AN INTUITIONISTIC THEORY OF LAWLIKE, CHOICE AND LAWLESS SEQUENCES

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Dedicated to Stephen Cole Kleene

Abstract

In [12] we defined an extensional notion of relative lawlessness and gave a classical model for a theory of lawlike, arbitrary choice, and lawless sequences. Here we introduce a corresponding intuitionistic theory and give a realizability interpretation for it. Like the earlier classical model, this realizability model depends on the (classically consistent) set theoretic assumption that a particular Δ^2_1 well ordered subclass of Baire space is countable.

§1. Introduction.

1.1. Background. Infinitely proceeding sequences of natural numbers are the fundamental objects of L. E. J. Brouwer's intuitionistic theory of the continuum. Choice sequences are generated by more or less freely choosing one integer after another; at each stage, the chooser *may* also specify restrictions on future choices (compatible with previous restrictions, if any, and with the indefinite continuation of the process).

Brouwer called "lawlike" or "a sharp arrow" any sequence *all* of whose values are completely determined (restricted) according to some fixed law at some finite stage in the generation of the sequence. G. Kreisel [9] called "lawless" any sequence for which (i) "the *simplest kind of restriction on restrictions is made*, namely some finite initial segment of values is prescribed, and beyond this, no restriction is to be made." Kreisel and A. S. Troelstra developed a theory of lawlike and *intensionally* lawless sequences, based on (i), for which they were able to prove that every formula without free lawless variables is equivalent to one without any lawless variables and hence "it is possible to regard lawless sequences as a 'figure of speech'."[2]

Alternatively a sequence could be called lawless if (ii) it successfully evades description by any fixed law. The assumption that lawless sequences are real

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[2] [15, p. 639]. Kreisel [9, p. 225] asserts however that the equivalence result is *not* to be interpreted in this way, but rather as "a complete analysis of all known properties of lawless sequences in the given context."

objects of the intuitionistic continuum, whose properties are determined by their relationship to the lawlike sequences as suggested by (ii), leads to an entirely *extensional* theory of lawlike, general choice, and lawless sequences reminiscent of the theory of generic real numbers.[3] A classical model for such a theory appears in [12], under the classically consistent assumption that a particular Δ^2_1 well ordered subclass of Baire space is countable. The class of "definably lawless sequences" studied there satisfies Kreisel's Axiom of Open Data (suggested by (i) above) and a strong continuity principle (but not bar induction) and is a comeager subset of the continuum. In another paper (now in preparation) we show that it has classical measure zero and is simply definable in terms of a notion of forcing.

This paper introduces intuitionistic theories of definably lawless sequences incorporating S. C. Kleene's fundamental axiomatization **FIM** [8] of Brouwer's theory of the continuum and extends Kleene's function realizability interpretation of **FIM** to the new systems under the set-theoretic assumption appealed to in [12]. Whenever possible the reasoning used is constructive; however the realizability of some of the new axioms will be established only classically.

1.2. Motivation. Before discussing lawless sequences in context (ii) we need to know something about the lawlike ones. According to [6] Kleene did not introduce a special type of lawlike sequences because the class of general recursive functions was adequate for his purposes and was definable in his theory. Here we need a broader interpretation of "lawlike" which we shall try to motivate constructively.

What assumptions can reasonably be made about *all* lawlike sequences? We propose the following:

1. If $P(x,y)$ is a definite property of ordered pairs of natural numbers such that for each x there is exactly one y which makes $P(x,y)$ true, then there is a lawlike function ϕ such that for all x, y :

$$\phi(x) = y \text{ if and only if } P(x,y).$$

2. The class of all lawlike sequences is countably infinite in the classical sense, but has no lawlike enumeration.

For (1) we accept as "definite" only properties P all of whose sequence parameters are lawlike, and whose constructive and classical meanings essentially coincide modulo Markov's Principle. Subject to this restriction P may involve quantification over all choice sequences and over all lawlike sequences, as well as over the natural numbers.

As in [12] we next define a notion of "lawless" relative to any given notion of "lawlike" satisfying (1) and (2). One possibility would be to adopt (ii) as the definition, so a sequence α is lawless if for no lawlike sequence ϕ and for no natural number x is it the case that $\lambda t\ \alpha(x+t)$ is ϕ; however then α might be lawless even though e.g. $\lambda t\ \alpha(2t)$ was lawlike. This objection suggests something like "α is lawless if and only if for each lawlike injection ϕ, $\alpha \circ \phi$ satisfies (ii)." What we

[3]The context of this theory is somewhat wider than Kreisel and Troelstra's since it includes arbitrary choice sequences as well as lawlike and lawless ones; however, some of the axioms concerning properties specific to lawless sequences will be restricted to the narrower context.

seem to need for the proofs is a stronger notion of "lawless" whose definition and key properties appear in Section 3.

1.3. Sources. This paper is intended to be a direct sequel to Kleene and Vesley's [8] and may be read independently of all other sources. However anyone interested in the subject should surely read Kreisel's [9] and consult Kreisel and Troelstra's [10]. One may also wish to consult [12], although there the viewpoint was classical, the formalization cumbersome, and the presentation uneven.

Especially since the publication twenty-five years ago of Kleene's and R. E. Vesley's metamathematical investigation [8], much effort has been devoted to axiomatizing parts of intuitionistic mathematics beyond number theory. Troelstra's and D. van Dalen's two recent volumes [15], taken together with Vesley's address [16] to the 1979 Kleene Symposium, provide an excellent guide to the history and current state of this work. In particular, Chapters 4 and 12 of [15] give the background of Kreisel and Troelstra's work on lawlike and lawless sequences; Chapter 12 also describes other special classes of choice sequences which have recently been studied by Troelstra, van Dalen, G. F. van der Hoeven, and others.

§2. The formal theories.

2.1. The basic theory BD. This will be an extension of Kleene's basic formal system **B** for the common portion of intuitionistic and classical analysis [8, Sections 4–6]. The main syntactic difference is that **BD** has *two* sorts of variables for functions, i.e., choice sequences of natural numbers; in the intended interpretation, one sort ranges over lawlike (or definable) sequences and the other over arbitrary choice sequences.[4] We use the letters a, b, c, d, e, g, h (with or without subscripts) to denote variables over lawlike sequences, and i, j, k, l, m, ... , i_1, ... as number variables. As in [12, 8] $\alpha, \beta, \gamma, \ldots, \alpha_1, \ldots$ denote variables over arbitrary choice sequences.

The language includes the numerical equality constant = , Church's λ, a finite list $\mathrm{f}_0, \mathrm{f}_1, \ldots, \mathrm{f}_p$ of constants for primitive recursive functions, and the logical constants $\&, \vee, \neg, \supset, \forall, \exists$. Each f_i expresses a function $f_i(x_1, \ldots, x_{k_i}, \alpha_1, \ldots, \alpha_{l_i})$ which, considered as a function of $x_1, \ldots, x_{k_i}$, is primitive recursive uniformly in $\alpha_1, \ldots, \alpha_{l_i}$. In particular, f_0 is 0, f_1 is $'$, f_2 is $+$, and f_3 is $\cdot$; see [8] and [7] for a suitable list.

Terms and **functors** are defined as in [8] except that now a, b, c, d, e, g, h, $\mathrm{a}_1, \ldots$ (as well as $\alpha, \beta, \ldots, \alpha_1, \ldots$) are functors while i, j, ... , i_1, ... are terms. Thus $\mathrm{f}_i(\mathrm{t}_1, \ldots, \mathrm{t}_{k_i}, \mathrm{u}_1, \ldots, \mathrm{u}_{l_i})$ is a term if t_j are terms and u_j are functors. If $k_i = 1$ and $l_i = 0$ then f_i is a functor. If u is a functor and t is a term, (u)(t) is a term. If x is a number variable and s is a term, λx(s) is a functor. A term t (functor u) is a **D-term** (**D-functor**) if it contains no occurrences of arbitrary function variables.

Prime formulas are of the form s = t where s, t are terms. **Formulas** are built up from these using the propositional connectives, and the quanti-

[4]While retaining Kleene's view of the primary importance of arbitrary choice sequences, we follow Kreisel [9] in adopting the notion of lawlike sequence as an additional primitive concept.

fiers $\forall \mathrm{x}, \exists \mathrm{x}, \forall \mathrm{a}, \exists \mathrm{a}, \forall \alpha, \exists \alpha$ over all three sorts of variables. A **D-formula** is one having free no arbitrary function variables. If u, v are functors, "u = v" abbreviates $\forall \mathrm{x}\, \mathrm{u}(\mathrm{x}) = \mathrm{v}(\mathrm{x})$ where x is not free in u or v. For any formulas A and B, "A $\sim$ B" abbreviates (A $\supset$ B) & (B $\supset$ A). "$\exists!\mathrm{y}\mathrm{A}(\mathrm{y})$" abbreviates $\exists \mathrm{y}\mathrm{A}(\mathrm{y})$ & $\forall \mathrm{y}\forall \mathrm{z}(\mathrm{A}(\mathrm{y})$ & $\mathrm{A}(\mathrm{z}) \supset \mathrm{y} = \mathrm{z})$ and similarly for $\exists!\mathrm{a}\mathrm{A}(\mathrm{a})$ and $\exists!\alpha\, \mathrm{A}(\alpha)$.

The substitution lemma (Lemma 3.1 of [8, p. 12]) has to be restated to allow substitution of D-functors (but not arbitrary functors) for definable function variables. In particular, if D-functors are substituted for *all* function variables occurring free in a term (functor) [formula], the result is a D-term (D-functor) [D-formula].

Lemma 3.3 of [8] has the following restatement: Let s be a term (u be a functor) [P be a prime formula] containing free no variables but $\mathrm{x}_1, \ldots, \mathrm{x}_k$, $\mathrm{a}_1, \ldots, \mathrm{a}_l$, $\alpha_1, \ldots, \alpha_m$. Then under the intended interpretation s (u(y) where y is another number variable) [P] expresses, as the ambiguous value, a function of $x_1, \ldots, x_k$, $a_1, \ldots, a_l$, $\alpha_1, \ldots, \alpha_m$ (function of $x_1, \ldots, x_k, y$, $a_1, \ldots, a_l$, $\alpha_1, \ldots, \alpha_m$) [predicate of $x_1, \ldots, x_k$, $a_1, \ldots, a_l$, $\alpha_1, \ldots, \alpha_m$] primitive recursive uniformly in $a_1, \ldots, a_l$, $\alpha_1, \ldots, \alpha_m$.

The new logical rules and axiom schemata needed are

9D. $\mathrm{C} \supset \mathrm{A}(\mathrm{a})$ / $\mathrm{C} \supset \forall \mathrm{a}\mathrm{A}(\mathrm{a})$.

10D. $\forall \mathrm{a}\mathrm{A}(\mathrm{a}) \supset \mathrm{A}(\mathrm{g})$.

11D. $\mathrm{A}(\mathrm{g}) \supset \exists \mathrm{a}\mathrm{A}(\mathrm{a})$.

12D. $\mathrm{A}(\mathrm{a}) \supset \mathrm{C}$ / $\exists \mathrm{a}\mathrm{A}(\mathrm{a}) \supset \mathrm{C}$.

For 9D and 12D, a is not free in C. For 10D and 11D, g is any D-functor free for a in A(a).

Using these we can easily derive, for all formulas C, $\mathrm{A}(\alpha)$ such that a is not free in $\mathrm{C} \supset \mathrm{A}(\alpha)$, α is not free in C, and a is free for α in $\mathrm{A}(\alpha)$:

$$\mathrm{C} \supset \mathrm{A}(\alpha) \;/\; \mathrm{C} \supset \forall \mathrm{a}\mathrm{A}(\mathrm{a}) \quad \text{and} \quad \mathrm{A}(\alpha) \supset \mathrm{C} \;/\; \exists \mathrm{a}\mathrm{A}(\mathrm{a}) \supset \mathrm{C}.$$

Notice also that $\forall \mathrm{a}\exists \alpha \forall \mathrm{x}\; \mathrm{a}(\mathrm{x}) = \alpha(\mathrm{x})$ is a formal theorem.

As in [12] we follow Kleene's conventions for coding finite sequences of numbers and functions, although our notation differs somewhat from his.[5] Here $\langle x_0, \ldots, x_{k-1}\rangle$ abbreviates $\prod_{i<k} p_i^{x_i+1}$ and $(x_0, \ldots, x_{k-1})$ is $\prod_{i<k} p_i^{x_i}$ where p_i is the $(i+1)^{st}$ prime; $(m)_i$ is the exponent of p_i in the prime factorization of m; $\langle \alpha_0, \ldots, \alpha_{l-1}\rangle$ is $\lambda t\ \langle \alpha_0(t), \ldots, \alpha_{l-1}(t)\rangle$ (similarly with ( ) instead of $\langle\ \rangle$); and $(\alpha)_i$ is $\lambda t(\alpha(t))_i$. We follow Kleene in writing $\overline{\alpha}(x)$ for the standard code $\langle \alpha(0), \ldots, \alpha(x-1)\rangle$ for the sequence of the first x values of α.[6]

If w codes a finite sequence, its length is the number $lh(w)$ of non-zero exponents in the prime factorization of w and for each $i < lh(w)$ the $(i+1)$st term of the sequence is $(w)_i - 1$. The code for the concatenation of finite sequences with codes u and v is $u * v$, and $u * \alpha$ is the infinite sequence defined by

$$(u * \alpha)(t) = \begin{cases} (u)_t - 1 & \text{if } t < lh(u), \\ \alpha(t - lh(u)) & \text{otherwise.} \end{cases}$$

[5]Kleene uses $\langle\ \rangle$, [ ] where we use ( ), $\langle\ \rangle$ respectively. Here [ ] will be given a different meaning.

[6]The notation $\tilde{\alpha}(x)$ for $(\alpha(0), \ldots, \alpha(x-1))$ is seldom used.

Seq(w) is an almost negative formula expressing the primitive recursive predicate $Seq(w)$, "w is the code of a finite sequence of numbers," and $\alpha \in$ w abbreviates $\overline{\alpha}(\text{lh}(\text{w})) = \text{w}$. The primitive recursive coding functions are among the initial functions $f_0, \ldots, f_p$ and their properties are assumed formally. For future applications we assume the characteristic functions of the primitive recursive predicates $T(e, \alpha, a, x, y)$, $T_1(e, w, z, x, y)$ and $U(y)$ are among the initial functions.[7]

We adapt the number-theoretic postulates and recursion equations for the initial functions f_i of [8, pp. 14, 19ff.] to the current situation by writing x, y, z in place of a, b, c. Similarly with Kleene's postulates concerning functions:

^{x}0.1. $\{\lambda \text{x}\ \text{r}(\text{x})\}(\text{t}) = \text{r}(\text{t})$.

^{x}1.1. $\text{x} = \text{y} \supset \alpha(\text{x}) = \alpha(\text{y})$.

^{x}2.1. $\forall \text{x} \exists \alpha \text{A}(\text{x}, \alpha) \supset \exists \alpha \forall \text{x} \text{A}(\text{x}, \lambda \text{y} \alpha((\text{x}, \text{y})))$.

For ^{x}0.1, r(x), t are terms such that t is free for x in r(x). For ^{x}1.1 and ^{x}2.1, x and y are distinct number variables and x is free for α in $\text{A}(\text{x}, \alpha)$.

A formula is **almost negative** if it contains no $\vee$ and no $\exists$ except in parts of the form $\exists$xP, $\exists$aP, $\exists\alpha$P with P prime, and $\exists \text{a} \forall \text{x}\ \text{a}(\text{x}) = \text{t}$ where t is a term not containing a free. Note that $\exists!$yB(y), $\exists!$aB(a) and $\exists!\alpha\text{B}(\alpha)$ are almost negative if B is prime, and then $\forall \text{x} \exists! \text{y} \text{B}(\text{x}, \text{y})$ is almost negative as well.

For each almost negative D-formula A(x, y) in which a and x are free for y we take as an axiom

^{x}2.2!D.$^{-}$ $\forall \text{x} \exists! \text{y} \text{A}(\text{x}, \text{y}) \supset \exists \text{a} \forall \text{x} \text{A}(\text{x}, \text{a}(\text{x}))$.

For any almost negative D-formula A(x, a) in which x is free for a it follows that[8]

*2.1!D.$^{-}$ $\forall \text{x} \exists! \text{a} \text{A}(\text{x}, \text{a}) \supset \exists \text{a} \forall \text{x} \text{A}(\text{x}, \lambda \text{y}\ \text{a}((\text{x}, \text{y})))$.

The Replacement Theorem (Lemma 4.2 of [8]) now holds with "$\text{x}_1, \ldots, \text{x}_n, \text{a}_1, \ldots, \text{a}_m, \alpha_1, \ldots, \alpha_l$" in place of "$\text{x}_1, \ldots, \text{x}_n, \alpha_1, \ldots, \alpha_m$" in the version for formulas. As in Lemma 4.3 of [8] each term, functor and formula has a normal form (without superfluous λs).

Lemmas 5.3 and 5.5 of [8] now have additional lawlike parts. Thus for Lemma 5.3, if y, z are distinct number variables, a is any lawlike function variable, and p(y), q, r(y, z), and r(z) are D-terms not containing a free, with a and y free for z in r(y, z) and r(z), then

(a) $\vdash \exists \text{a} \forall \text{y}\ \text{a}(\text{y}) = \text{p}(\text{y})$.

(b) $\vdash \exists \text{a}[\text{a}(0) = \text{q}\ \&\ \forall \text{y}\ \text{a}(\text{y}') = \text{r}(\text{y}, \text{a}(\text{y}))]$.

(c) $\vdash \exists \text{a} \forall \text{y}\ \text{a}(\text{y}) = \text{r}(\overline{\text{a}}(\text{y}))$ and $\vdash \exists \text{a} \forall \text{y}\ \text{a}(\text{y}) = \text{r}(\tilde{\text{a}}(\text{y}))$.

Lemma 5.5 extends to allow definitions of lawlike functions by cases, combined with primitive or course-of-values recursion, provided the case descriptions are almost negative D-formulas. As an example, for almost negative D-formulas Q_1, Q_2 not containing a free and D-terms r_1, r_2:

[7] See [5] and [7] for details. $T(e, \alpha, a, x, y)$ expresses "y is the Gödel number of a proof of $\{e\}(\alpha, a, x) = u$ for some u," and then $U(y)$ is the u.

[8] Formal theorems *n are distinguished notationally from axioms ^{x}m.

$\forall w[\mathrm{Seq}(w) \supset (Q_1(w) \vee Q_2(w)) \;\&\; \neg(Q_1(w) \;\&\; Q_2(w))] \vdash$

$$\exists a \forall y\; a(y) = \begin{cases} r_1(\bar{a}(y)) & \text{if } Q_1(\bar{a}(y)), \\ r_2(\bar{a}(y)) & \text{if } Q_2(\bar{a}(y)). \end{cases}$$

The last axiom schema of **BD** is the "Bar Theorem" in Kleene's form

$^{x}26.3.$ $\forall\alpha\exists!xR(\bar{\alpha}(x)) \;\&\; \forall w[\mathrm{Seq}(w) \;\&\; R(w) \supset A(w)]]$
$\;\&\; \forall w[\mathrm{Seq}(w) \;\&\; \forall sA(w * \langle s\rangle) \supset A(w)] \supset A(1).$

Here A(w) and R(w) may be any formulas satisfying the obvious restrictions on the variables α, x, w, s. Observe that R is assumed to "bar" *all* choice sequences, not just the lawlike ones.

2.2. The theory BDLS⁻. We now extend **BD** by adding axioms for lawless sequences. Here "DLS(α)" abbreviates a specific almost negative formula of the language of **BD** having free only the arbitrary function variable α; this formula will be given explicitly in the next section. (We purposely leave open the possibility of later interpreting "DLS(α)" as primitive, or as an abbreviation for another formula of this or an expanded language.) As in [12], $[\alpha,\beta]$ is the sequence defined by

$$[\alpha,\beta](2k) = \alpha(k), \quad [\alpha,\beta](2k+1) = \beta(k).$$

Similarly $[\alpha_1,\dots,\alpha_n]$ is the sequence obtained by meshing $\alpha_1,\dots,\alpha_n$. For future reference we introduce also the projection functions

$$^{k}[\beta]_i = \lambda t\; \beta(kt+i)$$

for $0 \le i < k$. These notions have the obvious formal equivalents.

The class of **restricted** formulas is defined by induction as follows. Prime formulas are **restricted**. If A, B are **restricted**, x is a number variable, and a is a definable function variable then A & B, A$\vee$B, $\neg$A, A $\supset$ B, $\forall$xA(x), $\exists$xA(x), $\forall$aA(a) and $\exists$aA(a) are all **restricted**. If $A(\beta,\gamma_1,\dots,\gamma_n)$ is **restricted** and contains free no arbitrary function variables but $\beta,\gamma_1,\dots,\gamma_n$ then $\forall\beta[\mathrm{DLS}([\beta,\gamma_1,\dots,\gamma_n]) \supset A(\beta,\gamma_1,\dots,\gamma_n)]$ and $\exists\beta[\mathrm{DLS}([\beta,\gamma_1,\dots,\gamma_n]) \;\&\; A(\beta,\gamma_1,\dots,\gamma_n)]$ are **restricted**.

The axioms for lawless sequences are then

^{x}DLS1.⁻ $\forall w[\mathrm{Seq}(w) \supset \neg\forall\alpha\neg(\mathrm{DLS}(\alpha) \;\&\; \bar{\alpha}(\mathrm{lh}(w)) = w)].$

^{x}DLS2.⁻ $\forall\alpha[\mathrm{DLS}(\alpha) \supset \forall w[\mathrm{Seq}(w) \supset \neg\forall\beta\neg(\mathrm{DLS}([\alpha,\beta]) \;\&\; \bar{\beta}(\mathrm{lh}(w))= w)]].$

^{x}DLS3.⁻ $\forall\alpha[\mathrm{DLS}(\alpha) \;\&\; A(\alpha) \supset \exists x\forall\beta[\bar{\beta}(x) = \bar{\alpha}(x) \;\&\; \mathrm{DLS}(\beta) \supset A(\beta)]].$

^{x}DLS4.⁻ $\forall\alpha[\mathrm{DLS}(\alpha) \supset \exists xA(\alpha,x)] \supset \exists e\forall\alpha[\mathrm{DLS}(\alpha) \supset \exists!y\; e(\bar{\alpha}(y)) > 0 \;\&$
$\forall y(e(\bar{\alpha}(y)) > 0 \supset A(\alpha, e(\bar{\alpha}(y)) - 1))].$

For ^{x}DLS3⁻ A(α) is restricted and almost negative and contains free no arbitrary function variables but α. For ^{x}DLS4⁻ A(α, x) satisfies the same conditions and in addition e, α, y are free for x in A(α, x).

2.3. The intuitionistic theory IDLS⁻. Kleene's basic theory **B** and his intuitionistic theory **FIM** differed only by a single continuity axiom, "Brouwer's Principle for Functions" [8, ^{x}27.1]. Similarly, but with an important difference: **IDLS⁻** comes from **BDLS⁻** by adjoining the axiom schema we call "Kleene's Principle for Functions":

^{x}KL1. $\forall\alpha[A(\alpha) \supset \exists\beta B(\alpha,\beta)] \supset \exists\tau\forall\alpha[A(\alpha) \supset \forall x\exists! y\ \tau(\langle x\rangle * \overline{\alpha}(y)) > 0\ \&$
$\forall\beta[\forall x\exists y\ \tau(\langle x\rangle * \overline{\alpha}(y)) = \beta(x)+1 \supset B(\alpha,\beta)]]$,

for all almost negative formulas $A(\alpha)$ and all formulas $B(\alpha,\beta)$ where α,β must be distinct arbitrary choice sequence variables.[9] An immediate consequence is "Kleene's Principle for Numbers" for A almost negative and τ, y, α free for x in $B(\alpha, x)$:

*KL2. $\forall\alpha[A(\alpha) \supset \exists x B(\alpha,x)] \supset \exists\tau\forall\alpha[A(\alpha) \supset \exists y\tau(\overline{\alpha}(y)) > 0\ \&$
$\forall y[\tau(\overline{\alpha}(y)) > 0 \supset B(\alpha, \tau(\overline{\alpha}(y)) - 1)]]$.

Kleene observed in [8, p. 74] that the special case *27.4 of ^{x}KL1 in which $A(\alpha)$ is $\forall x\ \sigma(\overline{\alpha}(x)) = 0$, with the additional assumption that σ is a spread-law [8, p. 56], follows from Brouwer's Principle for Functions; he also showed [8, p. 80, *27.16] that *KL2 (hence ^{x}KL1) fails if $A(\alpha)$ is not required to be almost negative.

Two important consequences of ^{x}KL1 are

*KL3. $\forall a\exists\beta B(a,\beta) \supset \exists\tau\forall a[\forall x\exists! y\tau(\langle x\rangle * \overline{a}(y)) > 0\ \&$
$\forall\beta[\forall x\exists y\tau(\langle x\rangle * \overline{a}(y)) = \beta(x)+1 \supset B(a,\beta)]]$

and the corresponding consequence *KL4 of *KL2, both proved by taking the almost negative formula $\exists a(a = \alpha)$ as the $A(\alpha)$. Since $DLS(\alpha)$ will be almost negative also, we conclude that in **IDLS$^-$** every function completely defined on either the species of all lawlike functions or the species of all lawless functions is continuous on that domain, though it may have no continuous extension to **B**.[10]

Since ^{x}KL1 is Kleene function-realizable it can consistently replace ^{x}27.1 in **FIM**. It is obvious, but worth emphasizing, that Kleene's formal systems **B** and **FIM** are subsystems of **BD** (*a fortiori* **BDLS$^-$**) and **IDLS$^-$** respectively.

2.4. Strengthening the density axioms. The theories **BDLS** and **IDLS** are extensions of **BDLS$^-$** and **IDLS$^-$** respectively, obtained by replacing ^{x}DLS1$^-$ and ^{x}DLS2$^-$ by

^{x}DLS1. $\forall w[Seq(w) \supset \exists\alpha(DLS(\alpha)\ \&\ \overline{\alpha}(lh(w)) = w\ )]$,

^{x}DLS2. $\forall\alpha[DLS(\alpha) \supset \forall w[Seq(w) \supset \exists\beta(DLS([\alpha,\beta])\ \&\ \overline{\beta}(lh(w)) = w)]]$.

Because of the "almost negativity" condition on ^{x}DLS3$^-$ and ^{x}DLS4$^-$, even **IDLS** is not an entirely satisfactory intuitionistic theory of lawlike, choice and lawless sequences, yet it is the strongest system whose consistency will be established in this paper. The condition can in fact be relaxed somewhat without strengthening the axioms.

2.5. Sidestepping almost negativity. A formula is **mildly assertive** if it is almost negative or obtainable from almost negative formulas using only disjunction and existential quantification over number and definable function variables; **feebly assertive** if only disjunction and existential number quantification are allowed. In **IDLS$^-$** we can prove the extension *DLS5 of ^{x}DLS3$^-$ to restricted mildly assertive

[9] This extension of Brouwer's Principle is called the "Generalized Continuity Principle" by Troelstra [14] who has used it to characterize Kleene's realizability. Brouwer's Principle follows trivially from it when $A(\alpha)$ is $0 = 0$.

[10] A similar situation arises in the theory of constructive real numbers, where local continuity holds but uniform continuity on $[0,1]$ (Brouwer's "Fan Theorem") may fail.

A(α) and the extension *DLS6 of $^{\times}$DLS4^{-} to restricted feebly assertive A(α, x), always assuming no arbitrary choice sequence variables but α are free in A.

A formula is **assertive** if it is almost negative or obtainable from almost negative formulas using only disjunction and existential quantification. In **IDLS** we can prove the extension *DLS7 of $^{\times}$DLS3^{-} to restricted assertive formulas A(α) containing free no arbitrary function variables but α.

$^{\times}$KL1 cannot consistently be similarly extended. For a counterexample let A(α) be $\forall x\alpha(x) = 0 \vee \neg\forall x\alpha(x) = 0$ and B(α,β) be $(\beta(0) = 0 \supset \forall x\alpha(x) = 0)\ \&\ (\beta(0) \neq 0 \supset \neg\forall x\alpha(x) = 0)$.

§3. Lawlessness relative to *D*.

3.1. The informal notion. Let N be $\{0,1,2,\dots\}$ and B be Baire space N^N. Assume D is a given subclass of B which is closed under relative recursion; we think of D as the class of **lawlike** sequences.

If $\beta \in B$ maps sequence numbers to sequence numbers, β is called a **predictor**. If $\gamma,\delta \in B$ and

$$\delta(n) = \begin{cases} 0 & \text{if } \gamma(m) \neq n \text{ for all } m, \\ \mu m(\gamma(m) = n) + 1 & \text{otherwise} \end{cases}$$

then δ is called the **converse** of γ. A sequence γ is **strongly lawlike** if both γ and its converse are lawlike.

A sequence $\alpha \in B$ will be called **lawless (relative to D)** if for each lawlike predictor π and each strongly lawlike injection γ, there is an x so that

$$\alpha \circ \gamma \in (\overline{\alpha \circ \gamma})(x) * \pi((\overline{\alpha \circ \gamma})(x)).$$

Here $\alpha \circ \gamma$ can be thought of as a **subpermutation** of α, so α is lawless if and only if every lawlike predictor is eventually correct (and hence very often wrong) on every strongly lawlike subpermutation of α.[11]

Note added in proof: A simpler, but equivalent, definition of "lawless (relative to D)" appears in [13].

A finite list of sequences $\alpha_0,\dots,\alpha_{k-1}$ is **independent** if $[\alpha_0,\dots,\alpha_{k-1}]$ is lawless. This convention, which was also used by Michael Fourman in [4], is incompatible with Kreisel and Troelstra's strongly intensional treatment of lawless sequences; however, it greatly simplifies the extensional theory.

All the lemmas of Section 1 of [12] hold for the present notion. We summarize them here, providing constructive proofs (modulo Markov's Principle, for the finite injury priority argument for Lemma 4) of the density lemmas.[12]

LEMMA 1. (Technical Lemma.) *Every strongly lawlike subpermutation of a lawless sequence is lawless. In particular:*

(a) If $[\alpha_0,\dots,\alpha_{k-1}]$ *is lawless, so is* $[\alpha_{\sigma(0)},\dots,\alpha_{\sigma(k-1)}]$ *where* σ *is any permutation of* $\{0,\dots,k-1\}$.

[11] Every strongly lawlike γ is lawlike with lawlike range, and conversely; thus this definition of "lawless relative to D" is equivalent to the one in [12].

[12] The proofs in [12] of the Technical and Uniformity Lemmas were already effective.

(b) If $\alpha_0, \dots, \alpha_{k-1}$ is an independent list, then each α_i is lawless, and if $0 \leq i < j < k$ then α_i and α_j are independent. If α is lawless so is ${}^k[\alpha]_i$ for each $k \in N$ and each $0 \leq i < k$.

(c) If α is lawless, so is $\lambda y\ \alpha((x, y))$ for each $x \in N$.

(d) If w is any sequence number and α is lawless then $w * \alpha$ is lawless.[13]

LEMMA 2. (First Density Lemma.) *Assume D is countable and let L be the class of all sequences lawless relative to D. Then L is dense in B.*

PROOF. By Lemma 1(d) we need only produce one lawless sequence β. Let $T = \{\tau_0, \tau_1, \dots\}$ be an enumeration of D^3 which is recursive in the given enumeration of D. Call a triple $\tau = ((\tau)_0, (\tau)_1, (\tau)_2)$ **good** if $(\tau)_0$ is a predictor and $(\tau)_1$ is a strongly lawlike injection with converse $(\tau)_2$.

Call a triple τ **nice at** w **for** n when both w and $(\tau)_0(w)$ are sequence numbers and if

$$p = lh(w * (\tau)_0(w)) \quad \text{and} \quad m = max(n, max\{(\tau)_1(i) : 0 \leq i < p\}) + 1$$

then for each $0 \leq i < j < p$:

$$(\tau)_1(i) \neq (\tau)_1(j)$$

and for each $0 \leq j < m$:

$$\text{if} \quad 0 < (\tau)_2(j) \quad \text{then} \quad (\tau)_1((\tau)_2(j) - 1) = j.$$

Observe that niceness (unlike goodness) is effectively decidable, and τ is good if and only if τ is nice at *every* sequence number w for *every* n.

By induction on k we define x_k, w_k, n_k (with $n_0 < n_1 < \cdots$) and $\overline{\beta}(n_k)$ as follows. For convenience set $n_{-1} = 0$. In general, let x_k be the least $x \geq 0$ such that for all $0 \leq j < n_{k-1}$, $(\tau_k)_2(j) \leq x$. (In particular, $x_0 = 0$.) Let w_k be the sequence number of length x_k such that for each $i < x_k$:

$$(w_k)_i = \begin{cases} \beta((\tau_k)_1(i)) + 1 & \text{if } (\tau_k)_1(i) < n_{k-1}, \\ 1 & \text{otherwise.} \end{cases}$$

If τ_k is not nice at w_k for n_{k-1}, let $n_k = n_{k-1} + 1$ and $\beta(n_k - 1) = 0$. Otherwise, let $p_k = x_k + lh((\tau_k)_0(w_k))$ and $n_k = max(n_{k-1}, max\{(\tau_k)_1(i) : 0 \leq i < p_k\}) + 1$, and for each $n_{k-1} \leq j < n_k$ define

$$\beta(j) = \begin{cases} (w_k * (\tau_k)_0(w_k))_i - 1 & \text{if } (\tau_k)_2(j) = i + 1 \text{ so } (\tau_k)_1(i) = j, \\ 0 & \text{if } (\tau_k)_2(j) = 0. \end{cases}$$

The reader may verify that if τ_k is nice at w_k for n_{k-1} then $(\overline{\beta \circ (\tau_k)_1})(p_k) = w_k * (\tau_k)_0(w_k)$, so β is lawless.

LEMMA 3. (Uniformity Lemma.) *If α is lawless, π is a lawlike predictor, and γ is a strongly lawlike injection, then for each $x_0 \in N$ there is some $x \geq x_0$ such that*

$$\alpha \circ \gamma \in (\overline{\alpha \circ \gamma})(x) * \pi((\overline{\alpha \circ \gamma})(x)).$$

[13] Troelstra's distinction between lawless and protolawless sequences is lost here.

LEMMA 4. (Second Density Lemma.) *If D is countable and α is lawless then the class of all β such that $[\alpha, \beta]$ is lawless is dense in B.*

PROOF. Assume α is lawless relative to D, and let T be as in the proof of Lemma 2. Call a triple τ **α-nice at w for n** if τ is nice at w for n, and if $i < lh(w * (\tau)_0(w))$ and $(\tau)_1(i) = 2q$ then $(w * (\tau)_0(w))_i = \alpha(q) + 1$. We define β in stages.

Stage 0. Let $n_0 = 0$ so $\overline{\beta}(n_0) = \langle\ \rangle = 1$. For notational convenience set $n_{-1} = 0$.

In general, at the conclusion of stage m we have $n_0 < n_1 < \cdots < n_m$ and values $\overline{\beta}(n_m)$. For $k \le m$ we say **$\overline{\beta}(n_k)$ is permanent at m** if for each $j \le k$ either

(i) for some sequence number $w \le m$, τ_j is not nice at w for m, or

(ii) for some $s \le m$, if $w = \overline{[\alpha, \overline{\beta}(n_k) * \lambda t\ 0] \circ (\tau_j)_1}(s)$ and $p = lh(w * (\tau_j)_0(w))$ then τ_j is α-nice at w for m and for each $i < p$: if $(\tau_j)_1(i) = 2q + 1$ then $q < n_k$ and $(w * (\tau_j)_0(w))_i = \beta(q) + 1$. Observe that in this case $w * (\tau_j)_0(w) = \overline{[\alpha, \overline{\beta}(n_k) * \gamma] \circ (\tau_j)_1}(p)$ for every $\gamma \in B$.

Stage $m+1$: Consider the least $k \le m+1$ such that $\overline{\beta}(n_k)$ is not permanent at m. *Case 1.* If for some $s \le m$ τ_k is α-nice at $w = \overline{[\alpha, \overline{\beta}(n_{k-1}) * \lambda t\ 0] \circ (\tau_k)_1}(s)$ for n_{k-1}, let w_k be the least such w and (re)define $n_k = max(n_{k-1}, max\{q : (\tau_k)_1(i) = 2q+1 \text{ for some } 0 \le i < p_k\}) + 1$ where $p_k = lh(w_k * (\tau_k)_0(w_k))$. For $n_{k-1} \le j < n_k$ (re)define

$$\beta(j) = \begin{cases} (w_k * (\tau_k)_0(w_k))_i - 1 & \text{if } (\tau_k)_2(2j+1) = i+1, \\ 0 & \text{if } (\tau_k)_2(2j+1) = 0. \end{cases}$$

If $n_k < m+1$, (re)define $n_{k+i} = n_k + i$ and $\beta(n_{k+i}) = 0$ for $i = 1, \ldots, m+1-k$. Observe that $\overline{\beta}(n_k)$ is permanent at $m+1$ in this case. *Case 2.* Otherwise, set $n_{m+1} = n_m + 1$ and $\beta(n_m) = 0$.

Relative to α and T the construction is effective and for each $k \le m$ one can decide effectively whether $\overline{\beta}(n_k)$ is permanent at m. By Markov's Principle with Lemma 3 and the lawlessness of α, for each k there is a stage at which $\overline{\beta}(n_k)$ becomes permanent, and if τ_k is good then $w_k * (\tau_k)_0(w_k) = \overline{[\alpha, \beta] \circ (\tau_k)_1}(p_k)$; so $[\alpha, \beta]$ is lawless. If u is any sequence number then $[\alpha, u * \beta]$ is lawless by Lemma 1, and the proof is complete.

3.2. The formal predicate. In the language of **IDLS** (or **IDLS$^-$**) we may express "α is lawless relative to D" by the almost negative formula

$$\mathrm{DLS}(\alpha) \equiv \forall b \forall c \forall d[\mathrm{Pred}(b) \ \&\ \mathrm{Inv}(c, d) \supset \exists x\ \alpha \circ c \in (\overline{\alpha \circ c})(x) * b((\overline{\alpha \circ c})(x))]$$

where

$$\mathrm{Pred}(b) \equiv \forall w[\mathrm{Seq}(w) \supset \mathrm{Seq}(b(w))]$$

and

$$\mathrm{Inv}(c, d) \equiv \forall x \forall y[c(x) = y \sim d(y) = x + 1].^{14}$$

The assumption "D is countable" may be expressed formally by $\exists\delta \mathrm{ED}(\delta)$ where

$$\mathrm{ED}(\delta) \equiv \forall n \exists a(a = (\delta)_n) \ \&\ \forall a \exists n(a = (\delta)_n).$$

[14]Note that ∃d Inv(c,d) economically expresses "c is a strongly lawlike injection."

We do *not* assume this formally. Eventually we will consider the weaker assertive assumption $\exists\delta\mathrm{ED}^-(\delta)$ ("D is weakly countable") where

$$\mathrm{ED}^-(\delta) \equiv \forall n \exists a(a = (\delta)_n) \;\&\; \forall a \neg\forall n \neg(a = (\delta)_n).$$

3.3. Consistency questions. By [12], under the assumption of a certain (classically consistent) set-theoretic axiom there is a classical model, with countably many lawlike sequences, for a theory **DLS** of which the current **BDLS** is (modulo notation) a proper subsystem.[15] Thus **BDLS** + $\exists\delta\mathrm{ED}(\delta)$ is classically consistent.

To verify the constructive content as well as the consistency of **IDLS**$^-$ and **IDLS** it is natural to look for realizability interpretations analogous to the one developed by Kleene in [8] for **FIM**. The next section provides a classical function-realizability interpretation for each of the new systems, relative to a defined class D of "lawlike" sequences, under the assumption that D is countable.

§4. The realizability interpretations.

4.1. Definition of D. Let $\mathrm{E}_0(x,y), \mathrm{E}_1(x,y), \ldots$ be an enumeration of all almost negative D-formulas having free no number variables except the distinct variables x and y; in particular let $\mathrm{E}_0(x, y) \equiv a(x) = y$. For each i let

$$\mathrm{F}_i \equiv \forall x \exists! y \mathrm{E}_i(x, y).$$

The primitive recursive function symbols $\lambda, 0, ', +, \cdot, \ldots, f_p, =$ will have their standard interpretations.

If $a_0, \ldots, a_{k-1}$ is the (possibly empty) list of the distinct variables occurring free in F_i in order of first free occurrence, and if $A \subset B$ and $\phi \in B$ and $\psi_0, \ldots, \psi_{k-1} \in A$, we say that E_i **defines** ϕ **over** A **from** $\psi_0, \ldots, \psi_{k-1}$ if and only if, when number variables range over N, definable function variables over A, and arbitrary function variables over B and $a_0, \ldots, a_{k-1}$ are interpreted by $\psi_0, \ldots, \psi_{k-1}$:

(i) F_i is true classically, and

(ii) for $x, y \in N$: $\phi(x) = y$ if and only if $\mathrm{E}_i(x, y)$ is true.

We say E_i **defines** φ **uniformly over** A if and only if for *all* $\psi_0, \ldots, \psi_{k-1} \in A$, E_i defines $\varphi[\psi_0, \ldots, \psi_{k-1}] = \varphi$ over A from $\psi_0, \ldots, \psi_{k-1}$. Observe that E_0 defines φ uniformly over A, where $\varphi[\phi] = \phi$.

Now let $\mathsf{Def}(A)$ be the class of all $\phi \in B$ which are definable over A by some E_i, from some $\psi_0, \ldots, \psi_{k-1} \in A$. Let

$$D_0 = \emptyset, \; D_{\eta+1} = \mathsf{Def}(D_\eta),$$

and for limit ordinals λ :

$$D_\lambda = \bigcup_{\zeta<\lambda} D_\zeta.$$

We want this induction to close off at a countable ordinal. The key is to observe that $\bigcup_{\zeta \in OR} D_\zeta$ has a natural definable well-ordering.

[15] **DLS** omits the "!" in ^{x}2.2!D$^-$ and the requirements of almost negativity from all axioms, strengthens the present ^{x}DLS4$^-$ and asserts the countable axiom of choice for the class of lawlike functions. The classical model naturally fails to satisfy ^{x}KL1.

In general, if $\prec$ well-orders A, and $\phi, \psi \in \mathsf{Def}(A)$, set $\phi \prec^* \psi$ if and only if either

(i) $\phi, \psi \in A$ and $\phi \prec \psi$, or

(ii) $\phi \in A$ and $\psi \notin A$, or

(iii) $\phi \notin A, \psi \notin A$, and $\Delta_A(\phi) < \Delta_A(\psi)$, where $\Delta_A(\phi)$ is the smallest tuple $(i, \psi_0, \dots, \psi_{k-1})$ in the lexicographic well-ordering $<$ of $N \cup \bigcup_{k>0}(N \times A^k)$ determined by $<$ on N and $\prec$ on A such that E_i defines ϕ over A from $\psi_0, \dots, \psi_{k-1}$.

Now let

$$\prec_0 = \emptyset, \qquad \prec_{\eta+1} = (\prec_\eta)^*,$$

and for limit ordinals λ :

$$\prec_\lambda = \bigcup_{\zeta<\lambda} \prec_\zeta .$$

Clearly $\prec_\eta$ well-orders D_η for each ordinal η. Since each $D_\eta \subset D_{\eta+1} \subset B$, by cardinality considerations there is a least ordinal ξ such that $D_\xi = D_{\xi+1}$; for this ξ let

$$D = D_\xi \quad \text{and} \quad \prec = \prec_\xi .$$

Then $\prec$ is a definable well-ordering of D. In fact, both D and $\prec$ are Δ^2_1 definable over B. If E_i defines $\varphi[\psi_0, \dots, \psi_{k-1}]$ uniformly over D we naturally call $\lambda\Psi\ \varphi$ a **definable operator on** D.

4.2. The countability assumption. We now *assume* that D is countable, in accord with Brouwer's assertion [1] (see also [3]) that every well-ordered species is countable and with the discussion in Section 1.2 above. Levy [11] proved the classical consistency with **ZFC** (relative to **ZF**) of the assumption that every definably well-ordered subclass of Baire space is countable; hence our assumption is classically consistent as well as constructively plausible.

No enumerating function can itself be lawlike, since D is closed under recursive operations and if δ enumerates D then for no $n \in N$ is $\lambda t((\delta(t))_t + 1) = (\delta)_n$. All we are assuming is that *some* enumerating function exists (i.e., that $\exists\delta\ \mathrm{ED}(\delta)$ is classically true) so the conclusions of the density lemmas hold.

4.3. Realizability/$_D$ and realizability//$_D$. Following Kleene, if $\tau, \alpha \in B$ we say $\{\tau\}[\alpha]$ is **properly defined** if and only if $(t)(E!y)\tau(\langle t \rangle * \overline{\alpha}(y)) > 0$, and then

$$\{\tau\}[\alpha](t) \simeq \tau(\langle t \rangle * \overline{\alpha}(\mu y \tau(\langle t \rangle * \overline{\alpha}(y)) > 0)) - 1.$$

If $x \in N$ then $\{\tau\}[x] \simeq \{\tau\}[\lambda t\ x]$; and $\{\tau\} \simeq \{\tau\}[0]$. If $x_1, \dots, x_k \in N$ and $\alpha_1, \dots, \alpha_m \in B$ then

$$\{\tau\}[x_1, \dots, x_k, \alpha_1, \dots, \alpha_m] \simeq \{\tau\}[(x_1, \dots, x_k, \alpha_1, \dots, \alpha_m)].$$

As in [8], if $\varphi[\Theta, \alpha]$ is partial recursive then there is a primitive recursive functional $\Lambda\alpha\ \varphi[\Theta, \alpha]$ such that

$$\{\Lambda\alpha\ \varphi[\Theta, \alpha]\}[\alpha] \simeq \varphi[\Theta, \alpha]$$

and if $\varphi[\Theta, \alpha]$ is completely defined then $\{\Lambda\alpha\ \varphi[\Theta, \alpha]\}[\alpha]$ is properly defined. Also $\Lambda x\ \varphi[\Theta, x] = \Lambda\alpha\ \varphi[\Theta, \alpha(0)]$ and $\Lambda\ \varphi[\Theta] = \Lambda x\ \varphi[\Theta]$, so $\{\Lambda x\ \varphi[\Theta, x]\}[x] \simeq \varphi[\Theta, x]$ and $\{\Lambda\ \varphi[\Theta]\} \simeq \varphi[\Theta]$.

Kleene's Λ thus incorporates the meaning of his S^m_n theorem. There is an obvious relativized notion $\Lambda^{\Psi}\alpha$ for functionals recursive in Ψ, where Ψ is any list of functions from B.

An **appropriate** interpretation of a list $\alpha_0, \ldots, \alpha_k, \mathrm{a}_0, \ldots, \mathrm{a}_m, \mathrm{x}_0, \ldots, \mathrm{x}_n$ of variables of the types indicated is any choice of functions $\alpha_0, \ldots, \alpha_k \in B$, $\phi_0, \ldots, \phi_m \in D$, and numbers $x_0, \ldots, x_n$. We now define when $\pi \in B$ **realizes-Ψ** a formula E all of whose distinct free variables are interpreted appropriately by the functions and numbers Ψ. The reader acquainted with Kleene's function-realizability interpretations need look only at the new Clauses 8 and 9.

1. π **realizes-Ψ** a prime formula P, if P is true-Ψ.

2. π **realizes-Ψ** A & B, if $(\pi)_0$ **realizes-Ψ** A and $(\pi)_1$ **realizes-Ψ** B.

3. π **realizes-Ψ** A $\vee$ B, if $(\pi(0))_0 = 0$ and $(\pi)_1$ **realizes-Ψ** A, or $(\pi(0))_0 \neq 0$ and $(\pi)_1$ **realizes-Ψ** B.

4. π **realizes-Ψ** A $\supset$ B, if, if σ **realizes-Ψ** A, then $\{\pi\}[\sigma]$ (is properly defined and) **realizes-Ψ** B.

5. π **realizes-Ψ** $\neg$A, if π **realizes-Ψ** A $\supset 1 = 0$.

6. π **realizes-Ψ** $\forall$xA(x), if, for each $x \in N$, $\{\pi\}[x]$ **realizes-Ψ, x** A(x).

7. π **realizes-Ψ** $\exists$xA(x), if $(\pi)_1$ **realizes-$\Psi, (\pi(0)_0)$** A(x).

8. π **realizes-Ψ** $\forall$aA(a), if, for each $\phi \in D$, $\{\pi\}[\phi]$ is completely defined and **realizes-Ψ, ϕ** A(a).

9. π **realizes-Ψ** $\exists$aA(a), if $\{(\pi)_0\} \in D$ and $(\pi)_1$ **realizes-$\Psi, \{(\pi)_0\}$** A(a).

10. π **realizes-Ψ** $\forall\alpha$A(α), if, for each $\alpha \in B$, $\{\pi\}[\alpha]$ is completely defined and **realizes-Ψ, α** A(α).

11. π **realizes-Ψ** $\exists\alpha$A(α), if ($\{(\pi)_0\}$ is properly defined and) $(\pi)_1$ **realizes-$\Psi, \{(\pi)_0\}$** A(α).

We say a closed formula E is **realizable/$_D$** [**realizable//$_D$**], if a function π general recursive in finitely many functions of [and finitely many definable operators on] D realizes E. An open formula is **realizable/$_D$** [**realizable//$_D$**] if and only if its closure is.

Note that a formula E all of whose free variables occur among Ψ is realizable/$_D$ [realizable//$_D$] if and only if there is a function φ partial recursive in finitely many functions of [and definable operators on] D such that, for each appropriate Ψ, $\varphi[\Psi]$ is completely defined and realizes-Ψ E. Such a φ is called a **realization/$_D$** [**realization//$_D$**] **function** for E.

LEMMA 5. *Let Ψ be any list of variables and let Ψ_1 be those of Ψ which occur free in* E. *For each ε and each appropriate interpretation of the variables: ε realizes-Ψ_1* E *if and only if ε realizes-Ψ* E.

LEMMA 6. *For each assertive formula* E *containing free only the variables Ψ and each appropriate Ψ:*

(i) If E *is realized-Ψ by some function ε then* E *is true-Ψ.*

To each almost negative formula E *containing free only the variables Ψ there is a partial recursive function $\varepsilon_{\mathrm{E}}[\Psi] = \lambda t \varepsilon_{\mathrm{E}}(\Psi, t)$ such that for each appropriate interpretation Ψ of the free variables:*

(ii) If E *is true-Ψ then $\varepsilon_{\mathrm{E}}[\Psi]$ is completely defined and realizes-Ψ* E.

The proof is like that of Lemma 8.4 of [8], with three new cases for (ii). If E is ∀aA(a) then $\varepsilon_E[\Psi]$ is $\Lambda\phi\varepsilon_{A(a)}[\Psi,\phi]$. If E is ∃aA(a) where A(a) is prime and Ψ is β, c, x then $\varepsilon_E[\Psi]$ is $(\Lambda\ \lambda t(\mu s[Seq(s)\ \&\ T_1^{2,1}(\overline{\beta}(lh(s)),\overline{\varsigma}(lh(s)),s,n,x)])_t - 1, \lambda t\, 0)$, where n is a Gödel number of the primitive recursive predicate $P(\phi,\beta,\varsigma,x)$ expressed by A(a). And if E is ∃a∀x(a(x) = t) where t = t[Ψ, x] is a term containing only Ψ, x free, representing the primitive recursive function $t[\Psi, x]$, then $\varepsilon_E[\Psi]$ is $(\Lambda\ \lambda x\ t[\Psi,x], \Lambda x\lambda s\ 0)$.

Since the predicate DLS(α) is almost negative, in particular α is lawless relative to D if and only if DLS(α) is realized-α by some function ε, if and only if $\varepsilon_{DLS(\alpha)}[\alpha]$ realizes-α DLS(α). This fact will be crucial to the proof of the main theorem.

LEMMA 7. *Lemma 8.5 of [8] on numeralwise representability (expressibility) of general recursive functions (predicates) is true for all the formal systems here, as are Lemmas 8.7 and 8.8 on formal decidability of the representing predicates. Hence D is closed under recursion.*

LEMMA 8. *Lemmas 9.1 and 9.2 of [8] on substitution of terms and functors carry over, with the new part for Lemma 9.1:*

(c) If g[Ψ_1, a] *is a D-functor, free for* a *in* A(a) *and containing free only the definable function and number variables* Ψ_1, a *where* $\Psi_1 \subset \Psi$, *then* g *represents a primitive recursive function* $g[\Psi_1,\phi]$, *and* ε *realizes-*Ψ, $g[\Psi_1,\phi]$ A(a) *if and only if* ε *realizes-*Ψ, φ A(g).

4.4. The consistency of ***IDLS⁻***.

THEOREM 1. *If* Γ ⊢ E *in* ***IDLS⁻*** *and the formulas* Γ *are all realizable*$//_D$ *then* E *is realizable*$//_D$.

PROOF. For each axiom E of **IDLS** which is "new" (in the sense that it is not an axiom of **FIM** extended to the language of **IDLS⁻**) we give a realization$//_D$ function $\varphi = \lambda\Psi\ \varphi[\Psi]$ where $\varphi[\Psi] = \lambda t\varphi(\Psi, t)$; and assuming that such a $\varphi'[\Psi']$ exists for each premise of a new rule of inference, we give a $\varphi[\Psi]$ for the conclusion.[16]

10D. ∀aA(a) ⊃ A(g) where g is a D-functor free for a in A(a). Let $\varphi[\Psi]$ be $\Lambda\sigma\ \{\sigma\}[g[\Psi_1,\phi]]$ where $g[\Psi_1,\phi]$ is the interpretation of g[Ψ_1, a] and φ interprets a in Ψ.

11D. A(g) ⊃ ∃aA(a) with the same conditions on g. Define $\varphi[\Psi]$ to be $\Lambda\sigma\ (\Lambda\ g[\Psi_1,\phi],\sigma)$.

Rule 9D. C ⊃ A(a) / C ⊃ ∀aA(a) where a is not free in C. If $\varphi'[\Psi,\phi]$ realizes-Ψ, φ C ⊃ A(a) for each $\phi \in D$, let $\varphi[\Psi]$ be $\Lambda\sigma\Lambda\phi\{\varphi'[\Psi,\phi]\}[\sigma]$.

Rule 12D. A(a) ⊃ C / ∃aA(a) ⊃ C. If $\varphi'[\Psi,\phi]$ realizes-Ψ, φ A(a) ⊃ C for each $\phi \in D$, let $\varphi[\Psi]$ be $\Lambda\sigma\{\varphi'[\Psi,\{(\sigma)_0\}]\}[(\sigma)_1]$.

^{x}2.2!D.⁻ ∀x∃!yA(x, y) ⊃ ∃a∀xA(x, a(x)) where A(x, y) is almost negative and all the free variables Ψ are lawlike function or number variables. If σ realizes-Ψ the hypothesis then ∀x∃!yA(x, y) is true-Ψ (using Lemma 6), so A(x, y) defines

[16]In the proof of Kleene's corresponding theorem for **FIM** (Theorem 9.3 of [8]), recursive realization functions were provided for all the "old" axioms and rules.

$\psi = \lambda x(\{\sigma\}[x](0))_{0,0}$ over D from Ψ, so $\psi \in D$. The axiom is realized$//_D$ by $\varphi[\Psi] = \Lambda\sigma(\Lambda\ \lambda x(\{\sigma\}[x](0))_{0,0}, \Lambda x(\{\sigma\}[x])_{0,1})$.

^{x}DLS1$^-$ and ^{x}DLS2$^-$ are almost negative, and true by the density lemmas with the countability assumption, so Lemma 6 provides recursive realization functions for them.

^{x}DLS3$^-$. $\forall\alpha[\mathrm{DLS}(\alpha)\ \&\ \mathrm{A}(\alpha) \supset \exists x\forall\beta[\overline{\beta}(x) = \overline{\alpha}(x)\ \&\ \mathrm{DLS}(\beta) \supset \mathrm{A}(\beta)]]$, where $\mathrm{A}(\alpha)$ is restricted and almost negative and contains free no arbitrary sequence variables but α. For convenience denote the almost negative subformula $\forall\beta[\overline{\beta}(x) = \overline{\alpha}(x)\ \&\ \mathrm{DLS}(\beta) \supset \mathrm{A}(\beta)]$ by $\mathrm{F}(\overline{\alpha}(x))$, so the axiom $\mathrm{E}[\Psi]$ is $\forall\alpha[\mathrm{DLS}(\alpha)\ \&\ \mathrm{A}(\alpha) \supset \exists x\mathrm{F}(\overline{\alpha}(x))]$. By Lemma 6, there is a partial recursive function $\varepsilon_{\mathrm{F}}[\Psi,\alpha,x]$ which realizes-Ψ,α,x $\mathrm{F}(\overline{\alpha}(x))$ if and only if $\mathrm{F}(\overline{\alpha}(x))$ is true. By induction on the logical form of $\mathrm{A}(\alpha)$, we provide a function $\xi(\Psi,\alpha,\sigma)$ partial recursive in functions from and definable operators on D so that if σ realizes-Ψ,α $\mathrm{DLS}(\alpha)\ \&\ \mathrm{A}(\alpha)$, then $\xi(\Psi,\alpha,\sigma)$ is defined and $\mathrm{F}(\overline{\alpha}(x))$ is true-$\Psi,\alpha,\xi(\Psi,\alpha,\sigma)$ and hence $\rho[\Psi,\alpha,\sigma] = (\lambda t\xi(\Psi,\alpha,\sigma), \varepsilon_{\mathrm{F}}[\Psi,\alpha,\xi(\Psi,\alpha,\sigma)])$ realizes-Ψ,α $\exists x\mathrm{F}(\overline{\alpha}(x))$. Then $\varphi[\Psi] = \Lambda\alpha\Lambda\sigma\ \rho[\Psi,\alpha,\sigma]$ realizes-Ψ the axiom.

1. $\mathrm{A}(\alpha)$ is $\mathrm{s} = \mathrm{t}$, where s expresses $s(\Psi,\alpha)$ and t expresses $t(\Psi,\alpha)$ and Ψ consists only of numbers $z_0,\dots,z_{l-1}$ and elements $\psi_0,\dots,\psi_{k-1}$ of D. Since s and t are primitive recursive the (representing function of the) predicate $\mathrm{s} = \mathrm{t}$ has a Gödel number e from Ψ,α. Let

$$\xi(\Psi,\alpha,\sigma) \simeq \mu x T_1^{k+1,l}(\overline{\psi_0}(x),\dots,\overline{\psi_{k-1}}(x),\overline{\alpha}(x),e,z_0,\dots,z_{l-1}).$$

If σ realizes-Ψ,α $\mathrm{DLS}(\alpha)\ \&\ \mathrm{s} = \mathrm{t}$ then $\alpha \in L$ and $s(\Psi,\alpha) = t(\Psi,\alpha)$ is true, so $\xi(\Psi,\alpha,\sigma)$ is defined and has an appropriate value.

2. $\mathrm{A}(\alpha)$ is $\mathrm{B}(\alpha)\ \&\ \mathrm{C}(\alpha)$. By the induction hypothesis there are realization$//_D$ functions $\chi_{\mathrm{B}},\chi_{\mathrm{C}}$ for the instances of ^{x}DLS3$^-$ with $\mathrm{B}(\alpha)$, $\mathrm{C}(\alpha)$ respectively as the $\mathrm{A}(\alpha)$. If σ realizes-Ψ,α $\mathrm{DLS}(\alpha)\ \&\ \mathrm{A}(\alpha)$, then $\nu_{\mathrm{B}} = ((\sigma)_0,(\sigma)_{1,0})$ realizes-Ψ,α $\mathrm{DLS}(\alpha)\ \&\ \mathrm{B}(\alpha)$ and $\nu_{\mathrm{C}} = ((\sigma)_0,(\sigma)_{1,1})$ realizes-Ψ,α $\mathrm{DLS}(\alpha)\ \&\ \mathrm{C}(\alpha)$, so take $\xi(\Psi,\alpha,\sigma)$ to be the larger of $(\{\{\chi_{\mathrm{B}}[\Psi]\}[\alpha]\}[\nu_{\mathrm{B}}](0)_0)$ and $(\{\{\chi_{\mathrm{C}}[\Psi]\}[\alpha]\}[\nu_{\mathrm{C}}](0)_0)$.

4. $\mathrm{A}(\alpha)$ is $\neg\mathrm{B}(\alpha)$. By the induction hypothesis there is a realization$//_D$ function χ_{B} for the instance of ^{x}DLS3$^-$ for $\mathrm{B}(\alpha)$. Recall that 1 is the smallest sequence number, coding the empty sequence $\langle\ \rangle$. Consider the almost negative predicates

$$\mathrm{D(w,v)} \equiv (\mathrm{Seq(w)} \supset \mathrm{Seq(v)})\ \&\ (\mathrm{v} = 1 \supset \forall\beta[\beta \in \mathrm{w}\ \&\ \mathrm{DLS}(\beta) \supset \neg\mathrm{B}(\beta)])$$
$$\&\ (\mathrm{v} > 1 \supset \forall\beta[\beta \in \mathrm{w} * \mathrm{v}\ \&\ \mathrm{DLS}(\beta) \supset \mathrm{B}(\beta)]),$$

$$\mathrm{E(w,v)} \equiv \mathrm{D(w,v)}\ \&\ \forall\mathrm{u}(\mathrm{u} < \mathrm{v} \supset \neg\mathrm{D(w,u)}).$$

Classically, $\forall\mathrm{w}\exists!\mathrm{vE(w,v)}$ is true-Ψ, by the following argument. If w is a sequence number and $\mathrm{DLS}(\gamma)\ \&\ \mathrm{B}(\gamma)$ is true-Ψ,γ for some γ with $\gamma \in w$ then by Lemma 6 some ε recursive in Ψ,γ realizes-Ψ,γ $\mathrm{DLS}(\gamma)\ \&\ \mathrm{B}(\gamma)$. Hence $\{\{\chi_{\mathrm{B}}[\Psi]\}[\gamma]\}[\varepsilon]$ realizes-Ψ,γ $\exists x\forall\beta[\overline{\beta}(x) = \overline{\gamma}(x)\ \&\ \mathrm{DLS}(\beta) \supset \mathrm{B}(\beta)]$, so this feebly assertive formula is true-Ψ,γ and there is a nontrivial sequence number v for which $\forall\beta[\beta \in \mathbf{w} * \mathbf{v}\ \&\ \mathrm{DLS}(\beta) \supset \mathrm{B}(\beta)]$ is true-Ψ. So $\mathrm{E(w,v)}$ defines a function ϕ uniformly from $\Psi \in D$, so $\phi \in D$. Since ϕ is a lawlike predictor, for each α making $\mathrm{DLS}(\alpha)$ true there is some x for which $\alpha \in \overline{\alpha}(x) * \phi(\overline{\alpha}(x))$, so if $\neg\mathrm{B}(\alpha)$ is true-Ψ then $\phi(\overline{\alpha}(x)) = 1$. Let

$$\xi(\Psi,\alpha,\sigma) \simeq \mu x\ \alpha \in \overline{\alpha}(x) * \phi(\overline{\alpha}(x)).$$

5. A(α) is B(α) $\supset$ C(α). Similarly, consider the almost negative predicates

$$G(w,v) \equiv (Seq(w) \supset Seq(v)) \ \& \ (v = 1 \supset \forall\beta[\beta \in w \ \& \ DLS(\beta) \supset (B(\beta) \supset C(\beta))])$$
$$\& \ (v > 1 \supset \forall\beta[\beta \in w * v \ \& \ DLS(\beta) \supset B(\beta) \ \& \ \neg C(\beta)]),$$
$$H(w,v) \equiv G(w,v) \ \& \ \forall u(u < v \supset \neg G(w,u)).$$

By Cases 2 and 4 (already established) with the induction hypothesis on B and C, classically H(w,v) defines a lawlike predictor ϕ uniformly from $\Psi \in D$. Define ξ from ϕ as in Case 4.

6. A(α) is $\forall$xB(α, x), where B(α, x) is almost negative. Consider the almost negative predicates

$$J(w, v, x) \equiv (v = 1 \supset \forall\beta[\beta \in w \ \& \ DLS(\beta) \supset \forall x B(\beta, x)])$$
$$\& \ (v > 1 \supset \forall\beta[\beta \in w * v \ \& \ DLS(\beta) \supset \neg B(\beta, x)]),$$
$$K(w,v) \equiv (Seq(w) \supset Seq(v))$$
$$\& \ \neg\forall x\neg J(w, v, x) \ \& \ \forall y(y < v \ \& \ Seq(y) \supset \forall x \neg J(w, y, x)).$$

Classically, by Case 4 and the induction hypothesis on B, K(w,v) defines a lawlike predictor ϕ uniformly from $\Psi \in D$. Define ξ from ϕ as in the preceding two cases.

7. A(α) is $\exists$xB(α, x) where B(α, x) is prime. Let f be a Gödel number of the (representing function of the) primitive recursive predicate expressed by B(α, x), and suppose Ψ consists of k lawlike function variables and l number variables. Let

$$\xi(\Psi, \alpha, \sigma) \simeq \mu x \ T_1^{k+1, l+1}(\overline{\psi_0}(x), \ldots, \overline{\psi_{k-1}}(x), \overline{\alpha}(x), f, z_0, \ldots, z_{l-1}, ((\sigma)_1(0))_0).$$

If σ realizes-Ψ, α DLS(α) & $\exists$xB(α, x) then B(α, x) is true-$\Psi, \alpha, ((\sigma)_1(0))_0$ so $\xi(\Psi, \alpha, \sigma)$ is defined and F($\overline{\alpha}$(x)) is true-$\Psi, \alpha, \xi(\Psi, \alpha, \sigma)$.

8. A(α) is $\forall$aB(α, a). Similar to Case 6.

9. A(α) is $\exists$aB(α, a) where B(α, a) is prime or of the form $\forall$x(a(x) = t(α, x)) for a term t not containing a free. The induction hypothesis gives a realization$//_D$ function χ_B for the instance of ^{x}DLS3$^-$ with B(α, a) as the A(α). If σ realizes-Ψ, α DLS(α) & $\exists$aB(α, a) then $\{(\sigma)_{1,0}\} \in D$ and $\nu_B = ((\sigma)_0, (\sigma)_{1,1})$ realizes-$\Psi, \{(\sigma)_{1,0}\}$ DLS(α) & B(α, a) so $\xi(\Psi, \alpha, \sigma) \simeq (\{\{\chi_B[\Psi, \{(\sigma)_{1,0}\}]\}[\alpha]\}[\nu_B](0))_0$.

10. A(α) is $\forall\gamma$[DLS([α, γ]) $\supset$ B(α, γ)]. By Case 4 with the induction hypothesis there is a realization$//_D$ function $\varphi[\Psi]$ for $\forall\gamma[DLS(\gamma) \ \& \ \neg B(^2[\gamma]_0, ^2[\gamma]_1) \supset \exists x \forall\delta[\overline{\delta}(x) = \overline{\gamma}(x) \ \& \ DLS(\delta) \supset \neg B(^2[\delta]_0, ^2[\delta]_1)]]$. Consider the almost negative predicates

$$L(w,v) \equiv (Seq(w) \supset Seq(v))$$
$$\& \ (v = 1 \supset \forall\beta[\beta \in w \ \& \ DLS(\beta) \supset \forall\gamma[DLS([\beta,\gamma]) \supset B(\beta,\gamma)]])$$
$$\& \ (v > 1 \supset \forall\beta[\beta \in w * v \ \& \ DLS(\beta) \supset \neg\forall\gamma[DLS([\beta,\gamma]) \supset B(\beta,\gamma)]]),$$
$$M(w,v) \equiv L(w,v) \ \& \ \forall y \ (y < v \supset \neg L(w,y)).$$

Then $\forall$w$\exists$!vM(w,v) is true-Ψ so M(w,v) uniformly defines a $\phi \in D$ from which ξ is determined as before.

Since A(α) is restricted and almost negative, no other cases can occur.

^{x}DLS4.$^-$ $\forall\alpha$[DLS(α) $\supset$ $\exists$xA(α, x)] $\supset$ $\exists$eB(e) where B(e) $\equiv$ $\forall\alpha$[DLS(α) $\supset$ $\exists$!y e($\overline{\alpha}$(y)) > 0 & $\forall$y(e($\overline{\alpha}$(y)) > 0 $\supset$ A(α, e($\overline{\alpha}$(y)) $-$ 1))] and A(α, x) is restricted and almost negative with no free function variables but α, so B(e) is almost negative also. Consider the almost negative predicates

$$\begin{aligned}P(w, x) \equiv (\mathrm{Seq}(w) \supset \mathrm{Seq}((x)_0)) \\ \& ((x)_0 > 1 \supset \forall\alpha[\mathrm{DLS}(\alpha) \& \alpha \in w * (x)_0 \supset A(\alpha, (x)_1)]) \\ \& ((x)_0 = 1 \supset \forall\alpha[\mathrm{DLS}(\alpha) \& \alpha \in w \supset \forall y \neg A(\alpha, y)]),\end{aligned}$$

$$Q(w, x) \equiv P(w,x) \& \forall u < x \neg P(w,u).$$

Classically $\forall w \exists ! x Q(w, x)$ is true-Ψ by Lemma 6 with the realizability of ${}^{\times}\mathrm{DLS3}^{-}$, just established, so Q(w, x) defines uniformly from Ψ some $\psi = \psi[\Psi]$ in D. Define χ from ψ by

$$\chi(w) = \begin{cases} (\psi(w))_1 + 1 & \text{if } Seq(w) \text{ and } (\psi(w))_0 > 1, \\ 0 & \text{otherwise,} \end{cases}$$

Then $\chi = \chi[\Psi] \in D$, uniformly in Ψ, and $\Lambda\sigma(\Lambda\, \chi, \varepsilon_{B(e)}[\chi])$ is a realization$//_D$ function for the axiom.

${}^{\times}$KL1. A realization$//_D$ function is $\Lambda\sigma(\Lambda\, \tau, \Lambda\alpha\Lambda\rho\, (\varepsilon_{\forall x \exists ! y \tau(\langle x \rangle * \overline{\alpha}(y)) > 0}, \pi))$, where $\tau = \Lambda\alpha\{(\{\{\sigma\}[\alpha]\}[\varepsilon_{A(\alpha)}])_0\}$ and $\pi = (\{\{\sigma\}[\alpha]\}[\varepsilon_{A(\alpha)}])_1$.

This completes the proof of the theorem.

*4.5. The consistency of **IDLS**.* Suppose δ enumerates the class D defined in Section 4.1 so δ classically satisfies ED(δ). The consistency of **IDLS** is an easy corollary of Theorem 1.

THEOREM 2. *If* $\Gamma \vdash$ E *in **IDLS** and the formulas* Γ *are all realizable*$//_{D\cup\{\delta\}}$ *then* E *is realizable*$//_{D\cup\{\delta\}}$.

PROOF. Realization$//_{D\cup\{\delta\}}$ functions must be provided for the axioms, and for the conclusions of the rules of inference (given realization$//_{D\cup\{\delta\}}$ functions for the hypotheses). Since we have not altered the definition of "ε realizes-Ψ E," lawlike function variables still range over D so by Theorem 1 we need only check the rules of inference and the new axioms. The rules present no problems, and the new axioms can be handled with the help of the density lemmas.

${}^{\times}$DLS1. $\varphi = \Lambda w \Lambda\sigma(\Lambda\, \psi_1[w], (\varepsilon_{DLS(\alpha)}[\psi_1[w]], \lambda t\, 0))$ realizes the axiom, where ψ_1 is recursive in δ by the proof of the first density lemma.

${}^{\times}$DLS2. $\varphi = \Lambda\alpha\Lambda\sigma\Lambda w\Lambda\rho(\Lambda\, \psi_2[\alpha], (\varepsilon_{DLS}([\alpha, \beta])[\alpha, \psi_2[\alpha]], \lambda t\, 0))$ realizes the axiom, where ψ_2 is recursive in δ by the proof of the second density lemma.

4.6. Remarks. No function can realize $\exists\delta \mathrm{ED}(\delta)$, since there is no continuous way of obtaining from an arbitrary $\phi \in D$ an n for which $\phi = (\delta)_n$. Thus $\neg\exists\delta\mathrm{ED}(\delta)$ is realizable (and hence realizable$//_{D\cup\{\delta\}}$) though false in our interpretation, while $\neg\exists$d ED(d) is provable in **BDLS**. On the other hand $\exists\delta\mathrm{ED}^{-}(\delta)$ *is* realizable$//_{D\cup\{\delta\}}$ (but not realizable$//_D$).

As usual Markov's Principle for decidable formulas is (classically) realizable$//_D$ and realizable$//_{D\cup\{\delta\}}$, as is $\neg\forall a \neg(a = u) \supset \exists a(a = u)$ for u any functor not containing a free. The Bar Theorem for lawless sequences and Troelstra's Extension Principle fail in **IDLS**; for counterexamples see [12].

Some of the anomalies of intuitionistic analysis are smoothed out by the lawless subspecies. For example, if B(α) is $\forall x\alpha(x) = 0 \vee \neg\forall x\alpha(x) = 0$ then **IDLS** $\vdash \neg\forall\alpha B(\alpha)$ and **IDLS** $\vdash \neg\forall a B(a)$ but **BDLS** $\vdash \forall\alpha[\mathrm{DLS}(\alpha) \supset B(\alpha)]$. However $\forall\beta[\mathrm{DLS}(\beta) \supset A(\beta) \vee \neg A(\beta)]$ is not in general realizable$//_D$ even for A(β) almost negative; for a trivial counterexample let A(β) be $\forall x(a(x) = 0)$.

The general form $\forall\beta\exists\alpha(A(\beta) \sim \exists x\alpha(x) = 0)$ of Kripke's Schema conflicts with ^{x}KL1, even for $A(\beta) \equiv \forall x\beta(x) = 0$. However, if $A(\beta)$ is almost negative and contains free no arbitrary function variables but β then $\forall\beta[\mathrm{DLS}(\beta) \supset \exists\alpha(A(\beta) \sim \exists x\alpha(x) = 0)]$ is realizable$//_D$ and hence (using Lemma 6) true under the interpretation. In fact, for such $A(\beta)$: $\exists a\forall\beta[\mathrm{DLS}(\beta) \supset (\exists x\, a(\overline{\beta}(x)) = 0 \sim A(\beta))]$ is realizable$//_D$ and hence true.

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