ABSOLUTENESS FOR PROJECTIVE SETS

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Abstract. We study an absoluteness argument for simple forcing in order to get the measurability of low projective sets of reals, such as Σ^1_3-sets.

§0. Introduction. In this note we will present absoluteness properties of the universe and show that they imply measurability (Baire property) of the low projective sets of reals. We will write $\Sigma^1_n(L)$ ($\Sigma^1_n(B)$) for the statement "Every Σ^1_n-set is Lebesgue measurable" (has the Baire property). By the construction of the Lebesgue measure we have that the Borel sets are measurable. The Δ^1_1-sets are exactly the Borel sets. So the first natural question is: Are the Σ^1_1-sets measurable? Sierpinski proved $\Sigma^1_1(L)$ and Luzin proved $\Sigma^1_1(B)$. Let us present the proof of this theorem by using absoluteness arguments.

THEOREM. $\Sigma^1_1(L)$ & $\Sigma^1_1(B)$.

Proof. Let $\varphi(x)$ be a Σ^1_1-formula. Let P be an Amoeba forcing. We know that in V^P we have that $\text{Ran}(V) = \{r : r \text{ is Random over } V\}$ is a measure one set.

Working in V^P, we let $A = \{x : \varphi(x)\}$. If $A \cap \text{Ran}(V) = \emptyset$, then A has measure zero ($\mu(A) = 0$). If $A \cap \text{Ran}(V) \neq \emptyset$ then let $r \in \text{Ran}(V)$ such that $V^P \models \varphi(r)$. But φ is a Σ^1_1-formula; therefore we have that $V[r] \models \varphi(r)$. Let $[\![\varphi(\underset{\sim}{r})]\!]_R$ be the Boolean value of $\varphi(\underset{\sim}{r})$, where $\underset{\sim}{r}$ is the canonical name for the random real. $[\![\varphi(\underset{\sim}{r})]\!]$ is an equivalence class of Borel sets. Let p be a representative.

Now $\mu(p) > 0$. Let $r_1 \in p \cap \text{Ran}(V)$. Then $V[r_1] \models \varphi(r_1)$, and by absoluteness of the Σ^1_1-formula we have that

$$V^P \models \varphi(r_1).$$

Similarly, if $r_1 \in \text{Ran}(V) - p$,

$$V^P \models \neg\varphi(r_1).$$

We actually proved that in V^P

$$\mu(A \,\triangle\, [\![\varphi(\underset{\sim}{r})]\!]) = 0.$$

Therefore we proved that A is a measurable set in V^P. But this is a Σ^1_2-sentence, and by Shoenfield's Lemma we have that this should be true in V. Thus A is measurable in V. To show $\Sigma^1_1(B)$ use $\text{Cohen}(V) = \{x : x \text{ is Cohen over } V\}$. ∎

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Throughout this paper we will generalize this idea to get models for $\Sigma^1_3(L)$ and $\Delta^1_3(L)$.

All the forcing notions in this work are defined in [S]. In [JS1] the following was proved.

THEOREM.

(i) $\Delta^1_2(L)$ *iff* $Ran(L[a]) \neq \emptyset, \forall a \in \mathbb{R}$.
(ii) $\Delta^1_2(B)$ *iff* $Cohen(L[a]) \neq \emptyset, \forall a \in \mathbb{R}$.

R. Solovay proved the following

THEOREM (see [JS1]).

(i) $\Sigma^1_2(L)$ *iff* $\mu(Ran(L[a])) = 1, \forall a \in \mathbb{R}$.
(ii) $\Sigma^1_2(B)$ *iff* $Cohen(L[a])$ *is comeager*, $\forall a \in \mathbb{R}$.

For the definition of Souslin forcing the reader should refer to [JS2].

§1. Δ^1_3-Stability. The main objective of this section is to find a characterization of the statement $\Delta^1_3(L)$ $(\Delta^1_3(B))$. We saw in the introduction that the measurability (like the Baire property) of the Δ^1_2-sets, as well as the Σ^1_2-sets, is intrinsically connected with properties of Random real forcing for the Δ^1_2-sets and Amoeba forcing for the Σ^1_2-sets, respectively.

It was proved by Gödel that if $V = L$ then the real numbers have a Δ^1_2-well order. Also, from this result, we can see that $V = L \models \neg\Delta^1_2(L)$ & $\neg\Delta^1_2(B)$. If we add a real to L then the old Δ^1_2-well order of the constructible reals is no longer Δ^1_2, but will be a Σ^1_2-well order in the big model. This kind of instability leads to the following

Definition. Let P be a forcing notion. We say that V is Δ^1_n-P-Stable if for every pair $(\varphi(x), \psi(x))$ of Σ^1_n-formulas, with parameters in V, we have

$$V \models \forall x(\varphi(x) \leftrightarrow \neg\psi(x)) \text{ iff } V^P \models \forall x(\varphi(x) \leftrightarrow \neg\psi(x))$$

This definition says that V is Δ^1_n-P-Stable when we cannot change the Δ^1_n-sets.

EXAMPLE 1. *V is always Δ^1_1-stable.*

EXAMPLE 2. *Let B be the measure product of $\aleph_1$-many random reals. Then $W = V^B$ is always Δ^1_n-Random-Stable.*

Proof. Let R be the random real forcing, and let $(\varphi(x), \psi(x))$ be a pair of Σ^1_n-formulas with parameters in W. It is well known that $R * B \cong B * R \cong B$. Therefore we may assume without loss of generality (w.l.o.g.) that the parameters in $(\varphi(x), \psi(x))$ are in V. We know, by assumption, that

$$W \models \forall x(\varphi(x) \leftrightarrow \neg\psi(x))$$

iff

$$[\![\forall x(\varphi(x) \leftrightarrow \neg\psi(x))]\!]_B = \mathbf{1} \text{ (by homogeneity of B)}$$

iff

$$[\![\forall x(\varphi(x) \leftrightarrow \neg\psi(x))]\!]_{B*R} = \mathbf{1}$$

iff

$$W^R \models \forall x(\varphi(x) \leftrightarrow \neg\psi(x)),$$

finishing the proof. ∎

The same example is true when Random is replaced by Cohen and Measure by Category, respectively.

FACT.

(i) *V is Δ^1_2-Random-Stable iff $\Delta^1_2(L)$.*
(ii) *V is Δ^1_2-Cohen-Stable iff $\Delta^1_2(B)$.*

Proof. (i) Assume V is Δ^1_2-Random-Stable, and $\neg\Delta^1_2(L)$. By §0, there is $a \in \mathbb{R}$ such that $\mathrm{Ran}(L[a]) = \emptyset$.

We define the following order on $\mathbb{R}$:

$x \prec y$ iff the first Borel measure zero set $A_x \in L[a]$ such that $x \in A_x$ has constructible order less than the first Borel measure zero set $A_y \in L[a]$ such that $y \in A_y$.

It is not hard to see that " $\prec$ " is a Σ^1_2-relation (see [JS1]).

Let us now define

$$x \preceq y \text{ iff } x \prec y \text{ or } A_x = A_y.$$

Also " $\preceq$ " is a Σ^1_2-relation.

By assumption ($\mathrm{Ran}(L[a]) = \emptyset$), we have that

$$(\forall x \forall y)(x \prec y \leftrightarrow \neg(y \preceq x)).$$

By Δ^1_2-Random-Stability we have that in $V[r]$

$$r \prec r \leftrightarrow \neg(r \preceq r),$$

when r is a Random real over V. But $V[r] \models \neg(r \prec r) \;\&\; \neg(r \preceq r)$, a contradiction.

Let us assume $\Delta^1_2(L)$, and assume also that $(\varphi(x), \psi(x))$ are two Σ^1_2-formulas satisfying

$$\forall x(\varphi(x) \leftrightarrow \neg\psi(x)).$$

Let a be such that the parameters of $(\varphi(x), \psi(x))$ are in $L[a]$.

Assume that

$$V^R \models \neg\forall x(\varphi(x) \leftrightarrow \neg\psi(x)).$$

Then we have two cases:

(a) $V^R \models \varphi(\tau(r)) \;\&\; \psi(\tau(r))$, for some $\tau \in V^R$. Then

$$L[a][\tau][r] \models \varphi(\tau(r)) \;\&\; \psi(\tau(r)) \text{ (by absoluteness of the } \Sigma^1_2\text{-formulas).}$$

Therefore there is $p \in R$ such that $L[a][\tau] \models p \Vdash_R \varphi(\tau) \;\&\; \psi(\tau)$. But $\mathrm{Ran}(L[a][\tau]) \neq \emptyset$, therefore for $s \in \mathrm{Ran}(L[a][\tau]) \cap p$, we have

$$L[a][\tau][s] \models \varphi(\tau(s)) \;\&\; \psi(\tau(s)).$$

Hence

$$V \models \varphi(\tau(s)) \;\&\; \psi(\tau(s)),$$

a contradiction.

(b) $V^P \models \neg\psi(\tau(s)) \;\&\; \neg\psi(\tau(s))$, we get an analogous contradiction. This finishes the proof of (i).

(ii) is proved similarly using Cohen instead of Random reals. ▮

FACT.

(i) $V \models \Delta^1_2(L) \to V[r] \models \Delta^1_2(L)$ *for* $r \in Ran(V)$.
(ii) $V \models \Delta^1_2(B) \to V[\check{r}] \models \Delta^1_2(B)$ *for* $\tau \in Cohen(V)$.

Proof. (i) Assume $\Delta^1_2(L)$ and let τ be an R-name for a real. Let r be Random over V. In V we have that there is a random real over $L[\tau]$. Let s be such a real. Then r is Random over $L[\tau][s]$. But two random reals commute. Therefore s is Random over $L[\tau][r]$. Therefore s is Random over $L[\tau(r)]$. This proves $\Delta^1_2(L)$ in $V[r]$.

(ii) is similar. ▮

In the model of example 2, we have that Δ^1_n-Random-Stable holds. But also in this model $\neg\Sigma^1_2(L)$. Thus if we are looking for a statement equivalent to $\Delta^1_3(L)$ we need to strengthen "Δ^1_3-Random-Stable." We don't have yet an exact equivalent statement for $\Delta^1_3(L)$ but the following is our best candidate.

THEOREM. *If* $V \models \Sigma^1_2(L)$ & Δ^1_3*-Random-Stable then* $V \models \Delta^1_3(L)$.

Proof. Let $(\varphi(x), \psi(x))$ be a pair of Σ^1_3-formulas satisfying

$$\forall x(\varphi(x) \leftrightarrow \neg\psi(x)).$$

Therefore, by Δ^1_3-Random-Stability, we have that

$$[\![\forall x(\varphi(x) \leftrightarrow \neg\psi(x))]\!]_R = \mathbf{1}.$$

Therefore, if $\underset{\sim}{r}$ is the canonical name for the random real, we have $\varphi(\underset{\sim}{r})$ or $\psi(\underset{\sim}{r})$ holds in the generic extension by random real forcing. Let us assume, w.l.o.g., that

$$p \Vdash_R \varphi(\underset{\sim}{r}),$$

for $p \in R$.

Let $\varphi_1(x, y)$ be a Π^1_2-formula such that

$$\varphi(x) = \exists y \varphi_1(x, y),$$

and let τ be an R-name such that

$$p \Vdash_R \varphi_1(\underset{\sim}{r}, \tau)$$

is true in V. But, by using absoluteness of Π^1_2-formulas, we have that

$$L[a][p][\tau] \models p \Vdash_R \varphi_1(\underset{\sim}{r}, \tau)$$

(when a encodes the parameters of φ_1). Let b encode a, p, τ. Thus

$$L[b] \models p \Vdash_R \varphi_1(\underset{\sim}{r}, \tau).$$

Now, by $\Sigma^1_2(L)$ we have $\mu(\mathrm{Ran}(L[b])) = 1$. If $x \in \mathrm{Ran}(L[b]) \cap p$, then $L[b][x] \models \varphi_1(x, \tau(x))$ and by absoluteness

$$\varphi_1(x, \tau(x)).$$

Therefore,

$$\exists y \varphi_1(x, y) = \varphi(x).$$

And this shows that

$$\mu(\{x : \varphi(x)\}) = \mu([\![\varphi]\!]_R). \quad \blacksquare$$

COROLLARY.

(i) *If* $V \models \Sigma^1_2(L)$ *then* $V[\aleph_1\text{-}Random] \models \Delta^1_3(L)$.
(ii) *If* $V \models \Sigma^1_2(B)$ *then* $V[\aleph_1\text{-}Cohen] \models \Delta^1_3(B)$.

Proof. (i) By example 2 we have that $V[\aleph_1\text{-Random}] \models \Delta^1_3$-$R$-Stable. We must prove that this model satisfies $\Sigma^1_2(L)$. Let a be a real. We must prove that $\mu(\mathrm{Ran}(L[a])) = 1$ in $V[\aleph_1\text{-Random}]$. It is well known that there is a random real $r \in \mathrm{Ran}(V)$ and an R-name τ such that $a = \tau(r)$. It is enough to show that $\mu(\mathrm{Ran}(L[a])) = 1$ in $V[r]$. Each measure zero set A in $L[a]$ is covered by a Borel measure zero set. Each Borel measure zero set in $L[a]$ has an R-name $\underset{\sim}{B}$ in $L[\tau]$ and also $\underset{\sim}{B}$ can be seen as a Borel set in $\mathbb{R}^2$, $\mu(\underset{\sim}{B}) = 0$. By $\Sigma^1_2(L)$ we have that there is a Borel set B of measure zero such that $\underset{\sim}{B} \subseteq B$ for every Borel set $\underset{\sim}{B} \in L[\tau]$, $\mu(B) = 0$. If r is random real over V, then $\underset{\sim}{B}(r) \subseteq B(r)$ for every $\underset{\sim}{B} \in L[\tau]$, $\mu(\underset{\sim}{B}) = 0$ and $B(r) = \{x : \langle r, x\rangle \in B\}$. Now, $\mathbb{R} - B(r) \subseteq \mathrm{Ran}(L[a])$.

(ii) is similar. $\blacksquare$

The following are open problems.

(1) $V \models \Delta^1_3(L) \to V \models \Delta^1_3$-Random-Stable?
(2) $V \models \Delta^1_3(L) \to V[r] \models \Delta^1_3(L)$, r Random?

§2. Σ^1_4-Absoluteness. In this section we are interested in the statement $\Sigma^1_3(L)$ ($\Sigma^1_3(B)$). Again we are looking for a forcing characterization of such statements; this means that we want to find a forcing statement which is equivalent to $\Sigma^1_3(L)$. Until now we only have sufficient conditions and we conjecture that they are also necessary.

In the study of measure and category on the projective sets, the first theorem about the Σ^1_3-sets was proved by S. Shelah. This results says that the measurability of the projective sets is intrinsically related to the existency of inaccessible cardinals. The theorem says

THEOREM (Shelah). $\Sigma^1_3(L) \to (\forall r \in \mathbb{R})(\omega_1^{L[r]} < \omega_1)$.

In other words, the theorem says that $\aleph_1$ is an inaccessible cardinal in $L[r]$ for every real r. Therefore if we want to understand the statement $\Sigma^1_3(L)$, we are forced to connect it with inner models with an inaccessible cardinal. Larger cardinal assumptions are artificial in this context.

Another interesting phenomenon is that at this stage of the development of the subject we are not able to see the difference, in ZFC, between "$\Sigma^1_3(L)$" and "$\Sigma^1_4(L)$." In other words, at present all the models for $\Sigma^1_3(L)$ are models for $\Sigma^1_4(L)$.

There is a program for building inner models for larger cardinals where $\Sigma^1_3(L)$ holds and $\Sigma^1_4(L)$ fails, but this answer, if it works, will only partially solve the question. We insist that in this context only inaccessible cardinals can be accepted. We can see this inner model development as evidence that eventually an answer in ZFC will be obtained.

It is also interesting to remark that there is a very deep asymmetry between $\Sigma^1_3(L)$ and $\Sigma^1_3(B)$, namely

THEOREM (Shelah). $Con(ZF) \to Con(ZFC + \forall n \Sigma^1_n(B))$.

Shelah started from L and in a ω_1-stage iteration with finite support, he got a model for $\forall n \Sigma^1_n(B)$. This is one of the most sophisticated forcing constructions and it is not really clear if further development can be done using these techniques. Anyway, if we assume $\Sigma^1_2(L)$, then there is no asymmetry between $\Sigma^1_3(L)$ and $\Sigma^1_3(B)$; both imply inner models for inaccessible cardinals. The most interesting open question in the study of the asymmetry between measure and category is the following:

Does $\Sigma^1_3(L) \to \Sigma^1_3(B)$?

Using Jensen's work on the core model it is possible to give a positive answer if we assume the existence of #'s and the non-existence of a proper class of measurable cardinals. But these are very restrictive assumptions. We are sure that this question has an answer in ZFC.

We will start by giving our main definition, and we will study it on the Σ^1_2-sets. After this, we will see what the connection is to the Σ^1_3-sets.

Definition. Let P be a forcing notion. We say that V is Σ^1_n-P-Absolute if for every Σ^1_n-sentence φ with parameters in V we have

$$V \models \varphi \text{ iff } V^P \models \varphi.$$

EXAMPLE. *V is always Σ^1_2-P-Absolute.*

Proof. By Shoenfield's Absoluteness Lemma. ∎

FACT. *The following are equivalent:*

(i) $\Sigma^1_2(L)$ $(\Sigma^1_2(B))$
(ii) *Σ^1_3-Amoeba-Absolute (Σ^1_3-Amoeba-Meager-Absolute)*

Proof. (ii) $\to$ (i) Let a be a real. Then

$$V \models \mu(\mathrm{Ran}(L[a])) = 1$$

is a Π^1_3-sentence in V.

Let P be Amoeba forcing. Then $V^P \models \mu(\mathrm{Ran}(L[a])) = 1$. We have by Σ^1_3-Absoluteness that $V \models \mu(\mathrm{Ran}(L[a])) = 1$.

(i) $\to$ (ii) Let $\varphi = \exists x\psi(x)$ be a Σ^1_3-sentence (so $\psi(x)$ is a Π^1_2-formula). Assume

$$V \models \exists x\psi(x).$$

Let $a \in V \cap \mathbb{R}$ such that

$$V \models \psi(a).$$

By Shoenfield's Absoluteness Lemma, $V^P \models \exists x\psi(x)$.

Assume $V^P \models \exists x\varphi(x)$. Let τ be a P-name for a real such that

$$V^P \models \psi(\tau).$$

Let a be a real encoding the parameters used in ψ. By Shoenfield's Absoluteness Lemma, we have

$$L[a][\tau]^P \models \psi(\tau).$$

Therefore we really have

$$L[a][\tau] \models \Vdash_P \text{“}\psi[\tau]\text{”}.$$

Now, in V we have $\mu(\mathrm{Ran}(L[a][\tau])) = 1$. Therefore there is $G \subseteq P$, an $L[a][\tau]$-generic filter, G in V. Therefore

$$L[a][\tau][G] \models \psi(\tau[G]).$$

By Shoenfield's Absoluteness Lemma

$$V \models \psi(\tau[G]).$$

Thus, $V \models \varphi$, finishing the proof. ∎

FACT. *The following are equivalent:*

(i) $\Delta^1_2(L)$ $(\Delta^1_2(B))$

(ii) Σ^1_3*-Random-Absolute* (Σ^1_3*-Cohen-Absolute*)

Proof. Similar. ∎

There is a very useful fact proved by Martin–Solovay about the Σ^1_3-formulas, namely

THEOREM (Martin–Solovay). *If $\varphi(x)$ is a Σ^1_3-formula and κ is a measurable cardinal and $|P| < \kappa$ and $\underset{\sim}{r}$ is a P-name for a real, then*

$$V[\underset{\sim}{r}] \models \varphi[\underset{\sim}{r}] \text{ iff } V^P \models \varphi(\underset{\sim}{r}).$$

We can improve this theorem in our context (remember that only inaccessible cardinals are accepted) to get

THEOREM. *Assume that $(\forall r \in \mathbb{R})(\omega_1^{L[r]} < \omega_1)$ and P is a Souslin forcing and τ is a P-name for a real, and $\varphi(x)$ is a Σ^1_3-formula. Then*

$$V[\tau] \models \varphi(\tau) \text{ iff } V^P \models \varphi(\tau).$$

Proof. Clearly $V[\tau] \models \varphi(\tau)$ implies $V^P \models \varphi(\tau)$, by using Shoenfield's Absoluteness Lemma. Let us assume $V^P \models \varphi(\tau)$. Let $\varphi(\tau) = \exists x\psi(x, \tau)$. And let $\theta \in V^P$ such that

$$V^P \models \psi(\theta, \tau).$$

Let $a \in \mathbb{R}$ encode the parameters of φ, and the parameters of the definition of P. Then by absoluteness of the maximal antichain of P we have θ, τ are P-names in $L[a][\tau][\theta]$, and also

$$L[a][\tau][\theta]^P \models \psi(\theta, \tau).$$

Now let $G \subseteq P$ be V-generic. Then we have that

$$L[a][\tau[G]][\theta[G]] \models \psi[\theta(G), \tau(G)].$$

Now, $L[a][\tau[G]][\theta[G]]$ is a forcing extension of $L[a][\tau[G]]$, and call this forcing Q. Then

$$L[a][\tau[G]] \models \text{“}(\exists q \in Q)(q \Vdash_Q \psi(\theta, \tau[G]))\text{”}.$$

Now we use the following

FACT. $\omega_1^{L[a][\tau[G]]} < \omega_1$ *in* V^P.

Proof. By [JS2].

But $L[a][\tau[G]] \subseteq V[\tau[G]]$. Therefore in $V[\tau[G]]$ we have that

$$\omega_1^{L[a][\tau[G]]} < \omega_1.$$

Thus $2^{|Q|} \cap L[a][\tau[G]]$ is countable in $V[\tau[G]]$, therefore we can get $H \subseteq Q$, in $V[\tau[G]]$, a generic filter over $L[a][\tau[G]]$ such that

$$L[a][\tau[G]][H] \models \psi(\theta(H), \tau[G]).$$

Now by using Shoenfield's Absoluteness Lemma we have

$$V[\tau[G]] \models \exists x \psi(\tau[G]),$$

finishing the proof of the theorem. ∎

Next we will study the connection between Σ^1_4-Amoeba-Absoluteness and Σ^1_3-Measurability.

Let us start with the following

FACT.

(i) Σ^1_4-*Random-Absoluteness* + $\Sigma^1_2(L) \to \Delta^1_3(L)$.
(ii) Σ^1_4-*Cohen-Absoluteness* + $\Sigma^1_2(B) \to \Delta^1_3(B)$.

Proof. (i) Let $(\varphi(x), \varphi(x))$ be a pair of Σ^1_3-formulas such that

$$\forall x(\varphi(x) \leftrightarrow \neg\psi(x)).$$

But this is a Π^1_4-statement, thus is true after forcing with Random real forcing.

(ii) is similar. ∎

FACT. Σ^1_4-*Random-Absoluteness* + $\Sigma^1_2(L)$ *does not imply* $\Sigma^1_3(L)$.

Proof. Let $M \models MA + \omega_1^L = \omega_1$. By adding $\aleph_1$-many random reals to M we get a model for $\Sigma^1_2(L) + \Sigma^1_4$-Random-Absoluteness. In this model $\omega_1^L = \omega_1$, therefore $\neg\Sigma^1_3(L)$. ∎

THEOREM.

(i) Σ^1_4-*Amoeba-Absoluteness* + $(\forall r \in \mathbb{R})(\omega_1^{L[r]} < \omega_1) \to \Sigma^1_3(L)$.

(ii) Σ^1_4-*Amoeba-Meager-Absoluteness* + $(\forall r \in \mathbb{R})(w_1^{L[r]} < \omega_1)$ $\to \Sigma^1_3(B)$.

Proof. Let $\varphi(x)$ be a Σ^1_3-formula. We want to show that $A = \{x : \varphi(x)\}$ is a measurable set. This is a Σ^1_4-statement. Therefore it is enough to show that A is measurable in V^P, when P is Amoeba forcing. In V^P we have that $\mu(\text{Ran}(V)) = 1$. If $\text{Ran}(V) \cap A = \emptyset$, then

$$V^P \models \mu(A) = 0,$$

and we finish. Therefore we may assume $A \cap \text{Ran}(V) \neq \emptyset$. Let $r \in A \cap \text{Ran}(V)$. Then

$$V^P \models \varphi(r).$$

By the previous theorem

$$V[r] \models \varphi(r).$$

Now in V there is $p \in R$ such that

$$V \models p \Vdash_B \varphi(r),$$

where $\underset{\sim}{r}$ is the canonical R-name for the random real. Let $r_1 \in V^P \cap \text{Ran}(V) \cap p$. Then we have

$$V[r_1] \models \varphi(r_1),$$

and thus $V^P \models \varphi(r_1)$, proving that $\mu(A) = \mu([\![\varphi(\underset{\sim}{r})]\!]_R)$.

(ii) is similar. ▮

Let us introduce a stronger principle, namely

Definition. We say that V is Σ^1_3-P-Correct if for every P-name τ for a real and $\varphi(x)$ a Σ^1_3-formula, we have $V[\tau] \models \varphi(\tau)$ iff $V^P \models \varphi(r)$.

COROLLARY. *If P is Souslin and $(\forall r \in \mathbb{R})(\omega_1^{L[r]} < \omega_1)$, the V is Σ^1_3-P-correct.*

FACT. *If V is Σ^1_3-Amoeba-correct and Σ^1_4-Amoeba-Absolute, then $\Sigma^1_3(L)$.*

Proof. The proof is similar to that of the previous theorem. ▮

CONJECTURE.

(i) Σ^1_4-*Amoeba-Absolute* $\to$ Σ^1_3-*Amoeba-Correctness*

(ii) Σ^1_{n+1}-*Amoeba-Meager-Absoluteness* + Σ^1_n-*Amoeba-Meager-Correctness* $\to \Sigma^1_n(B)$.

FACT.

(i) Σ^1_{n+1}-*Amoeba-Absolute* + Σ^1_n-*Amoeba-Correctness* $\to$ $\Sigma^1_n(L) + \Sigma^1_n(B)$.

(ii) Σ^1_{n+1}-*Amoeba-Meager-Absoluteness* + Σ^1_n-*Amoeba-Meager-Correctness* $\to \Sigma^1_n(B)$.

Proof. We will prove only $\Sigma^1_n(B)$ of (i). Let $\varphi(x)$ be a Σ^1_n-formula. Forcing with P-Amoeba, we have that Cohen(V) is a comeager set in V^P. We leave the rest of the details to the reader. ▮

We suggest that the reader check that Solovay's model satisfies Σ^1_n-Amoeba-Correctness and Σ^1_{n+1}-Amoeba-Absoluteness.

The last result connects Δ^1_n-Stability with Σ^1_{n+1}-Absoluteness. Really they are equivalent.

FACT (Bagaria). *V is Δ^1_n-P-Stable iff V is Σ^1_{n+1}-P-Absolute.*

Proof. One direction is trivial. We will show only the nontrivial part. Let $\exists x \varphi(x)$ be a Σ^1_{n+1}-formula. If

$$V \models \exists x \varphi(x) \text{ then } V \models \varphi(r) \text{ for some } r \in \mathbb{R},$$

by the induction hypothesis $V^P \models \varphi(r)$ then $V^P \models \exists x \varphi(x)$. Assume now $V^P \models \exists x \varphi(x)$, and $V \models \neg\exists x \varphi(x)$. Then $V \models \forall x(\neg\varphi(x) \leftrightarrow x = x)$ by Δ^1_n-Stability $V^P \models \forall x(\neg\varphi(x) \rightarrow x = x)$. Therefore $V^P \models \neg\exists x \varphi(x)$, a contradiction. ▮

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