A TRANSFINITE VERSION OF PUISEUX'S THEOREM, WITH APPLICATIONS TO REAL CLOSED FIELDS

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Abstract. Extending the effective version of Puiseux's theorem, we compute the roots of polynomials inside all fields of the form $k((x))^{\Gamma}$, where k is real closed and Γ divisible. We use this computation to prove that every real closed field has an integer part, that is, a discrete subring which plays for the field the same role as $\mathbf{Z}$ plays for $\mathbf{R}$.

Puiseux's theorem in its detailed form allows one to compute the roots of a polynomial, inside the field of "Puiseux series"; Lemma 3.6 and Remark 3.7 below generalize this computation to the case of a field $k((x))^{\Gamma}$, where k is a real closed field and Γ a divisible ordered abelian group, even if Γ is non-archimedian.

Two applications are given: the truncation lemma of F. Delon (see 3.5 below), and the existence of an "integer part" in every real closed field (see 1.4 below). Another proof of these results appears in [MR], but without the generalization of Puiseux's theorem which is one of the chief interests of this paper. The first author has developed extensions and other applications of such computations in [M].

§1. Definitions, remarks.

1.1. Let us denote by $\widetilde{K}$ the real closure of a totally ordered field K.

1.2. We say that a subring Z of a ring A is an *integer part* of A if it is discrete and if for any $x \in A$, there is $z \in Z$ such that $z \leq x < z+1$. We call this unique element z the *integer part* of x and write $z = [x]$.

1.3. S. Boughattas showed in [B] that on the one hand, every totally ordered field has an ultrapower endowed with an integer part, and on the other hand, there are ordered fields without integer parts. In fact, he has for every integer p, a p-real closed field with no integer part. We show that these examples are optimal, in the sense that every real closed field has an integer part.

1.4. In fact, we will prove a stronger result:

Let A be a convex subfield of $K = \widetilde{K}$. Then any integer part Z of A can be extended to an integer part Z_K of K.

§2. Convex valuation in a totally ordered field.

The following results are classical results of valuation theory (cf. [K], [KW], [R]). Let K be a (totally) ordered field and v a convex valuation on K. We denote by k the residue field and by $\Gamma = v(K)$ the abelian totally ordered group of valuations. When $y \in K$ and $v(y) = 0$, $\overline{y}$ will be the residue image of y in k.

2.1. PROPOSITION. *If K is real closed, then k is real closed and Γ is divisible. Moreover, k can be embedded in K, and there exists a cross section, i.e., a family $\{x^\gamma : \gamma \in \Gamma\} \subset K^+$ that satisfies*

$$\forall \gamma, \gamma' \in \Gamma\ [v(x^\gamma) = \gamma \text{ and } x^\gamma \cdot x^{\gamma'} = x^{\gamma+\gamma'}].$$

2.2. The field of formal series $k((x))^\Gamma$: Given an ordered field k and an ordered abelian group Γ, we let $k((x))^\Gamma = \{0\} \cup \{\sum_{i<\mu} a_i x^{\gamma_i} : \mu \text{ is an ordinal}$, $(\gamma_i)_{i<\mu}$ a strictly increasing family of Γ, $(a_i)_{i<\mu}$ a family of elements from $k^*\}$.

Then $k((x))^\Gamma$ is a field with the usual sum and with the product induced by $x^\gamma \cdot x^{\gamma'} = x^{\gamma+\gamma'}$. Moreover, the order on k can be extended to an order on $k((x))^\Gamma$ thus: $x^\gamma > x^{\gamma'} > k$ iff $\gamma < \gamma' < 0$. In addition, if k is real closed and Γ is divisible then $k((x))^\Gamma$ is real closed. The map $d : k((x))^\Gamma \to \Gamma \cup \{\infty\}$, defined by $d(s) = \gamma_0$, is a convex valuation on $k((x))^\Gamma$ with valuation group Γ and residue field isomorphic to k.

The following result is a version of the theorem: "any henselian subfield of $k((x))^\Gamma$ is real closed" [R]. In the following L will be a subfield of $k((x))^\Gamma$ such that $d(L) = \Gamma$ and $k \subset L$.

2.3. DEFINITION. *Let y be algebraic over L with $d(y) = 0$. We say that y satisfies condition (H) if there is a polynomial $P(X) = \sum_{k=0}^{n} A_k X^k \in L[X]$ such that*

(i) $P(y) = 0$ and P is primitive (i.e., $\min d(A_k) = 0$),

(ii) $\overline{P}'(\overline{y}) \neq 0$ (where $\overline{P}$ is the image of P in $k[X]$).

2.4. PROPOSITION. *Let $y \in \widetilde{L}$ with $d(y) = 0$. Then there is $y_1, \dots, y_k$ belonging to $\widetilde{L}$ such that $y = y_1 + \cdots + y_k$ and $y_i/x^{d(y_i)}$ satisfies the condition (H) over $L(y_1, \dots, y_{i-1})$ for all i.*

Proof. See [MR].

§3. Integer part in subfields of $k((x))^\Gamma$, closure under truncation.

In the following k will be a real closed field and Γ a totally ordered abelian group.

3.1 DEFINITION. Let $s = \sum_{i<\mu} a_i x^{\gamma_i} \in k((x))^\Gamma$. An *initial segment* of s (abbreviated by I.S.) is any element $\sum_{i<\lambda} a_i x^{\gamma_i}$ with $\lambda < \mu$. We use $(s)_{<\gamma}$ to denote the I.S. $\sum_{i<\lambda} a_i x^{\gamma_i}$ of s where $i < \lambda$, $\gamma_i < \gamma$, and $\gamma_\lambda \geq \gamma$. Note that $(s)_\gamma = 0$ when $\gamma \leq \gamma_0$. We say that the subfield L of $k((x))^\Gamma$ is *closed under truncation* if every I.S. of any element of L also belongs to L.

3.2. LEMMA. *Assume that k has an integer part Z. Then every subfield $L \supset k$ of $k((x))^\Gamma$ which is closed under truncation, has an integer part $Z_L \supset Z$.*

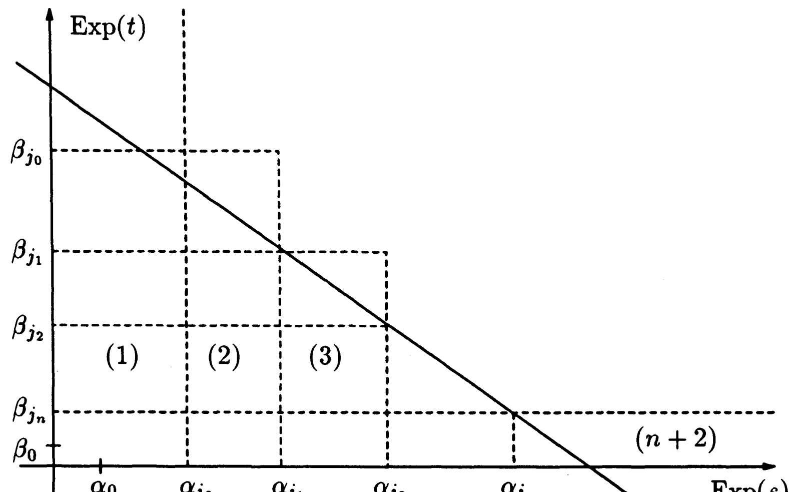
Proof. Let $Z_L = \{\sum_{i\leq\mu} a_i x^{\Gamma_i} \in L : \gamma_i \leq 0 \text{ and } (\gamma_i = 0 \Rightarrow a_i \in Z)\}$. It is easy to see that Z_L is a discrete subring of L. Let $y \in L$ and let y' be the initial segment of y such that every exponent of y' is strictly negative. Let a_{i_0} be the term of y with exponent 0. Then if we let $[y] = y' + [a_{i_0}] - 1$ or $y = y' + [a_{i_0}]$ (depending on y), we get $[y] \in Z_L$, and $[y]$ is the integer part of y. □

Next we will study preservation of closure under truncation by field extensions.

3.3. Lemma. *Let $s = \sum_{i<\mu} a_i x^{\alpha_i}$ and $t = \sum_{j<\nu} b_j x^{\beta_j}$ be two elements of $k((x))^\Gamma$ and $(s \cdot t)_{<\delta}$ a strict I.S. of $s \cdot t$. Then there is a unique strictly increasing subsequence $(\alpha_{i_0}, \ldots, \alpha_{i_n})$ of $(\alpha_i)_{i<\mu}$ and a unique strictly decreasing subsequence $(\beta_{i_0}, \ldots, \beta_{i_n})$ of $(\beta_j)_{j<\nu}$ such that $\alpha_{i_1} + \beta_{j_1} \geq \delta$ and*

(E) $(s\cdot t)_{<\delta} = t\cdot(s)_{<\alpha_{i_0}} + (t)_{<\beta_{j_0}}\cdot((s)_{<\alpha_{i_1}} - (s)_{<\alpha_{i_0}}) + \cdots + (t)_{<\beta_{j_n}}\cdot(s - (s)_{<\alpha_{i_n}})$.

Subsequently we will say that $(s \cdot t)_{<\delta}$ is written in form (E).



$$(s \cdot t)_{<\delta} = \overbrace{t \cdot (s)_{<\alpha_{i_0}}}^{(1)} + \overbrace{(t)_{<\beta_{j_0}}\left[(s)_{<\alpha_{i_1}} - (s)_{<\alpha_{i_2}}\right]}^{(2)} + \cdots + \overbrace{(t)_{<\beta_{j_n}}\left[s - (s)_{<\alpha_{i_n}}\right]}^{(n+2)}$$

Proof. Let Exp(s) denote the set of exponents of the series s. Since $(s\cdot t)_{<\delta}$ is a strict I.S. of $s \cdot t$, there exist $\alpha \in \text{Exp}(s)$ and $\beta \in \text{Exp}(t)$ with $\alpha + \beta \geq \delta$. Let α_{i_0} be the smallest exponent of s satisfying this property, and let β_{j_0} be the smallest exponent β of t such that $\alpha_{i_0} + \beta \geq \delta$. If $\alpha + \beta < \delta$ for all $\alpha \in \text{Exp}(s)$ and for all $\beta \in \text{Exp}(t)$ such that $\alpha > \alpha_{i_0}$ and $\beta < \beta_{j_0}$, then $(s\cdot t)_{<\delta} = t\cdot(s)_{<\alpha_{i_0}} + (t)_{<\beta_{j_0}}\cdot(s - (s)_{<\alpha_{i_0}})$ and Lemma 3.3. holds. In the other case let α_{i_1} the smallest exponent α of s strictly larger that α_{i_0} such that there is $\beta \in \text{Exp}(t)$, $\beta < \beta_{j_0}$, with $\alpha + \beta \geq \delta$, and let β_{j_1} be the smallest exponent $\beta < \beta_{j_0}$ of t such that $\alpha_{i_1} + \beta \geq \delta$. If $\alpha + \beta < \delta$ for all exponents $\alpha > \alpha_{i_1}$ of s

and all $\beta > \beta_{j_1}$, then

$$(s \cdot t)_{<\delta} = t \cdot (s)_{<\alpha_{i_0}} + (t)_{<\beta_{j_0}} \cdot ((s)_{<\alpha_{i_1}} - (s)_{<\alpha_{i_0}}) + (t)_{<\beta_{j_1}} \cdot (s - (s)_{<\alpha_{i_1}})$$

and Lemma 3.3. holds again.

Going on inductively, we build two sequences (β_{j_l}) and (α_{i_l}). Since $\operatorname{Exp}(t)$ is a well-ordered set, the strictly decreasing sequence (β_{j_l}) is finite, hence so is the sequence (α_{i_l}). The uniqueness of the sequences $(\beta_{j_l})_{l=0}^{n}$ and $(\alpha_{i_l})_{l=0}^{n}$ is easy to prove. So Lemma 3.3. is proved. □

NOTES. (1) $(s \cdot t)_{<\delta}$ is the sum of the terms $a_i b_j x^{\alpha_i+\beta_j}$ of $s \cdot t$ whose exponents (α_i, β_j) are in the area $\alpha + \beta < \delta$ (see the picture).

(2) We can see from the picture that (E) is in fact the expression of $(s \cdot t)_{<\delta}$ as an exact and finite Riemann sum over this area.

(3) Note that in (E), β_{j_n} is the smallest exponent of t such that there is an $\alpha \in \operatorname{Exp}(s)$ with $\beta + \alpha \geq \delta$.

3.4. LEMMA. *Let L be a subfield of $k((x))^{\Gamma}$ closed under truncation and let $y \in k((x))^{\Gamma}$ be such that every I.S. of y belongs to L. Then $L(y)$ remains closed under truncation.*

Proof. In the following, H will be any subfield of $k((x))^{\Gamma}$, and $s = \sum_{i<\mu} a_i x^{\alpha_i}$, $t = \sum_{j<\nu} b_j x^{\beta_j}$ will be any two elements of H.

FACT 1. If every I.S. of s and t belongs to H, then every I.S. of $s \cdot t$ again belongs to H. Fact 1 follows from Lemma 3.3.

FACT 2. Let $t' = (t)_{<\beta_\lambda}$ be a strict I.S. of t. Then there is a unique exponent $\gamma \leq \alpha_0 + \beta_\lambda$ of $\operatorname{Exp}(s) + \operatorname{Exp}(t)$ and two finite sequences as in Lemma 3.3 with

(a) $(s \cdot t)_{<\gamma} = t' \cdot (s)_{<\alpha_{i_1}} + (t)_{<\beta_{j_1}} \cdot ((s)_{<\alpha_{i_2}} - (s)_{<\alpha_{i_1}}) + \cdots + (t)_{<\beta_{j_n}} \cdot (s - (s)_{<\alpha_{i_n}})$.

(b) For any exponent γ' of $\operatorname{Exp}(s) + \operatorname{Exp}(t)$ the form (E) of $(s \cdot t)_{<\gamma'}$ uses only *strict* I.S. of t'.

Proof. Let γ be the smallest element of the set $\{\delta \in \operatorname{Exp}(s \cdot t) : \forall j < \lambda\ (\alpha_0 + \beta_j < \delta)\}$. Since $\operatorname{Exp}(s) + \operatorname{Exp}(t)$ is a well-ordered set, γ exists and $\gamma \leq \alpha_0 + \beta_\lambda$. Let $(\alpha_{i_l})_{l=0}^{n}$ and $(\beta_{j_l})_{l=0}^{n}$ the two sequences for $(s \cdot t)_{<\gamma}$ given by Lemma 3.3. By the definition of γ, we get $\forall j < \lambda\ (\alpha_0 + \beta_j < \gamma \leq \alpha_0 + \beta_\lambda)$. Thus by the definition of α_{i_0} and β_{j_0}, $\alpha_{i_0} = \alpha_0$ and $\beta_{j_0} = \beta_\lambda$. Hence $(s)_{<\alpha_{i_0}} = (s)_{<\alpha_0} = 0$ and $(t)_{<\beta_{j_0}} = (t)_{<\beta_\lambda}$, whence (a) holds. Moreover, let $\gamma' < \gamma$ be an element of $\operatorname{Exp}(s) + \operatorname{Exp}(t)$. Then by definition of γ, there exists $j < \lambda$ such that $\gamma' < \alpha_0 + \beta_j$. Hence if we denote by $(\alpha'_{i_l})_{l=0}^{m}$ and $(\beta'_{j_l})_{l=0}^{m}$ the two sequences given by Lemma 3.2 for $(s \cdot t)_{<\gamma'}$, we get $\alpha'_{i_0} = \alpha_0$, $\beta'_{j_0} = \beta_j < \beta_\lambda$; and since the sequence $(\beta'_{j_l})_{l=0}^{m}$ is decreasing, $(t)_{<\beta_{j_l}}$ is a strict initial segment of t' and (b) holds.

FACT 3. If every I.S. of s belongs to H then every I.S. of $1/s$ again belongs to H.

Proof. If not, let $t' = (1/s)_{<\beta_\lambda}$ be the shortest I.S. of $1/s$ which does not belong to H. Then by Fact 2 applied to $t = 1/s$

$$(\exists \gamma \leq \alpha_0 + \beta_\lambda)[1 = (s \cdot 1/s)_{<\gamma} = t' \cdot (s)_{<\alpha_{i_1}} + C]$$

where C is a finite sum of finite products of I.S. of s and strict I.S. of t'. Thus $C \in H$ and $t' \in H$, contradicting the definition of t'.

Lemma 3.4 follows obviously from Facts 1 and 3. □

3.5 LEMMA (F. Delon). *Let L be a subfield of $k((x))^{\Gamma}$ such that $k \subset L$ and $d(L) = \Gamma$. If L is closed under truncation then so is $\widetilde{L}$.*

Remark. This lemma has for us an eventful history. We gave a proof in which D. Marker pointed out an important error. In order to correct it, we proved Facts 4 and 5 below, from which the lemma follows if you know Ribenboim's theorem as stated in 2.3. But we did not know it, and it is only after F. Delon outlined for us a proof of 3.5 based on her work [D] that we completed the proof presented below, which is different from Delon's proof, since it provides an explicit computation that does not follow from Delon's work; see Remark 3.7 below.

Proof of 3.5: It is a straightforward consequence of Proposition 2.4 and Lemma 3.6 below. □

3.6. LEMMA. *Let L be a subfield of $k((x))^{\Gamma}$ closed under truncation with $k \subset L$ and $d(L) = \Gamma$. Let y be any element of $\widetilde{L}$ satisfying $d(y) = 0$ and the statement (H). Then any I.S. of y belongs again to $\widetilde{L}$.*

Proof. We shall prove Lemma 3.6 by way of contradiction: if it fails, there is an I.S. of y that does not belong to L. Let $y' = (y)_{<\beta_\lambda}$ be the shortest I.S. of t satisfying this property. Let $P(X) = X^n + A_{n-1}X^{n-1} + \cdots + A_0$ be given by hypothesis and let κ be the set of exponents $k \neq 0$ of P such that $d(A_k) = 0$ (note that κ has at least one element). We have to make a distinction between two cases and get a contradiction in each of them.

First case. Assume that there exists a convex subgroup Γ_0 of Γ such that $\mathrm{Exp}(y')$ is cofinal in Γ_0 (in other words $\forall \alpha, \beta \in \mathrm{Exp}(y')\ \exists \gamma \in \mathrm{Exp}(y')\ (\alpha + \beta \leq \gamma)$). Let $(A_k)_{<\Gamma_0}$ denote the largest initial segment of A_k such that $\mathrm{Exp}((A_k)_{<\Gamma_0}) \subset \Gamma_0$ (this definition is available since Γ_0 is a convex subgroup) and let $Q(X) = \sum_{k=0}^{n} (A_k)_{<\Gamma_0} X^k \in L[X]$ (recall that L is closed under truncation). Q is not constant since $\forall k \in \kappa\ d(A_k) = 0$. So $P(X) = Q(X) + R(X)$ where $R(X) = \sum_{k=0}^{n} (A_k - (A_k)_{<\Gamma_0}) X^k = \sum_{k=0}^{n} B_k X^k$ with $d(B_k) > \Gamma_0$. Hence $d(R(y)) \geq \min(d(B_k)) > \Gamma_0$. Moreover, Taylor's formula gives $Q(y) = Q(y' + y'') = Q(y') + T$, where $d(T) > \Gamma_0$. So $0 = P(y) = Q(y') + T + R(y) = Q(y') + S$ with $\mathrm{Exp}(Q(y')) \subset \Gamma_0$ and $d(S) > \Gamma_0$. Therefore $Q(y') = 0$, contradicting the choice of y'.

Second case. Assume the negation for the hypothesis of the first case: i.e., $\exists \alpha, \beta \in \mathrm{Exp}(y')\ \forall \beta_j \in \mathrm{Exp}(y')\ (\beta_j < \alpha + \beta)$.

We prove first two facts:

FACT 4. Let $\oplus_n \mathrm{Exp}(y)$ denote the set $\{\alpha_0 + \alpha_1 + \cdots + \alpha_n \colon \forall i,\ \alpha_i \in \mathrm{Exp}(y)\}$. Let γ be the smallest element of the set $\Delta = \{\delta \in \oplus_n \mathrm{Exp}(y) : \forall j < \lambda,\ \beta_j < \delta\}$. Then

(a) $$\forall k \geq 1, (y^k)_{<\gamma} = y' \cdot B_k + C_k$$

where B_k and C_k belong to the ring generated by the strict I.S. of y'. Moreover $d(B_k) = 0$ and $\overline{B}_k = k\overline{y}^{k-1}$.

(b) If $\gamma' < \gamma$ belongs to $\oplus_n \operatorname{Exp}(y)$, then $(y^k)_{<\gamma'}$ belongs to the ring generated by the strict I.S. of y'.

Proof of Fact 4. First, we can see that Δ is not an empty set; indeed by hypothesis there are α and β belonging to $\operatorname{Exp}(y')$ such that $\forall j < \lambda$, $\beta_j < \alpha+\beta$, hence Δ contains an element of $\oplus_2 \operatorname{Exp}(y)$ (since $d(y) = 0 \Rightarrow \oplus_2 \operatorname{Exp}(y) \subset \oplus_n \operatorname{Exp}(y)$). So, since $\oplus_n \operatorname{Exp}(y)$ is a well-ordered set, γ is well defined. Since $\forall k \leq n$, $\operatorname{Exp}(y^{k-1}) \subset \oplus_n \operatorname{Exp}(y)$, by the same argument as in Fact 2(a), we can easily prove that there are a sequence $\gamma > \alpha_1 > \cdots > \alpha_m > 0$ of $\operatorname{Exp}(y)$ and a sequence $0 < \gamma_1 < \cdots < \gamma_m < \gamma$ of $\oplus_n \operatorname{Exp}(y)$ such that, $\forall k$, $1 \leq k \leq n$:

(1)

$$(y^k)_{<\gamma} = (y \cdot y^{k-1})_{<\gamma} = (y)_{<\gamma} \cdot (y^{k-1})_{<\gamma_1} + (y)_{<\alpha_1} \cdot [(y^{k-1})_{<\gamma_2} - (y^{k_1})_{<\gamma_1}] + \cdots + (y)_{<\alpha_m} \cdot [(y^{k-1})_{<\gamma} - (y^{k-1})_{<\gamma_m}]$$

By definition of γ, $(y)_{<\gamma} = y'$, moreover $\forall i$, $(y)_{<\alpha_i}$ is a strict initial segment of y'. Hence we get, $\forall k \leq n$:

$$(2) \qquad (y^k)_{<\gamma} = y' \cdot (y^{k-1})_{<\gamma_1} + y_1 \cdot [(y^{k-1})_{<\gamma_2} - (y^{k-1})_{<\gamma_1}] + \cdots + y_m \cdot [(y^{k-1})_{<\gamma} - (y^{k-1})_{<\gamma_m}]$$

where $y_1, \ldots, y_m$ are strict I.S. of y'. Now we are able to end the proof of Fact 4 by induction on k:

(i) $k = 1$. By definition of γ, $(y)_{<\gamma} = y'$, then Fact 4(a) holds with $C_1 = 0$ and $B_1 = 1$; moreover for any γ' in $\oplus_n \operatorname{Exp}(y)$, strictly less than γ, there is β_j in $\operatorname{Exp}(y')$ such that $\gamma' \leq \beta_j$, so $(y)_{<\gamma'}$ is a strict I.S. of y', then Fact 4(b) holds.

(ii) Assume that Fact 4 holds for $k-1$. Recall (2) above. By induction hypothesis $(y^{k-1})_{<\gamma} = y' \cdot B_{k-1} + C_{k-1}$, and $\forall i < m$, $\gamma_i < \gamma$, then $(y^{k-1})_{<\gamma_i}$ belongs to the ring generated by the strict I.S. of y'. Let $B_k = (y^{k-1})_{<\gamma_1} - y_m \cdot B_{k-1}$ and

$$C_k = y_1 \cdot [(y^{k-1})_{<\gamma_2} - (y^{k-1})_{<\gamma_1}] + \cdots + [C_{k-1} - (y^{k-1})_{<\gamma_m}],$$

then (2) becomes

$$(y^k)_{<\gamma} = y' \cdot B_k + C_k$$

where B_k and C_k belong to the ring generated by the strict I.S. of y'. So Fact 4(a) holds. Fact 4(b) holds using the same argument as in Fact 2(b) and induction hypothesis.

FACT 5. Consider the truncation of $P(y)$ at the exponent γ given by Fact 4. Then $0 = (P(y))_{<\gamma} = y'A + B$, where $A, B \in L$. Moreover $d(A) \geq 0$ and we have $\overline{A} = \sum_{k\in\kappa} k\overline{A}_k(\overline{y})^{k-1}$.

Proof of Fact 5.

(i) First suppose that $k \in \kappa$. Lemma 3.3 with $s = A_k$ and $t = y^k$ gives

$$(1) \qquad (A_k \cdot y^k)_{<\gamma} = y^k \cdot (A_k)_{<\alpha_{i_0}} + (y^k)_{<\beta_{j_0}} \cdot ((A_k)_{<\alpha_{i_1}} - (A_k)_{<\alpha_{i_0}}) + (y^k)_{<\beta_{j_n}} \cdot (A_k - (A_k)_{<\alpha_{i_n}}).$$

By the definition of α_{i_0}, β_{j_0} and since $d(A_k) = 0$, we have $\alpha_{i_0} = 0$ and $(y^k)_{<\beta_{j_0}} = (y^k)_{<\gamma}$. So (1) becomes

$$(2) \quad (A_k \cdot y^k)_{<\gamma} = (y^k)_{<\gamma} \cdot (A_k)_{<\alpha_{i_1}} + \cdots + (y^k)_{<\beta_{j_n}} \cdot (A_k - (A_k)_{<\alpha_{i_n}}).$$

Since the sequence (β_{j_l}) is decreasing, all I.S. of y^k in (2) are I.S. of $(y^k)_{<\gamma}$. By Fact 4(b) they belong to the ring generated by the strict I.S. of y'. Hence they belong to L. Therefore we can write:

$$(3) \qquad (A_k \cdot y^k)_{<\gamma} = (y^k)_{<\gamma} \cdot (A_k)_{<\alpha_{i_1}} + C$$

where $C \in L$. Now Fact 4 gives

$$(4) \quad (A_k \cdot y^k)_{<\gamma} = (y'B_k + C_k) \cdot (A_k)_{<\alpha_{i_1}} + C = y' \cdot B_k(A_k)_{<\alpha_{i_1}} + S'_k,$$

with $S'_k \in L$. So $(A_k \cdot y^k)_{<\gamma} = y' \cdot S_k + S'_k$, where $S_k, S'_k \in L$. Moreover $d(S_k) = 0$ and $\overline{S}_k = \overline{B}_k \cdot \overline{A}_k = \overline{A}_k \cdot k(\overline{y})^{k-1}$.

(ii) We now consider $k \neq 0$ and $k \notin \kappa$. By Lemma 3.3 we get

$$(1) \quad (A_k \cdot y^k)_{<\gamma} = y^k \cdot (A_k)_{<\alpha_{i_0}} + (y^k)_{<\beta_{j_0}} \cdot ((A_k)_{<\alpha_{i_1}} - (A_k)_{<\alpha_{i_0}}) + \cdots + (y^k)_{<\beta_{j_n}} \cdot (A_k - (A_k)_{<\alpha_{i_n}}).$$

We have $d(A_k) + \gamma > \gamma$, and in this case, $\alpha_{i_0} = d(A_k)$ and $\beta_{j_0} \leq \gamma$.

Therefore exactly as in (i) we get $(A_k \cdot y^k)_{<\gamma} = y' \cdot S_k + S'_k$, where $S_k, S'_k \in L$. But then $d(S_k) = d(B_k) + d(A_k) > 0$, or $S_k = 0$. Note that $(A_0)_{<\gamma} \in L$.

(iii) Let $A = \sum_{k=0}^n S_k$ and $B = \sum_{k=0}^n S'_k$. Then (i) and (ii) give us $(P(y))_{<\gamma} = \sum_{k=0}^n (A_k y^k)_{<\gamma} = y' \cdot A + B$, where $A, B \in L$, and $d(A) \geq \min(d(S_k)) \geq 0$. Moreover, (i) and (ii) yield $\overline{A} = \sum_{k\in\kappa} k\overline{A}_k(\overline{y})^{k-1}$. So Fact 5 holds.

We next prove that $A \neq 0$. Observe that $\sum_{k\in\kappa} k\overline{A}_k(\overline{y})^{k-1} = \overline{P}'(\overline{y})$. By hypothesis, $\overline{P}'(\overline{y}) \neq 0$. Thus $\overline{A} \neq 0$, hence $A \neq 0$. Finally, $y' = -B/A \in L$, contradicting the choice of y'. So Lemma 3.6 is proved in this case too. □

3.7. *Remark*. The above proof computed y' from its strict initial segments; this computation actually holds for every I.S. y' of y, and together with Proposition 2.4 and Lemma 3.4, it provides a computation, as effective as can be in the most general case, of any roots of a polynomial inside $k((x))^\Gamma$. This is developed in [M].

§4. Every real closed field has an integer part.

Let K be closed, A a convex subring, and Z an integer part of A. Let v be the convex valuation given by A. Then the residue field k can be embedded in K in such a way that Z becomes an integer part of k. Let $\Gamma = v(K)$, $\{x^\gamma : \gamma \in \Gamma\}$ as in 2.2, and denote by H_0 the subfield generated by k and $\{x^\gamma : \gamma \in \Gamma\}$.

4.1. LEMMA. *Let H be a real closed subfield of K, $H_0 \subset H$, and assume $f : H \to k((x))^\Gamma$ with the following properties:*

(a) the restriction of f to H_0 is Id

(b) f is an injective homomorphism of ordered fields

(c) if $y \in H$, then $v(y) = d(f(y))$

(d) $f(H)$ is closed under truncation.
Then for any $y \in K - H$, f can be extended to $f' : \widetilde{H}(y) \to k((x))^{\Gamma}$ which satisfied the conditions (a), (b), (c), (d).

Proof. By hypothesis we may identify H with $f(H)$ in $k((X))^{\Gamma}$. Let us say that the series $\sum_{i<\alpha} a_i x^{\gamma_i} \in H$ is a ***development at order α of y with respect to H***, if $v(y - \sum_{i<\alpha} a_i x^{\gamma_i}) > \gamma_i$ for all $i < \alpha$. By 2.3, since $H_0 \subset H$, y has a development at any finite order with respect to H. Let $S(y)$ denote the set of all the developments of y with respect to H. $S(y)$ is totally ordered by the relation "initial segment of." Let $f'(y) \in k((x))^{\Gamma}$ be the least upper bound of $S(y)$ for the relation "I.S. of". We can see that $f'(y) \notin H$. Moreover, we have:

FACT. $\forall z \in H\ (z < y \Leftrightarrow f(z) < f'(y))$.

Proof. $z < y \Rightarrow v(z) \geq v(y)$. If $v(z) > v(y)$, then $d(z) > d(f'(y))$ and thus $f(z) < f'(y)$. In the other case $v(z) = v(y)$. Then $y - z = ax^{\alpha} + y'$, where $a > 0$ and $v(y') > \alpha = v(y - z) \geq v(y) = v(y)$. Let $z' = (z)_{<\alpha+1} + ax^{\alpha}$, then since H is closed under truncation, $z' \in H$, and z' is the development of y at order α with respect to H. We get

$$z < (z)_{\alpha+1} + (a/2)x^{\alpha} < (z)_{<\alpha+1} + ax^{\alpha} + s$$

for any s such that $v(s) > \alpha$. Hence z is strictly smaller than every element of $S(y)$, and finally $z < f'(y)$.

Thus we can extend f' to an isomorphism of $\widetilde{H}(y)$ onto $\widetilde{L}$ where L is the field generated over H by $f'(y)$ in $k((x))^{\Gamma}$. In addition, by the construction of $f'(y)$ all strict I.S. of $f'(y)$ belong to H. Thus L and $\widetilde{L}$ are closed under truncation, by Lemmas 3.4 and 3.5. □

4.2. COROLLARY. *Under the same hypotheses as in 4.1, there is $f' : K \to k((x))^{\Gamma}$ with the properties (a), (b), (c), (d).*

Proof. Transfinite iteration of 4.1, beginning with $H = \widetilde{H}_0$. □

From Lemma 3.2 applied to the image of the application f given by 4.2, we have finally:

4.3. THEOREM. *K has an integer part $Z_K \supset Z$.*

4.4. COROLLARY. *Every real closed field admits an integer part.*

Proof: Apply the preceding theorem with $Z = \mathbb{Z} \subset K$. □

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