongly abelian varieties, we have the following theorem:

THEOREM 4. (See [H,V].) *Let $\mathcal{V}$ be a strongly abelian variety. Then $\mathcal{V}$ has few models iff $\mathcal{V}$ is bi-interpretable with a multi-sorted unary variety which is linear and has the ascending chain condition.*

So for a strongly abelian locally finite variety, the properties of having few models and being decidable coincide.

Let $R$ be a countable ring and $\mathcal{M}_R$ be the variety of all unitary left $R$-modules. Baldwin and McKenzie showed that if $I(\mathcal{M}_R, \lambda) < 2^\lambda$ for some $\lambda$, then every left $R$-module is $\omega$-stable. (See [Ba,M].) This implies that every left module is a direct sum of indecomposable submodules that are finitely generated (see [G]). If the ring $R$ is an Artin algebra (in particular if either $R$ is finite or if $R$ is a finite-dimensional algebra) then this implies that there are only finitely many indecomposable modules. (See [Pr1].) (An Artin algebra is a ring which is finitely generated as a module over its center and its center is an artinian ring).

So we see that having few models has some relationships with being decidable. But in case of modules as we shall see, it is a too strong property.

From now on, we will concentrate on the following question: for which finite rings $R$ is the theory of unitary left $R$-modules decidable?

## §2. Locally finite affine varieties.

Let $R$ be a ring with unity 1. Let $L = \{+, -, 0, r; r \in R\}$, $r$ is viewed as a unary function symbol. Let $T_R$ be the theory of unitary left $R$-modules.

Baur gave the first example of a finite ring $R = \mathbb{Z}/p^9\mathbb{Z}[x]/(x^2)$ for which $T_R$ is undecidable ($p$ is any prime number). (See [Ba1].) So the theory of $\mathbb{Z}[x]$-modules is undecidable. (The theory of $\mathbb{Z}$-modules is decidable; see [Sz].)

THEOREM 5. (See [Ba].) *The theory $T$ of pairs of abelian groups of exponent $p^9$ is undecidable.*

The idea of the proof is to interpret in a finite extension of $T$ the word problem for a finitely presented semi-group on two generators.

Let $R = \mathbb{Z}/p^9\mathbb{Z}[x]/(x^2)$.

COROLLARY. *$T_R$ is undecidable.* (See [Ba1].)

PROOF. Let $M \models T_R$. Define $\mathcal{A}_M = (\ker x, \operatorname{im} x)$. Then $\mathcal{A}_M \models T$. Let $(\mathcal{A}, B) \models T$. Let $\mathcal{B}_1$ be an isomorphic copy of $\mathcal{B}$. Set $M = \mathcal{A} \times \mathcal{B}_1$. Define $x.a = 0$ for all $a \in \mathcal{A}, x.b_1 = b$ where $b_1$ is sent to $b$ by the isomorphism between $\mathcal{B}_1$ and $\mathcal{B}$. So we have a faithful interpretation of $T$ in $T_R$.

COROLLARY. (See [Ba1].)

1. *The theory of pairs of $k[x]$-modules is undecidable, where $k$ is a finite field.*
2. *The theory $T_{k[x,y]}$ of $k[x,y]$-modules is undecidable, where $k$ is a finite field.*

Let $\mathcal{C} = \{(V, W, f) : V, W$ are $k$-vector spaces, $V \supseteq W, f \in \mathrm{End}(V)\}$. Then $\mathrm{Th}(\mathcal{C})$ is undecidable. (This follows from the point 1 of the Corollary above.) Let $k$ be a field.

DEFINITIONS. *Let $A$ be a $k$-algebra i.e., $A$ is a ring with unity and a left $k$-module satisfying:*

$$r(x.y) = (r.x).y = x.(r.y) \quad \text{for all } x, y \in A, r \in k.$$

*Let $J$ be the Jacobson radical of $A$ i.e., $J = \bigcap\{M : M$ is a maximal left ideal of $A\}$.*
*$A$ is local if $A/J \cong k$.*
*$A$ is left artinian if $A$ has the descending chain condition on left ideals.*

PROPOSITION 1. (See [P1].) *Let $A$ be a commutative artinian ring. Let $\mathcal{I}$ be the set of maximal ideals of $A$ and $A_I$ the localization of $A$ by $I, I \in \mathcal{I}$. Suppose that $A_I$ is a $k$-algebra over some field $k$. Then $T_A$ is decidable iff each $T_{A_I}$ is decidable, for all $I \in \mathcal{I}$.*

PROPOSITION 2. (See [P1].) *Let $A$ be a local artinian commutative $k$-algebra. Suppose that characteristic of $k \neq 2$ or that $k$ is finite.*
*Then, either*

1. *$A$ has a residue ring isomorphic to $k\langle x, y, z\rangle/I_1$, where $I_1$ is the two-sided ideal generated by all monomials of degree 2, and $T_A$ is undecidable.*

2. *$A$ has a residue ring isomorphic to $k\langle x, y\rangle/(x^2, xy - yx, y^2x, y^3)$ and $T_A$ is undecidable.*

3. *$A$ is isomorphic to $B_{n,m} = k\langle x, y\rangle/I$ where $I$ is the two-sided ideal generated by $x.y$, $y.x$, $x^n$, $y^m$ with $n + m \geq 5$.*

4. *Let $\overline{k}$ be a quadratic extension of $k$ and $\overline{A} = A \otimes \overline{k}$. Then $\overline{A}$ is isomorphic to $\overline{B}_{n,m} = \overline{k}\langle x, y\rangle/I$.*

5. *$A$ is isomorphic to $B_2 = k\langle x, y\rangle/I_2$ where $I_2$ is generated by $x^2$, $y^2$, $xy - yx$.*

6. *$A$ is isomorphic to $B_{2,2} = k\langle x, y\rangle/I_{2,2}$ where $I_{2,2}$ is generated by $x^2, y^2, x.y, y.x$.*

7. *$A$ is isomorphic to $B_1(n) = k\langle x\rangle/I_3$ where $I_3$ is the two-sided ideal generated by $x^n$.*

*In cases 5, 6 and 7, $T_A$ is decidable.*

The idea of the proof of cases 1 and 2 is the following one:
Let $T$ be the theory of pairs $(V, W)$ of $k[x]$-modules with $V \supseteq W$. We show that there exists a finite extension $T^\star$ of $T_A$ such that in any model $M$ of $T^\star$ we can define a pair $(V, W)$ of $k[x]$-modules with $V \supseteq W$, where the action of $x$ is definable and every model of $T$ is of that form.

This proof is inspired by the proof of Drozd of wildness of these algebras (see [D1]). He showed that there is an exact and faithful embedding of the finite-dimensional elements of $\mathcal{C}$ into the class of finitely generated modules over each of these algebras.

Since there is a link between the representation type of a finite-dimensional algebra and the decidability of the theory of modules over it, for sake of completeness, we shall give the definition of the various kinds of representation types for a finite-dimensional $k$-algebra. (If $k$ is not algebraically closed, tensor up with the algebraic closure of $k$.)

DEFINITION. (See [R1].) *Let $R$ be a finite-dimensional $k$-algebra and let $R$-mod be the class of finitely generated left $R$-modules. Assume $k$ is algebraically closed.*

1. *$R$ is of finite representation type if there are only finitely many indecomposable elements (up to isomorphism) in $R$-mod.*

2. *$R$ is of tame representation type provided $R$ is not of finite representation type and for any dimension $d$ there is a finite number of embedding functors $F_i : k[x]$-mod $\to R$-mod such that all but a finite number of indecomposable finitely generated $R$-modules of dimension $d$ are of the form $F_i(M)$ for some $i$ and for some indecomposable finitely generated $k[x]$-module $M$.*
   *In case there exists (independently of $d$) a finite number of such embedding functors $F_i$, then $R$ is domestic.*
   *If $R$ is tame, not domestic but there is a finite bound on the number of functors needed, then $R$ is of finite growth.*
   *If $R$ is tame, not domestic and not of finite growth, then $R$ is infinite growth.*

3. *$R$ is of wild representation type if there is a functor from $k\langle x,y\rangle$- mod which preserves and reflects indecomposability and isomorphism.*

(See also [Pr1]. This definition can be phrased for an arbitrary $k$-algebra.)

REMARKS.

1. Let $R$ be a finite-dimensional algebra over an algebraically closed field $k$. Then either $R$ is of finite representation type or else there are infinitely many $d_i$'s such that there are infinitely many pairwise non isomorphic indecomposable $R$-modules of dimension $d_i$, for all $i$. (Those indecomposable are indexed by the elements of $k$). (See [N,R].)

2. Every finite-dimensional $k$-algebra of infinite representation type is either tame or wild but not both. (See [D2].)

EXAMPLES. Let $\tilde{k}$ be the algebraic closure of $k$.

1. Then $\tilde{k} \otimes B_1(n)$ is of finite representation type.
   (The indecomposable finitely generated modules are isomorphic to $\tilde{k}[x]/(x)^m$, $m \leq n$.
2. Then $\tilde{k} \otimes B_2$ and $\tilde{k} \otimes B_{2,2}$ are of tame domestic representation type. (See [G,P2].)
3. Then $\tilde{k} \otimes B_{n,m}$, $n + m \geq 5$, are of tame, infinite growth representation type. (See [G,P1].)

Using the classification of Ringel of the representation type of local $k$-algebras, $k$ an algebraically closed field, one shows:

PROPOSITION 3. (See [P1].) *Let $R$ be a local, complete $k$-algebra with $k$ an algebraically closed field. If $R$ is of wild representation type and $R$ is not a residue ring of the group algebra of the generalized quaternion algebra, then $T_R$ is undecidable.*

Recent work by A. Marcja, M. Prest and C. Toffalori showed undecidability results for wild classes of modules over group rings of the form $\mathbf{Z}/p^k\mathbf{Z}[G]$, where $G$ is a $p$-group. (See [M,P,T].) Their notion of a wild class is the following: for some field $K$ one can interpret in it a class of $K\langle x,y\rangle$-modules including the finite-dimensional $K\langle x,y\rangle$-modules in such a way that finite-dimensional $K\langle x,y\rangle$-modules are interpreted in structures of the class which are "finite-dimensional" in some sense (e.g. finitely generated).

Now we come to the decidability proofs which go deeper in the structure of the models.

First we are going to recall some notions of module theory.

DEFINITIONS. *A module $M$ is pure-injective (p.i.) if every system of equations with parameters in $M$ which is finitely satisfiable in $M$, has a solution in $M$. A p.p.-formula $\varphi(y,\bar{b})$ is a formula of the form*

$$\exists \bar{z} \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} \bar{z} \\ \\ y \end{pmatrix} = \begin{pmatrix} \\ \bar{b} \\ \\ \end{pmatrix},$$

*where $A$ is a matrix with coefficients in $R$ and $\bar{b}$ are parameters from a module $M$. If $\bar{b} = \bar{0}$, the set $\{y \in M : M \models \varphi(y,\bar{0})\}$ is a (p.p. definable) subgroup of $M$; $\varphi(y,\bar{0})$ asserts the solvability of a finite system of linear equations with parameter $y$.*
*An invariant sentence is of the form $(\varphi,\psi) \geq k$, where $k \in \mathbf{N} - \{0\}$, $\varphi, \psi$ are p.p. formulas without parameters, $\psi \rightarrow \varphi$ and $(\varphi,\psi) \geq k$ means that the subgroup defined by $\psi$ has index $\geq k$ in the subgroup defined by $\varphi$.*

To prove decidability of $T_R$, you may expect to use the result of Ziegler which roughly says that if you know the space of pure-injective indecomposable $R$-modules with its topology, then $T_R$ is decidable.

THEOREM 6. (See [Z].) *Let $R$ be a ring which is finitely presented as an $R$-module with decidable word problem. Suppose that there is a recursive enumeration of all those conditions of the form $\bigwedge_i(\varphi_i, \varphi_j) \in [m, n)$ which are satisfied by some pure-injective indecomposable $R$-module, where $n, m \in \mathbb{N} \cup \{+\infty\}, m \neq +\infty$ and $(\varphi_i)$ a recursive enumeration of all the p.p. formulas. Then $T_R$ is decidable.*

Going back to Proposition 2, one has for cases (5) and (6):

(a) $T_{B_{2,2}}$ is decidable iff $T_{B_2}$ is decidable.

(b) $T_{B_{2,2}}$ is decidable since it is interpretable in the theory of quadruples (i.e., the theory of a vector space and four subspaces).

THEOREM 7. (See [Ba2].) *The theory of $k$-vector spaces with four subspaces is decidable, whenever the theory of $k[x]$-modules is decidable.*

THEOREM 8. (See [E,F].) *Let $k$ be a recursive field with decidable word problem and a splitting algorithm (i.e., an algorithm which determines for any element of $k[x]$ its irreducible factors). Then the theory of $k[x]$-modules is decidable.*

In his proof of decidability of the theory of quadruples, Baur described the $\aleph_1$-saturated models and used the description of finitely generated indecomposable models given by Gelfand and Ponomarev (see [G,P2]).

This case is the cutting line. The theory of a vector space and five subspaces is undecidable, and the theory of a vector space and three subspaces is of course decidable (the proof in this case is much simpler). (See [Ba3].)

Extending the result of Baur on quadruples, Prest proved:

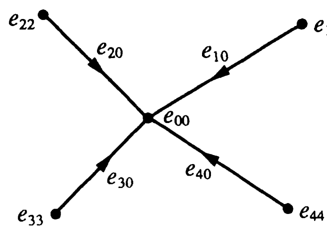
THEOREM 9. (See [Pr2].) *Let $\Delta$ be a quiver without relations, the underlying diagram of which is extended Dynkin. Then the theory of modules over the path algebra $k[\Delta]$ is decidable.*

DEFINITIONS.

1. *A quiver $\Delta$ is a finite directed graph with no oriented cycles.*

2. *The basis of $k[\Delta]$ as a $k$-vector space is the set of all oriented paths of $\Delta$ which includes the path of length zero at each vertex. The product of two basis elements is composition of paths when defined and zero otherwise.*
   *The paths of length zero compose as "local identities".*

The theory of quadruples over the field $k$ can be translated into the theory of modules over the path algebra of the quiver:




or over the algebra

$$\begin{pmatrix} k & k & k & k & k \\ & k & \cdot & \bigcirc & \cdot \\ & & k & \cdot & \cdot \\ & \bigcirc & & k & \cdot \\ & & & & k \end{pmatrix}$$

and vice-versa.

Roughly, the decidability proof of Prest for $k[\Delta]$ is done taking the proof that $k[\Delta]$ is of tame domestic representation type, replacing indecomposable finitely generated modules by pure-injective indecomposable modules (to prove that he obtains all of them he uses the density of the set of regular indecomposable modules) and showing that the functors respects p.p. definable subgroups. So the fact that the topology of the space of indecomposable pure-injective modules over $k[x]$ is given explicitly transfers to the space of indecomposable pure-injective modules over $k[\Delta]$.

Gabriel established the connection between finite-dimensional algebras and path algebras with relations (a relation is a linear combination of sums of paths with same starting point and same end point which is declared to be zero).

THEOREM 10. (See [Ga].) *Over an algebraically closed field, any finite-dimensional algebra is Morita equivalent to a path algebra over a quiver with relations.*

THEOREM 11. (See [P,Pr].) *Over sufficiently decidable rings, decidability of the theory of modules is a Morita invariant.*

Do there exist a transfer theorem linking the representation type of a finite-dimensional algebra $R$ and the decidability of $T_R$?

A more modest question is: since the representation type concerns the class of finitely generated modules or of finitely presented modules, do we have $T_R = T_R^{f.p.}$, where $T_R^{f.p}$ is the theory of finitely presented $R$-modules? (If $R$ is artinian, then $T_R^{f.p.} = T_R^{f.g.}$, where $T_R^{f.g.}$ is the theory of finitely generated $R$-modules.

PROPOSITION 4. (See [P,Pr].) *If $R$ is of finite representation type, then $T_R = T_R^{f.p.}$.*

THEOREM 12. (See [P,Pr].) *If $R$ is the path algebra over a quiver without relations, of infinite representation type, then $T_R \neq T_R^{f.p.}$.*

THEOREM 13. (See [P,Pr].) *Let $R$ be a noetherian local $k$-algebra of infinite representation type. Then $T_R \neq T_R^{f.p.}$.*

COROLLARY. *Let $R$ be a finite-dimensional commutative $k$-algebra of infinite representation type. Then $T_R \neq T_R^{f.p.}$.*

The motivating idea is the following: Let $R = \mathbf{Z}$, let $\sigma =$

$$\exists v(p.v = 0 \wedge v \neq 0) \wedge \forall v \, \exists \omega(v = p.\omega).$$

$\mathbf{Z}_{p^\infty} \models \sigma$ and if an abelian group satisfies $\sigma$, it is infinitely generated. Thus, $T_{\mathbf{Z}} \neq T_{\mathbf{Z}}^{f.g.}$ Replacing $p$ by $x - \alpha$, $\alpha \in k$, we have the same result for $R = k[x]$ i.e., $T_{k[x]} \neq T_{k[x]}^{f.g.}$.

Returning to Proposition 2 and the decidability question for $T_R$ with $R$ a commutative artinian local $k$-algebra, we have

(a) in cases (1), (2) of Proposition 2, both $T_R$ and $T_R^{f.g.}$ are undecidable;

(b) in cases (5), (6), both $T_R$ and $T_R^{f.g.}$ are decidable. (See [P2].)

For (a), we use a theorem of Slobodskoi (see [S]) that the word problem for the class of finite groups is undecidable. This implies that the theory of finite-dimensional vector-spaces with two endomorphisms is undecidable. This implies that the theory of pairs of finite-dimensional $k$-vector spaces with an endomorphism is undecidable (see [Pr1], Corollary 17.7).

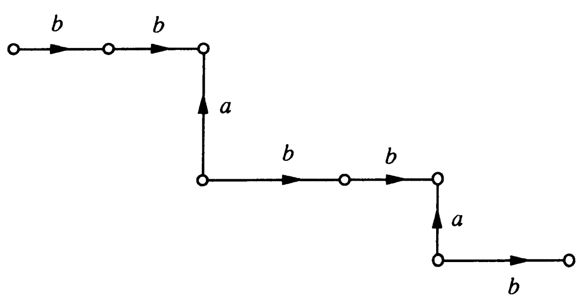
For (b), we use the fact that the theory of finitely generated $k[x]$-modules is decidable and the proof of Prest of Baur's result on the theory of quadruples.

To finish, we would like to make some comments about case (3) of Proposition 2. In that case, there are $2^{\aleph_0}$ pure-injective indecomposable $R$-modules (see [Pr1]).

Let $R = k\langle a, b\rangle/(a^n, b^m, ab, ba)$ with $n + m \geq 5$. The finitely generated indecomposable modules have been described by Gelfand and Ponomarev (see [G,P1]). They are of two types: string and band.

Let us give an example of each type (suppose $n = 2, m = 3$).

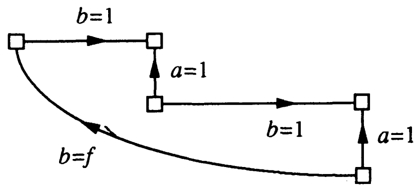
A *string* module $M(C)$ where $C = b^{-2}ab^{-2}ab^{-1}$:




The points represent isomorphic copies of an one dimensional $k$-subspace $V$,

$M$ is the direct sum of those and the actions of $a$ and $b$ are represented by the arrows.

A *band* module $M(C)$ where $C = b^{-1}ab^{-1}ab^{-1}$:




The boxes represent isomorphic copies of a $q$-dimensional $k$-subspace $V$ and $M$ is the direct sum of those; $f$ is an automorphism of $V$ such that $(V, f)$ is an indecomposable $k[x, x^{-1}]$-module. The actions of $a$ and $b$ are represented by the arrows.

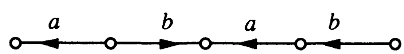
An analogous case is the case of the dihedral algebra:

$$R = k\langle a, b\rangle / (a^2, b^2, (ab)^n, (ba)^m)\,.$$

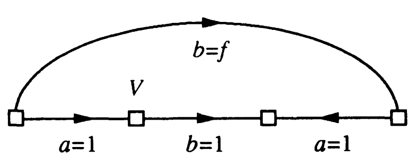
The finitely generated indecomposable have been described by Ringel (see [R2]). They are of two types: string and band.

We give an example of each type (with the same interpretation of the diagrams).

A *string* module $M(C)$ where $C = ab^{-1}ab$:




A *band* module $M(C)$ where $C = a^{-1}b^{-1}ab$:




$f \in \mathrm{Aut}(V)$ and $(V, f)$ is an indecomposable $k[x, x^{-1}]$-module.

Now we would like to give an idea of a tentative decidability proof of the theory of $R$-modules in both cases. (See [P3].)

A word will be a finite or infinite one-sided sequence of letters belonging to $\{a, a^{-1}, b, b^{-1}\}$. Let $W_a$ be the set of words beginning by an $a$ or $a^{-1}$.

In any $R$-module, one can attach to an element $x$ an ordered pair of two infinite one-sided words belonging to $W_b \times W_a$ (see [R2]). One can introduce a partial order on the words (see [R2]) and so on the pairs of words.

An element $x$ is maximal in $M$ if one cannot decompose $x$ into a sum of 2 elements $x_1, x_2$ which have a strictly smaller pair of words associated to them.

NOTATION. Let $(D, C) \in W_b \times W_a$. We write $D$ from the right to the left and we replace each letter by its inverse. We denote this new sequence by $D^-$. $D^- \frown C$ is the concatenation of $D^-$ and $C$.

The strategy of the proof in both cases is the following:

1. We show that each module is elementarily equivalent to a direct sum of pure-injective indecomposable modules containing a maximal element.
2. We give two criteria of independence:
   Let $x, y$ be two maximal elements belonging to a pure-injective module.
   (a) Suppose that for all $(\alpha, \beta) \in k^2 - \{(0,0)\}$, the pair associated with $\alpha x + \beta y$ is $(D, C)$ and that $\alpha x + \beta y$ is maximal. Suppose that $D^- \frown C$ is not an infinite periodic two-sided word. Then $x$ and $y$ belong to distinct summands of $M$.
   (b) Let $(F, E)$ be the pair of words associated with $y$ and $(D, C)$ be the pair of words associated with $x$.
   Suppose that $F^- \frown E \neq D^- \frown C$, then $x$ and $y$ belong to distinct summands of $M$.
3. We describe the pure-injective indecomposable modules containing a maximal element. There are 2 types of them, the string ones associated with a word (finite, one-sided infinite, two-sided infinite), the band ones associated with a finite word of even length and an indecomposable $k[x, x^{-1}]$-module where the automorphism $x$ is given by the action of the word.
4. Then we show that we only need invariant sentences of a certain kind (i.e., those associated with a pair of finite words) to distinguish, up to elementary equivalence, between two pure-injective non-periodic indecomposable modules containing a maximal element.

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