TEMPORAL EXPRESSIVE COMPLETENESS IN THE PRESENCE OF GAPS

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Abstract

It is known that the temporal connectives *until* and *since* are expressively complete for Dedekind complete flows of time but that the Stavi connectives are needed to achieve expressive completeness for general linear time which may have "gaps" in it. We present a full proof of this result.

We introduce some new unary connectives which, along with *until* and *since* are expressively complete for general linear time. We axiomatize the new connectives over general linear time, define a notion of complexity on gaps and show that *since* and *until* are themselves expressively complete for flows of time with only isolated gaps. We also introduce new unary connectives which are less expressive than the Stavi connectives but are, nevertheless, expressively complete for flows of time whose gaps are of only certain restricted types. In this connection we briefly discuss scattered flows of time.

§1. Introduction: the problem of expressive completeness.

This section will present the problem of expressive completeness of temporal connectives within the more general model theoretic concept of the existence of a finite G-basis for m-adic theories. The known results in this area will then be outlined.

We begin with the ordinary propositional temporal logic. Assume we are given a flow of time $(T, <)$, where T is the set of moments of time and $<$ is a transitive and irreflexive relation on T, thought of as the earlier-later relation. We define the notion of m-dimensional temporal logic on $(T, <)$. An m-dimensional atomic proposition q on $(T, <)$ can be associated with a subset Q of T^m, representing the set of all m-tuples of moments of time where q is true. The boolean logical operations on temporal formulas, such as $\wedge$, $\vee$, $\sim$ and $\rightarrow$ correspond naturally to operations on these subsets. It is clear that a temporal assignment h to the atoms associating with atoms q_i subsets $h(q_i) \subseteq T^m$, gives rise to an ordinary model for $(T, <, Q_i, =)$. To be able to express formally the connections between propositional temporal formulas and subsets of T^m, we need to use the m-adic language with $(T, <, Q_i, =)$, where $Q_i \subseteq T^m$ are m-place predicates and $=$ is equality.

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DEFINITION 1.1.

1. *We define the temporal propositional language* $L[C_1, \ldots, C_n]$, *with connectives* $C_1, \ldots, C_n$ *as follows:*

   (a) *Any atom* q *is a wff.*

   (b) *If* A *and* B *are wffs so are* $A \wedge B$, $A \vee B$, $\sim A$ *and* $A \rightarrow B$.

   (c) *If* C_i *is* n_i*-place and* $A_1, \ldots, A_{n_i}$ *are wffs so is* $C_i(A_1, \ldots, A_{n_i})$.

2. *Let* $(T, <)$ *be a flow of time. Let* Π *be a set of* m*-place predicates. The* m*-adic theory* $(T, <, \Pi, =)$ *is defined as the language with* $(<, =, Q_i \in \Pi)$ *and wffs as follows:*

   (a) $Q_i(x_1, \ldots, x_m), x_i = x_j$ *and* $x_i < x_j$ *are wffs, for* x_j *variables and* $Q_i \in \Pi$.

   (b) *If* φ *and* ψ *are wffs so are* $\varphi \wedge \psi$, $\varphi \vee \psi$, $\sim\varphi$, $\varphi \rightarrow \psi$, $\forall x \varphi$ *and* $\exists x \varphi$.

3. *The temporal language and the* m*-adic language can be connected in the following manner.*

   (a) *Enumerate the atomic propositions of* $L[C_1, \ldots, C_n]$ *as* $q_1, q_2, \ldots$ *and enumerate the* m*-adic predicates of* Π *as* $Q_1, Q_2, \ldots$ *and associate* q_i *with* Q_i.

   (b) *Associate with the connective* $C(p_1, \ldots, p_n)$, *where* $p_1, \ldots, p_n$ *are propositional variables, a formula* $\psi_C(t_1, \ldots, t_m, P_1, \ldots, P_n)$ *with* m *free variables* $t_1, \ldots, t_m$ *and* n m*-adic variables* $P_1, \ldots, P_n$. ψ_C *is called a table for* C.

   (c) *Any model* $(T, <, \Pi)$ *of the* m*-adic language will now give rise to an* m*-dimensional temporal model as follows. Let the assignment* h *be*
   $$h(q_i) = \{(s_1, \ldots, s_m) \mid (T, <, \Pi) \models Q_i(s_1, \ldots, s_m)\}.$$
   Extend h *to all wff by the equations:*
   $$\begin{array}{rcl} h(A \wedge B) & = & h(A) \cap h(B) \\ h(\sim A) & = & T^m - h(A) \\ h(C(A_1, \ldots, A_n)) & = & \{(t_1, \ldots, t_m) \mid (T, <, \Pi) \models \\ & & \psi_C(t_1, \ldots, t_m, h(A_1), \ldots, h(A_n))\} \end{array}$$
   for any connective C.

It is obvious from Definition 1.1 that any formula $\psi(t_1, \ldots, t_m, Q_1, \ldots, Q_n)$ defines an n-place connective $C_\psi(q_1, \ldots, q_n)$ via the following truth table: $C_\psi(q_1, \ldots, q_n)$ holds at $t_1, \ldots, t_m$ iff $\psi(t_1, \ldots, t_m, Q_1, \ldots, Q_n)$ holds, where $Q_i = \{(s_1, \ldots, s_m) \mid q_i \text{ holds at } (s_1, \ldots, s_m)\}$.

In particular the connectives *since* (S) and *until* (U) correspond to the monadic tables:
$$\psi_S(t, Q_1, Q_2) = \exists s < t(Q_1(s) \wedge \forall u(s < u < t \rightarrow Q_2(u)))$$

and

$$\psi_U(t, Q_1, Q_2) = \exists s > t(Q_1(s) \wedge \forall u(s > u > t \to Q_2(u))).$$

Clearly we can use the connectives $S(p,q)$ and $U(p,q)$ to build arbitrary wffs $A(q_1, \ldots, q_n)$. It is easy to see that for each A, there exists a formula $\psi_A(t, Q_1, \ldots, Q_n)$ of the monadic language such that for all t and $q_1, \ldots, q_n$, $A(q_1, \ldots, q_n)$ holds at t iff $\psi_A(t, Q_1, \ldots, Q_n)$ holds, where $Q_i = \{s \mid q_i \text{ holds at } s\}$.
The family of all ψ_A can be defined inductively as follows:

DEFINITION 1.2. *Let $W_1(\{\psi_S, \psi_U\})$ be the smallest set of well formed formulas of the monadic language with one free variable satisfying the following conditions:*

1. *$Q_i(t) \in W_1$ for Q_i atomic.*
2. *If φ, $\psi \in W_1$ so are $\varphi \wedge \psi$, $\sim\varphi$, $\varphi \vee \psi$ and $\varphi \to \psi$.*
3. *$\psi_U, \psi_S \in W_1$.*
4. *If $\psi(t, Q_1, \ldots, Q_n) \in W_1$ with t the free variable and Q_i the monadic letters in ψ and if $\psi_i(t) \in W_1$, for $i = 1, \ldots, n$ then $\psi(t, \psi_1, \ldots, \psi_n)$ is also in W_1, where $\psi(t, \psi_1, \ldots, \psi_n)$ is obtained from $\psi(t, Q_1, \ldots, Q_n)$ by substituting simultaneously $\lambda t\psi_i(t)$ for $\lambda t Q_i(t)$, $i = 1, \ldots, n$.*

DEFINITION 1.3. *In general given formulas $\psi_1(t_1, \ldots, t_m), \ldots, \psi_k(t_1, \ldots, t_m)$ with m free variables the set $W_m(\{\psi_1, \ldots, \psi_k\})$ can be defined in the m-adic language as follows:*

- *$Q_i(t_1, \ldots, t_m) \in W_m$ for Q_i atomic.*
- *If φ, $\psi \in W_m$ so are $\varphi \wedge \psi$, $\sim\varphi$, $\varphi \vee \psi$ and $\varphi \to \psi$.*
- *$\psi_1, \ldots, \psi_k \in W_m$.*
- *If $\psi(t_1, \ldots, t_m, Q_1, \ldots, Q_n) \in W_m$ with $t_1, \ldots, t_m$ exactly the free variables of ψ and Q_i exactly the m-adic predicates in ψ and if $\psi_i(t_1, \ldots, t_m) \in W_m$, for $i = 1, \ldots, n$ then*

  $$\psi(t_1, \ldots, t_m, \psi_1, \ldots, \psi_n)$$

  is also in W_m, where $\psi(t_1, \ldots, t_m, \psi_1, \ldots, \psi_n)$ is obtained from $\psi(t_1, \ldots, t_m, Q_1, \ldots, Q_n)$ by substituting simultaneously $\lambda t_1, \ldots, t_m \psi_i(t_1, \ldots, t_m)$ for $\lambda t_1, \ldots, t_m Q_i(t_1, \ldots, t_m)$, $i = 1, \ldots, n$.

DEFINITION 1.4.

1. *The problem of expressive completeness for a set of m-adic wffs*

   $$\{\psi_1, \ldots, \psi_k\}$$

   over a class $\mathcal{K}$ of flows of time is the question of whether $W_m(\{\psi_1, \ldots, \psi_k\})$ is essentially the set of all m-adic wffs over $\mathcal{K}$: namely whether for any ψ there exists $\phi \in W_m$ such that $\mathcal{K} \models \psi \leftrightarrow \phi$.

2. *The problem of* finite G_m-basis *for the m-adic language over a class $\mathcal{K}$ of flows $(T,<)$ is whether the m-adic language can be represented as equal to a $W_m(\{\phi_1,\ldots,\phi_k\})$ for some finite set $\{\phi_1,\ldots,\phi_k\}$.*

3. *The problem of expressive completeness of* since *and* until *over a class $\mathcal{K}$ is whether $\{\psi_U,\psi_S\}$ form a finite G_1-basis for all monadic wffs over the class $\mathcal{K}$ of models $(T,<,\Pi)$.*

The problem of finite basis is a general model theoretic one. Let $\mathcal{K}$ be a class of models in some language, e.g. it might be the class $\mathcal{G}$ of all groups.

Let $(\mathcal{G},Q_1,Q_2,\ldots,=)$ be the m-adic theory of $\mathcal{G}$ where Q_i are new additional m-ary relational variables. Let $\phi_1,\ldots,\phi_k$ be m-adic formulas with m free variables. We can still define $W_m(\{\phi_1,\ldots,\phi_k\})$ and ask whether W_m essentially equals the set of all m-adic wffs over $\mathcal{G}$. We can thus ask whether the theory of groups admits a finite G_m-basis for its m-adic theory.

H. Kamp in [K] has shown that *since* and *until* form a finite G_1-basis for the monadic theory of Dedekind complete linear orderings. J. Stavi put forward two additional connectives which are shown in Theorem 3 to be a finite G_1-basis for general linear time. A first complete proof of this result is given in this paper. Schlingloff [S] has produced a finite G_1-basis for binary trees. The current paper studies finite bases for linear orderings with manageable gaps.

The problem of the existence of a finite G_m-basis for a class of models $\mathcal{K}$ is related to the notion of Gabbay's H_m-dimension.

DEFINITION 1.5.

1. *A theory $\mathcal{T}$ is said to have a finite H_m-dimension $\leq n$ over a class of models $\mathcal{K}$ iff every wff $\phi(t_1,\ldots,t_m,Q_1,\ldots,Q_k)$ with at most m free variables t_i and arbitrary number k of m-adic predicates is equivalent over $\mathcal{K}$ to a wff $\psi(t_1,\ldots,t_m,Q_1,\ldots,Q_k)$ where ψ uses no more than n distinct bound variable letters.*

2. *The minimal n satisfying* (a) *above is called the H_m-dimension of $\mathcal{T}$.*

THEOREM 1 [GHR].

- *A class $\mathcal{K}$ of models has a finite H_m-dimension if it has a finite G_m-basis.*
- *Let the class $\mathcal{K}$ have H_m-dimension n, then it has a finite G_{m+n}-basis.*
- *There is a class $\mathcal{K}$ of models with H_1-dimension 3 but with no finite G_1-basis.*

Another notion of interest is that of weak m/m'-dimensional logic where $1 \leq m' < m$. This notion arises from m dimensional logic where the atoms $Q_i(t_1,\ldots,t_m)$ depend only on the first m' places. For example for $m' = 1$ the weak $(m/1)$ m-dimensional temporal logic has the Q_i unary. In this case the existence of a finite G_1-basis implies the existence of a finite $G_{m/1}$-basis for any m.

Of special interest for applications are one or two dimensional temporal logics over a linear flow of time. In intuitive terms we are evaluating formulas at points or at intervals (or pairs of points). The problem of finding an expressively complete set of connectives is of special importance. Such connectives are extensively studied in [GHR]. We quote one theorem here of relevance.

DEFINITION 1.6. *Let $\mathcal{K}$ be a class of linear flows of time.*

1. *A formula A of a one-dimensional temporal logic is said to be pure future (past) iff its truth value at a point of any (T, h) for any $T \in \mathcal{K}$ and any h, depends only on the value of the atoms at the future (past) of that point.*

2. *A set of one-dimensional connectives is said to have the separation property over $\mathcal{K}$ iff every formula A can be rewritten equivalently (over $\mathcal{K}$) as a boolean combination of pure past, atomic and pure future formulas.*

THEOREM 2. *A set of one-dimensional connectives $\{C_1, \ldots, C_k\}$ has the separation property over $\mathcal{K}$ iff it forms a G_1-basis over $\mathcal{K}$.*

Separation can be combinatorially checked by trying actually to rewrite any formula into a separated boolean combination. In the case of linear ordering the presence of gaps seems to be of combinatorial importance. As atoms are true or false over stretches of time, the first or last point of truth is very useful. If no such point exists we have a gap. We therefore need to study temporal behaviour around gaps. The case of Dedekind complete flows is simple. *Since* and *until* form a G_1-basis.

If the flow allows for gaps then a lot depends on the kind of gaps allowed. It is clear that in the general case new connectives are needed. It is not hard to show, and indeed our Lemma 3 below shows, that U and S are then not adequate to express some first-order connectives. However, as mentioned in [GPSS], Stavi was able to introduce two new connectives U' and S' so that the set $\{U, S, U', S'\}$ is expressively complete over all linear time. We present what we believe is the first full published proof of this result in Section 8.

For the sake of completeness, we consider the question of whether there are intermediate connectives appropriate for structures in which the gaps are in certain senses nice. In this paper we classify the gaps appearing in linear orders and are then able to introduce new connectives to talk about the behaviour of atoms in the neighbourhood of such gaps. Natural questions arise about the expressive power of sets of these connectives and we are able to present a fairly comprehensive (although by no means complete) range of answers.

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§2. Gaps in the flow of time.

We identify gaps in a flow of time with supremum-less non-empty proper initial segments of the order and insert the gap in the appropriate place in the order. Dedekind complete orders then are those without gaps. The completion T^* of an order T is another order consisting of T and all the gaps in the right places and is Dedekind complete.

The simplest kind of gap imaginable is an *isolated* gap which exists in an open interval of time which is otherwise gap-free. Taking one point out of the reals or sticking two copies of the integers together are two straightforward ways of producing an isolated gap.

We are going to define a hierarchy of kinds of gaps. For any (zero, successor or limit) ordinal α, an αth order gap is a gap which is not of lesser order but lies in an open interval which contains, apart from itself, only gaps of order less than α. So a zero order gap is just an isolated gap.

Of course this hierarchy does not include all the gaps possible. For example, nowhere in the rationals is there a gap of any order at all.

We will use the game characterisation of unranked gaps. Let γ_0 be a gap of T. Players $\forall$ and $\exists$ move alternately, defining a sequence γ_i $(0 < i < \omega)$ of gaps. In each round, $\forall$ chooses an open interval I_i containing γ_i, and $\exists$ chooses $\gamma_{i+1} \in I_i$ with $\gamma_{i+1} \neq \gamma_i$. $\exists$ wins iff the game goes on for ω moves. γ_0 is unranked iff $\exists$ has a winning strategy for the game.

To see this one can employ a straightforward transfinite induction to show that if γ_0 is ranked then $\forall$ has a winning strategy. This simply involves continually choosing open intervals around gap γ_i which contain, apart from γ_i itself, only gaps of lesser ranks. Conversely it can be seen that if γ_0 is unranked then every open interval containing it also contains other unranked gaps. $\exists$ can win by always choosing unranked gaps.

It is interesting to note that if all gaps in a flow of time have ordinal order then the cardinality of the flow is at least as great as the cardinality of any of those orders and for every infinite ordinal α, there exists a flow of time of cardinality the same as α with a gap of order α. Let us prove the first of these statements.

DEFINITION 2.1. *Let T be a linear order of cardinality κ, and suppose that Γ is a set of gaps of T. A gap γ of T is said to be Γ-rich if every open interval I of T containing γ contains $\geq \kappa^+$ gaps from Γ. Here, κ^+ is the next largest cardinal after κ.*

PROPOSITION 1. *Let T be a linear order of cardinality κ. Suppose that Γ is a set of gaps of cardinality $\geq \kappa^+$. Then there is a Γ-rich gap $\gamma \in \Gamma$.*

PROOF. If not, for each $\gamma \in \Gamma$ choose an open interval I_γ with endpoints $a_\gamma < b_\gamma$ in T, such that $\gamma \in I_\gamma$ and $| I_\gamma \cap \Gamma | \leq \kappa$. As $| \Gamma | > \kappa$, there is $\Gamma' \subseteq \Gamma$ with $| \Gamma' | > \kappa$ and $I_\gamma = I$ say, for all $\gamma \in \Gamma'$. Then $\Gamma' \subseteq I$, a contradiction. □

COROLLARY 1. *Let T be a linear order of cardinality κ. Then T has at most κ ranked gaps.*

PROOF. Assume not. Let Γ be a set of κ^+ ranked gaps of T. We will show that any Γ-rich gap is unranked; this will contradict the proposition.

Let γ_0 be an Γ-rich gap. $\forall$ and $\exists$ will play the game above, starting with γ_0. $\exists$ will privately construct sets $\Gamma_i (i < \omega)$ of κ^+ ranked gaps, so that each γ_i is

Γ_i-rich. She begins by defining $\Gamma_0 = \Gamma$.

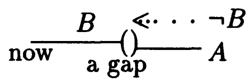
Inductively assume that $i < \omega$, and γ_i is a Γ_i-rich gap. $\forall$ chooses an interval $I_i = (a, b)$ say, around γ_i. I_i contains κ^+ gaps from Γ_i. Let Γ_{i+1} be the gaps from Γ_i contained in (a, γ_i) if this set has cardinality κ^+; otherwise let Γ_{i+1} be the gaps from Γ_i contained in (γ_i, b). So in any case, $|\Gamma_{i+1}| = \kappa^+$. By the proposition, $\exists$ can choose a Γ_{i+1}-rich gap $\gamma_{i+1} \in \Gamma_{i+1}$. If she does this, the game goes on forever and she wins. Hence γ_0 was unranked, as required. □

COROLLARY 2. *Let T be a linear order of cardinality κ and let γ be a ranked gap of T. Then $|rank\ (\gamma)| \leq \kappa$.*

PROOF. Any gap γ of rank α has gaps of rank β arbitrarily close, for all $\beta < \alpha$. So if T has a gap of rank α with $|\alpha| > \kappa$, then T has more than κ ranked gaps. The result follows from the previous corollary now. □

§3. Connectives to talk about gaps.

Recall that $U'(A, B)$ is as pictured:



S' is defined dually i.e., with past and future swapped. Despite involving a gap, U' is in fact a first-order connective and its table is given by:

$$
\begin{array}{llll}
U'(p,q) \equiv & & & \\
\exists s & t < s & & \\
\quad \wedge & \forall u & (& t < u < s \rightarrow \\
& & ([& \exists v(u < v \wedge \forall w(t < w < v \rightarrow q(w)) \quad] \\
& \vee & [& \forall v(u < v < s \rightarrow p(v)) \\
& & \wedge & \exists v(t < v < u \wedge \neg q(v)) \quad])) \\
\quad \wedge & \exists u[t < u < s \wedge \neg q(u)] & & \\
\quad \wedge & \exists u[t < u < s \wedge \forall v(t < v < u \rightarrow q(v))] & &
\end{array}
$$

By presenting our new connectives below in terms of U, S, U' and S' we thus guarantee that they are also first-order.

We start off with some new unary connectives which talk about a single gap located by the vicissitudes of a single temporal formula. First we need to know that there is a gap coming up.

$$
\begin{array}{rcl}
\gamma^+(A) & = & U(\neg A, \top) \wedge U(A, A) \\
& & \wedge \neg U(\neg A, A) \wedge \neg U(\neg U(\top, A), A)
\end{array}
$$

This is true whenever A holds up until a gap but fails to hold arbitrarily soon afterwards. We call such a gap an A left gap: A is true on the left of the gap. Dually we can define γ^- and A right gaps. Notice that $\gamma^\pm$ are expressible in $\{U, S\}$.

Next we specify that the gap coming up is isolated, as far as gaps definable by the same formula and in the same direction go.

$$\gamma_0^+(A) = \gamma^+(A) \wedge U'(\neg\gamma^+(A), A)$$

Dually we can define γ_0^-. Notice that we use the Stavi connectives here.

Now we can recursively define a hierarchy of connectives. For every $n \geq 0$, define

$$\gamma_{\leq n}^+(A) = \gamma_0^+(A) \vee \cdots \vee \gamma_n^+(A)$$

and

$$\begin{aligned} \gamma_{n+1}^+(A) \;=\; & \gamma^+(A) \\ & \wedge\neg\gamma_{\leq n}^+(A) \\ & \wedge U'(\gamma^+(A) \to \gamma_{\leq n}^+(A), A) \end{aligned}$$

$\gamma_{\leq n}^-$ and γ_n^- are defined dually.

Notice that there is a distinction between gaps in the flow of time and gaps definable by a particular temporal formula or even by any temporal formula. Thus we need to define another hierarchy of gaps—this time within a temporal structure rather than just in a flow of time. Let A be a temporal formula. For any ordinal α, an αth order A left gap is an A left gap which is not of lesser order but begins an interval containing only A left gaps of lesser order. Dually we can define A right gaps of each order.

For $\alpha < \omega$, gap γ is an αth order A left gap if and only if $\gamma_\alpha^+(A)$ holds in an interval on the left of γ. We consider the possibility of $\gamma_\alpha^+(A)$ for $\alpha \geq \omega$ later.

We have mentioned the distinction between αth order A gaps and αth order gaps in the flow of time. Nevertheless, it is clear that there is only an A gap when there is a gap in time at the right place and that it is an A gap of order α when that gap in time is at least of order α or possibly of non-ordinal order.

Let us finish this section by demonstrating the existence of definable gaps which do not fit into our scheme of classification. The idea is Robin Hirsch's.

We create a flow of time from a certain subset of the set Q^* of finite sequences of rational numbers. Let T consist of those non-empty sequences in which every rational number but the last is a power of $1/2$ and the last number in the sequence is neither a power of $1/2$ nor zero. We order the sequences as follows: $(a_0, a_1, \ldots, a_m) < (b_0, b_1, \ldots, b_n)$ iff there is some $k \geq 0$ such that $k \leq n$, $k \leq m$, for all $i < k$, $a_i = b_i$ and $a_k < b_k$.

We turn $(T, \leq)$ into a $\{p\}$-structure by making $T \models p(t)$ if and only if the last number in the sequence t is negative.

Each p left gap in T occurs just after the segment $(a\hat{}(-1), a\hat{}0)$ where a is a sequence of powers of $1/2$ (possibly the empty sequence).

It is easy to prove that none of these gaps is isolated. Let $(a\hat{}0, t)$ be any open interval after a gap. If t is not of the form $a\hat{}q\hat{}b$ for some possibly empty sequence b and some rational q then we show that there is a left gap in $(a\hat{}0, a\hat{}1)$ and note that $a\hat{}1$ which is of that form must be less that t. So wlog t is $a\hat{}q\hat{}b$. Let r be any power of $1/2$ less than q. Clearly all elements of T of the form $a\hat{}r\hat{}s$ are in the interval $(a\hat{}0, t)$ and so is the gap at $a\hat{}r\hat{}0$.

Now a very straight forward transfinite induction proves that

LEMMA 1. *For any formula A and any non-zero ordinal α, each αth order A left gap is followed arbitrarily soon by zero order A left gaps.*

So if a flow has no isolated p left gaps then it has no p left gaps of any order at all and we have our result.

§4. Expressive power.

Before we state our new results we mention the theorem which makes our job a lot easier.

THEOREM 3 [GPSS]. *$\{U, S, U', S'\}$ is expressively complete over all linear time.*

We will prove this in Section 8. From this theorem, our first result falls out easily:

LEMMA 2. *Over flows of time with only isolated gaps, $\{U, S\}$ is expressively complete.*

This is because, over such flows,

$$U'(A, B) \equiv \gamma^+(B) \wedge U(\neg B, \gamma^+(B) \vee A).$$

Kamp's pioneering theorem is then a special case of this lemma.

Our next lemma shows that gaps don't have to get much more complicated before *until* and *since* are not sufficient.

LEMMA 3. *In general linear time, $\{U, S\}$ is not expressively complete. There are even flows of time with a single non-isolated gap on which γ_0^+ is not expressible in terms of $\{U, S\}$.*

PROOF. We take a flow of time $(T, \leq)$ with a single non-isolated gap and show that there is no temporal formula built from $\{U, S\}$which is equivalent to $\gamma_0^+(p)$ on all temporal structures over T.

T is constructed in two successive parts: the first one is got by taking a copy of $\mathbb{Z}$ for each negative integer and joining them into one long line and the second has a copy of $\mathbb{Z}$ for each integer arranged in order. There is a gap at the beginning of the whole order and a gap at the end of each copy of $\mathbb{Z}$. The only non-isolated gap is that between the two parts.

A p-structure over T will be called nice iff

- on each little copy of the integers, either p is always true or always false and
- every point has both p and $\neg p$ true in both its past and future.

An easy induction with several cases shows that for any formula ϕ constructed from p in the language $\{U, S\}$, there is a formula p, $\neg p$, $\top$ or $\perp$ which is uniformly equivalent to ϕ everywhere in all nice structures over T. It is easy to show, though, that these four formulae are all distinct in their truth conditions. For example, $U(p, p)$ is always equivalent to p.

So suppose, for contradiction, that we can express $\gamma_0^+(p)$ in $\{U, S\}$. By the above argument, we have a formula ψ always equivalent to it over nice structures.

Look now at a particular nice p-structure in which p alternates in truth on copies of $\mathbb{Z}$ but is true in the last copy of the first part. Here $\gamma_0^+(p)$ is false. Thus ψ must be either $\neg p$ or $\perp$.

Look next at a structure in which p alternates in the first part, is true in the last copy of $\mathbb{Z}$ there, is false for an initial segment of the second part and then alternates again. Here $\gamma_0^+(p)$ is true in the end of the first part. Thus ψ must be either p or $\top$ and we have our contradiction. □

A similar proof to the above readily shows that

LEMMA 4. *If for all i, $m < n_i$ then $\gamma_m^\pm$ is not expressible over all linear flows by any formula built from U,S and any (finite) number of $\gamma_{n_i}^\pm$.*

It is a bit harder to prove that any $\gamma_m^\pm$ can be expressed in terms of $\gamma_n^\pm$ (in combination with U and S) for any $n < m$.

LEMMA 5. *For any temporal formula P and any $n \geq 0$,*

$$\gamma_{n+1}^+(P) = \gamma_n^+(P \wedge \gamma^+(P) \wedge \neg\gamma_n^+(P))$$

The dual result also holds.

PROOF. This is immediate from the more informative lemma which follows the next:

LEMMA 6. *Let $n \geq 0$ and P be any temporal formula. We write Q for*

$$P \wedge \gamma^+(P) \wedge \neg\gamma_n^+(P)$$

and consider the left P gaps in a structure.

- *Every left Q gap is a left P gap.*
- *No order n left P gap is a left Q gap.*
- *All the other left P gaps are left Q gaps.*

The dual result also holds.

PROOF.

- To prove the first claim let us examine a left Q gap α say. Q is true in an interval, containing a point t say, on the left of α and false arbitrarily soon after. P, as a conjunct of Q, is thus true from t until α. If P is false arbitrarily soon after α then we have a left P gap at α as required. Suppose for contradiction that P is instead true for a while after α. Thus, like P , $\gamma^+(P)$ must stay true for a while after α. Finally look at the third conjunct, $\neg\gamma_n^+(P)$, of Q. Since it is also true at t, β can not be an order n left P gap and again the conjunct remains true after α at least as far as β. We have shown that Q remains true before and after α and we have our desired contradiction.
- The second observation is clear as $\gamma_n^+(P)$ is true arbitrarily recently before an order n left P gap.

- For the third let us look at a non-nth order left P gap. For a while, on the left, $P\ \gamma^+(P)$ and $\neg\gamma_n^+(P)$ are all true. Since P, and hence Q, is false arbitrarily soon after the gap, we have a left Q gap.

Now we can actually be more specific about the orders of the gaps involved:

LEMMA 7. *Let k and n be whole numbers and P be any temporal formula. We write*

$$Q = P \wedge \gamma^+(P) \wedge \neg\gamma_n^+(P)$$

and consider the left P gaps in a structure.

Any order k left P gap is

- *an order k left Q gap if $k < n$*
- *not a left Q gap at all if $k = n$ and*
- *an order $k-1$ left Q gap if $k > n$.*

Any order k left Q gap is

- *an order k left P gap if $k < n$ and*
- *an order $k+1$ left P gap if $k \geq n$.*

The dual result with right substituted for left also holds.

PROOF. Fix n. Now we proceed by induction on k.

First part. Suppose that we have an order k left P gap at α. If $k = n$ then the previous lemma gives us our result. So suppose not. Thus α is a left Q gap. We will show that α is an order K left Q gap where

$$K = \begin{cases} k & \text{if } k < n \\ k-1 & \text{if } k > n. \end{cases}$$

Now for a while after α any left P gaps are of order less than k. Any left Q gaps which are in this interval are then by the previous lemma, left P gaps and so of order less than k as left P gaps. If $k < n$ then these gaps are by the inductive hypothesis, left Q gaps of the same order less than $k = K$. If $k > n$ then these gaps are, also by the inductive hypothesis, left Q gaps of order one less than their order as P gaps which is less than $K = k - 1$. In either case, for a while after α all left Q gaps are of order less than K.

If $K = 0$ then we have shown than α is an isolated left Q gap.

Otherwise, since α is an order k left P gap it must have order $k-1$ left P gaps arbitrarily soon afterwards. There are three cases:

- if $k < n$ then by the inductive hypothesis these are order $K-1 = k-1$ left Q gaps;
- if $k > n+1$ then these are order $k-2 = K-1$ left Q gaps and

- if $k = n + 1 > 1$ then the order $k - 1 = n > 0$ left P gaps are also followed arbitrarily closely by order $k - 2$ left P gaps which are by the inductive hypothesis also order $k - 2 = K - 1$ left Q gaps.

- if $k = n + 1 = 1$ then $K = k - 1 = 0$ which we have supposed to not be the case.

This proves that α is a left Q gap of order K.

Second part. Suppose that α is an order k left Q gap. It is also a left P gap. We will show that it is also an order K left P gap where

$$K = \begin{cases} k & \text{if } k < n \\ k+1 & \text{if } k \geq n. \end{cases}$$

For a while after α all left Q gaps are of orders less than k. By our inductive hypothesis they will also be left P gaps of various finite orders. Thus α must be a finite order left P gap say of order l. We know that l is not n for then α wouldn't be a left Q gap at all.

By part one, if $l < n$ then $k = l$ so $K = k = l$ as required.

If $l > n$ then $k = l - 1 \geq n$ so $K = k + 1 = l$ as required. □

Now what if we can use $\gamma_0^{\pm}$? Let us consider the new set of connectives $\{U, S, \gamma_0^{\pm}\}$ and ask about its expressive power. In fact, the connectives which talk of higher order gaps are redundant. In expressive power, the γ_i^+ hierarchy collapses: for each $n \geq 0$,

$$\begin{array}{rcc} \gamma_{n+1}^+(p) & \equiv & \gamma^+(p) \wedge \neg\gamma_{\leq n}^+(p) \\ & \wedge & \gamma_0^+(\gamma^+(p) \wedge \neg\gamma_{\leq n}^+(p)) \\ & \wedge & U(\gamma_{\leq n}^+(p), p \vee U(\gamma_{\leq n}^+(p), \neg\gamma^+(p) \vee \gamma_{\leq n}^+(p))) \end{array}$$

Thus one might think that higher order gaps hold no surprises for $\{U, S, \gamma_0^{\pm}\}$. In fact we do not even need to stop at finite orders.

LEMMA 8. *$\{U, S, \gamma_0^{\pm}\}$ is expressively complete over general linear time.*

PROOF. We will exhibit a $\{U, S, \gamma_0^{\pm}\}$ formula which is equivalent to $U'(p, q)$ in any $\{p, q\}$-structure. Because the dual formula will be equivalent to $S'(p, q)$, we will have shown that $\{U, S, \gamma_0^{\pm}\}$ is expressively complete over such structures.

Let ϕ be:

$$\begin{array}{cl} & \gamma^+(q) \wedge U(U(\neg q, p), q) \\ \\ \vee & \gamma^+(q) \\ \wedge & U(\neg q, \gamma^+(\neg U(\neg q, p)) \vee p) \\ \wedge & \neg U(\neg q, \neg U(\neg q, p)) \\ \wedge & \gamma_0^+(\neg U(\neg q, p)) \end{array}$$

Suppose that B is a $\{p, q\}$-structure. We will show that for any $b \in B$,

$$\begin{array}{cc} & B \models U'(p, q)(b) \\ \Leftrightarrow & B \models \phi(b) \end{array}$$

($\Longleftarrow$) Let us assume that ϕ holds at b. We must show that $U'(p, q)$ is true at b. There are two cases.

If the first disjunct holds then it is clear that $U'(p,q)$ does too.

Now suppose that the second disjunct of ϕ holds at b but that the first does not. The first conjunct guarantees that q is true from b up until a gap which we can call β. $\neg q$ is true arbitrarily soon after β.

For contradiction we also suppose that $U'(p,q)$ does not hold at b. Thus p is false arbitrarily soon after β. Since $U(\neg q, \gamma^+(\neg U(\neg q,p)) \vee p)$ holds at b, we have $\gamma^+(\neg U(\neg q,p))$ true arbitrarily soon after β.

Since q is true up until β but p is false arbitrarily soon afterwards, we must have $\neg U(\neg q,p)$ holding from b at least up until β. But

$$\neg U(\neg q, \neg U(\neg q,p))$$

holds at b so $U(\neg q,p)$ must be true arbitrarily soon after β.

So $\neg U(\neg q,p)$ is true up until β but false arbitrarily soon afterwards. Thus $\gamma^+(\neg U(\neg q,p))$ holds at b and β is the $\neg U(\neg q,p)$ left gap involved.

Knowing that both $U(\neg q,p)$ and $\gamma^+(\neg U(\neg q,p))$ are true arbitrarily soon after β tells us that there are $\neg U(\neg q,p)$ left gaps arbitrarily soon after β.

Thus β is not an isolated $\neg U(\neg q,p)$ left gap and this contradicts

$$\gamma_0^+(\neg U(\neg q,p))$$

holding at b. We are done.

($\Longrightarrow$) Suppose that $B \models U'(p,q)(b)$. So q is true for a while after b up until a gap, called β say. We must show that ϕ is true at b. There are two cases.

If p is true for a while before β as well as after then it is clear that the first disjunct of ϕ holds at b.

So let us assume that that p is false arbitrarily soon before β.

In this case it is not hard to see that the second disjunct of ϕ holds at b. To see that those conjuncts involving $\neg U(\neg q,p)$ hold one need only notice that $\neg U(\neg q,p)$ holds from b up until β and is false for a while afterwards. It is false after the gap, i.e., $U(\neg q,p)$ holds, because p is true for a while and $\neg q$ is true arbitrarily soon after β. □

§5. Other connectives.

The connective

$$\rho^+(q) = U'(\neg q, q)$$

and its dual ρ^- could equally well have been used in this paper instead of $\gamma_0^\pm$. We can define

$$\gamma_0^+(p) = \rho^+(\gamma^+(p))$$

so that the set $\{U, S, \rho^\pm\}$ is expressively complete.

$Ung(p,q)$ iff q holds until a gap after which it is arbitrarily soon false and after which there are no left p gaps for a while. Dually we define Sng. Note that $\gamma_0^+(p)$ equals $Ung(p,p)$. Thus $\{U, S, Ung, Sng\}$ is expressively complete.

We say that a p left gap is pure if it is not itself an isolated p left gap but there are no isolated p gaps for a while after the gap. Lemma 1 above shows that pure gaps have non-ordinal order. The non-ordinal order gaps constructed in the example above are pure. We can define a new connective using purity: $\pi^+(q)$ holds iff $\gamma^+(q) \wedge U(\neg q, \neg\gamma_0^+(q))$ does.

An argument similar to the proof of Lemma 2 shows that π^+ is not expressible in terms of U and S. We compare the truth of $\pi^+(p)$ before gaps in two different structures. In one p is true up until a gap after which p is false for a while. In the other p is true up until a gap after which open intervals of p being true, and open intervals of p being false replace rational numbers in an interval from that ordering.

The proof of Lemma 2 can also be employed to show that γ_0^+ can not be expressed in terms of U, S or $\pi^\pm$. Clearly $\pi^\pm(p)$ is always false in the structures defined there.

§6. An axiomatisation of $U, S, \gamma_0^\pm$ using the irreflexivity rule.

We first axiomatise U, S and $\gamma_0^\pm$ over arbitrary linear flows of time using the irreflexivity rule of [G1]. This rule allows simple axiomatisations of many temporal connectives over irreflexive flows of time. We derive some simple consequences and list some open questions. In the next section we will relate some of these questions to the class of scattered flows of time.

In this section, unless otherwise stated a *temporal formula* will mean one written with the connectives U, S, γ_0^+ and γ_0^-. We will use the standard abbreviations F, P, H and G: Fp abbreviates $U(p, \top)$ etc. Recall also that $K^+(q)$ abbreviates $\neg U(\top, \neg q)$ and $\gamma^+(q)$ abbreviates $F\neg q \wedge U(q,q) \wedge \neg U(\neg q \vee K^+(\neg q), q)$; and similarly for K^- and γ^-.

We adopt as axioms the following:

1. All truth functional tautologies.

2. $G(p \to q) \to (Gp \to Gq)$
   $H(p \to q) \to (Hp \to Hq)$

3. $q \to GPq, q \to HFq$

4. $FFq \to Fq$ [transitivity]

5. $G(p \wedge Gp \to q) \vee G(q \wedge Gq \to p)$
   $H(p \wedge Hp \to q) \vee H(q \wedge Hq \to p)$ [linearity]

6. $r \wedge H\neg r \to [U(p,q) \leftrightarrow F(p \wedge H(Pr \to q))]$
   $r \wedge H\neg r \to [S(p,q) \leftrightarrow P(p \wedge G(F(r \wedge H\neg r) \to q))]$

7. $r \wedge H\neg r \to [\gamma_0^+(q) \leftrightarrow (\gamma^+(q) \wedge F(\neg q \wedge H(P(\neg q \wedge Pr) \to \neg\gamma^+(q))))]$
   $r \wedge H\neg r \to$
   $[\gamma_0^-(q) \leftrightarrow (\gamma^-(q) \wedge P(\neg q \wedge G(F(\neg q \wedge F(r \wedge H\neg r)) \to \neg\gamma^-(q))))]$.

The rules of inference are:

modus ponens
substitution
generalisation: $\vdash A \Rightarrow \vdash GA \wedge HA$
irreflexivity: $\vdash r \wedge H\neg r \rightarrow A \Rightarrow \vdash A$
(for all A and atoms r not occurring in A).

These axioms and rules are valid over irreflexive linear time.

DEFINITION 6.1. *If A is a temporal formula, N a temporal structure, and t a point of the flow of time of N (for short, "$t \in N$"), we write $N \models A(t)$ if A holds at t in N.*

Take any set Σ of temporal formulas. A model *of Σ will be an irreflexive linear temporal structure N such that for some $t \in N, N \models A(t)$ for all $A \in \Sigma$.*

THEOREM 4. (Completeness.) *Given any countable consistent set Σ of formulas, there is a countable model N of Σ in which all instances of the axioms are valid at every point.*

PROOF (sketch; see e.g. [GH] for details). Using standard techniques we can obtain a countable irreflexive linear temporal structure N whose points are maximal consistent sets of temporal formulas. The irreflexivity rule allows us to assume that for each $t \in N$ there is an atom r with $r \wedge H\neg r \in t$. Further:

- there is $t_0 \in N$ with $\Sigma \subseteq t_0$.
- for all atoms q and all $t \in N, N \models q(t)$ iff $q \in t$.
- for each formula A there is an atom q such that $A \leftrightarrow q \in t$ for all $t \in N$.
- for all formulas A built using only F and P, and all $t \in N, A \in t$ iff $N \models A(t)$.

It now easily follows that for all $t \in N$ and all temporal formulas $A, N \models A(t)$ iff $A \in t$. The proof is by induction on the structure of A using axioms 6 and 7. Hence as $\Sigma \subseteq t_0$, we have constructed a model of Σ. □

QUESTION. Is there an axiomatisation of U, S and $\gamma_0^{\pm}$ without using the irreflexivity rule? Burgess axiomatises U and S over arbitrary linear time in [B], without using this rule.

Even if the answer is negative, we still obtain the following corollaries, whose statements do not mention the irreflexivity rule.

COROLLARY 3. (Compactness.) *Let Σ be a set of temporal formulas (of U, S, γ_0^+ and γ_0^-). Suppose that every finite subset of Σ has a model. Then Σ has a model.*

PROOF. With the given axioms and finitary rules, no contradiction is derivable from Σ. Hence by Theorem 4 Σ has a model as stated. □

COROLLARY 4.

1. *The connective $\gamma^+_{\geq\omega}(-)$, saying that there is a gap of rank at least ω coming up on the right, is not definable by any first order formula.*
2. *Not both of the connectives $\gamma^+_\omega(-)$ and $\gamma^+_{\text{ordinal}}(-)$, saying that there is coming up on the right a gap of rank ω, or (respectively) ordinal rank, are first order definable.*

PROOF.

1. Assume for contradiction that $\gamma^+_{\geq\omega}(q)$ has a first order table. Hence by expressive completeness of $\{\bar{U}, S, \gamma_0^+, \gamma_0^-\}$ (Lemma 8 above) there is already a temporal formula equivalent to $\gamma^+_{\geq\omega}(q)$. So consider $\Sigma = \{\neg\gamma^+_{\geq\omega}(q) \wedge \gamma^+_{\geq n}(q) : n < \omega\}$. Every finite subset of Σ has a model, but Σ does not. This contradicts the previous corollary.
2. We have $\gamma^+_{\geq\omega}(q) \equiv \gamma^+(q) \wedge [\neg\gamma^+_{\text{ordinal}}(q) \vee \gamma^+_\omega(q) \vee \neg U'(\neg\gamma^+_\omega(q), q)]$, so the definability of both of γ^+_ω and $\gamma^+_{\text{ordinal}}$ would contradict (1). □

QUESTIONS.

1. Is γ^+_ω definable? Note that γ^+_ω is definable from $\gamma^+_{\geq\omega}$ by $\gamma^+_\omega(q) \equiv \gamma^+_{\geq\omega}(q) \wedge U'(\neg\gamma^+_{\geq\omega}(q), q)$.
2. Is $\gamma^+_{\text{ordinal}}$ first order definable?

By Corollary 4(2), relevant to the definability of γ_ω is the fact that the flows of time in which there are essentially no unranked gaps are essentially exactly the scattered flows: those that do not embed the rationals. They are our next topic.

§7. Unranked gaps and scattered flows of time.

We will observe that any temporal logic with first order connectives over the class of all scattered flows of time is decidable. This gives a weak recursive axiomatisation of the temporal structures with scattered flows of time, though a strong axiomatisation is not possible.

Recall that a q-definable gap (one where $\gamma^+(q)$ holds on some interval to the left) is of rank ∞ ('*unranked*') if it is not of rank α for any ordinal α. An example of such gaps was given in Section 3.1. They can also be exhibited by first defining N_i ($i = 0, 1$) to be a structure with flow of time $\mathbf{Q}$, on which q is always true ($i = 1$) or always false ($i = 0$), and then replacing each $i \in \mathbf{Q}$ by a copy of N_0 or N_1 in such a way that any interval of $\mathbf{Q}$ contains copies of both structures. Let Q be the resulting temporal structure. Each $i \in \mathbf{Q}$ that is given a copy of N_1 yields a pure unranked q-gap in Q corresponding to the 'right hand end' of that copy. Note that the flow of time of Q is isomorphic to $\mathbf{Q}$.

We defined unranked gaps of a flow of time in Section 2. As an example, all gaps in $\mathbf{Q}$ are unranked. Flow-of-time gaps may not be 'definable' by a temporal formula (i.e., detectable by γ). However, note that an unranked definable gap is also an unranked flow-of-time gap.

DEFINITION 7.1.

1. *If I is a linear order and $x, y \in I$ we will write $[x, y]$ for the closed interval of I with endpoints x, y. This extends the usual notation to the case where $x \geq y$.*

2. *An equivalence relation $\equiv$ on a linear ordering I is called* a condensation *if the $\equiv$-classes are convex (i.e., are intervals, but possibly one-point intervals or with gaps for endpoints). Note that if $\equiv$ is a condensation, the ordering of I induces a canonical linear ordering of $I/\equiv$. Strictly speaking, the condensation is this linear ordering, and not the corresponding relation $\equiv$.*

3. *Recall that I is said to be* scattered *if $\mathbf{Q}$ does not embed into I. See [Ro] for general information on scattered orderings.*

PROPOSITION 2 (cf. [D], Lemma 2.3). *A linear ordering I is scattered iff whenever $\equiv$ is a condensation of I, $I/\equiv$ is not dense.*

PROOF.

$\Rightarrow$ If $\equiv$ is a dense condensation of I, we can use the axiom of choice to choose a set of representatives of the $\equiv$-classes. Some subset of this will have order type $\mathbf{Q}$.

$\Leftarrow$ If $\mathbf{Q} \subseteq I$ define $\equiv$ on I by $x \equiv y$ iff $[x, y] \cap \mathbf{Q}$ is finite. Clearly $I/\equiv$ is dense.

□

THEOREM 5. *Let I be a linear ordering.*

1. *Suppose that I is scattered. Then there are no unranked flow-of-time gaps in I.*

2. *Assume that I is countable and that no temporal structure M with flow of time I has unranked definable gaps. Then I is scattered.*

PROOF.

1. Clearly $(*)$ any open interval of I containing an unranked gap contains infinitely many unranked gaps. Suppose that γ is an unranked gap of I. We define a chain of finite sets $S_n \subseteq I$ by induction on n so that for all adjacent points $i < j$ in S_n, the open interval (i, j) contains (a) an unranked gap, and (b) a point of S_{n+1}.

   Choose $i_0 < \gamma < i_1$ arbitrarily and let $S_0 = \{i_0, i_1\}$. Let $S_n = \{s_0, \ldots, s_k\}$ be given, satisfying (a) and with $s_0 < s_1 < \cdots < s_k$. By $(*)$, for each $i < k$ we can take $s_i < t_i < s_{i+1}$ such that both (s_i, t_i) and (t_i, s_{i+1}) contain unranked gaps. Define $S_{n+1} = S_n \cup \{t_i : i < k\}$. Clearly (b) holds now for S_n, and (a) holds for S_{n+1}.

Having defined the S_n, we observe that $\bigcup_{n<\omega} S_n$ has order type $\mathbf{Q} \cap [0,1]$, so that $\mathbf{Q}$ embeds into I. Hence I is not scattered.

Note that in the case where I is already a temporal structure and γ is a q-definable gap, the same argument shows that the extensions (truth sets) in I of q and of $\neg q$ both embed $\mathbf{Q}$.

2. The example $I = \mathbf{R}$ shows that the theorem can fail if the assumption of countability is discarded. Assume that I is not scattered. Let $\equiv$ be a condensation of I such that $(I/\equiv) \cong \mathbf{Q} \cap [0,1]$ (use Proposition 2, the countability of I and Cantor's theorem). Let Q^* be obtained from the structure Q made from N_0 and N_1 as above, by adding left and right endpoints at which q is false (say). Hence there is an order isomorphism $\theta : I/\equiv \to Q^*$. Define I as a q-structure M by: if $m \in I, M \models q(m)$ iff $Q^* \models q(\theta(m/\equiv))$. Then each unranked q-definable gap of Q^* gives rise to a similar gap in M. □

If the compactness theorem held for the scattered orderings, non-definability of γ_ω (even in the class of scattered orderings) would again follow. For the previous argument using compactness would show that $\gamma_{\geq\omega}$ is not definable even over the scattered orderings. But $\gamma^+_{\geq\omega}(q) \equiv \gamma^+(q) \wedge [\neg\gamma^+_{\text{ordinal}}(q) \vee \gamma^+_\omega(q) \vee \neg U'(\neg\gamma^+_\omega(q), q)]$, as above. In scattered orderings, because of Theorem 5 we have $\gamma^+_{\geq\omega}(q) \equiv \gamma^+_\omega(q) \vee \gamma^+(\neg\gamma^+_\omega(q))$, so that γ_ω's being definable would force $\gamma_{\geq\omega}$ to be definable, a contradiction.

However, we now show that this is not the case.

PROPOSITION 3. *The compactness theorem fails for the class of scattered orderings.*

PROOF. Introduce propositional atoms $q_i (i \in \mathbf{Q})$. Let $\Sigma = \{P(q_i \wedge H\neg q_i \wedge Pq_j) : j < i \text{ in } \mathbf{Q}\}$. Then any finite subset of Σ has a scattered model. But if M were a scattered model of Σ, then $\mathbf{Q}$ would embed into M via $i \mapsto m_i$ where $m_i \in M$ satisfies $M \models (q_i \wedge H\neg q_i)(m_i)$. □

Now the rules of inference are finitary, so completeness implies compactness. Hence, for the class of scattered orderings, there is no completeness theorem of the form: Σ is consistent iff Σ has a scattered model. However, there is a weak completeness theorem that deals with the case where Σ is finite. That is, there is a recursive set of axioms such that $\vdash A$ iff $\models A$ for all temporal formulas A. This follows trivially from the following decidability result.

PROPOSITION 4.

1. *The monadic second order theory of the class of countable scattered linear orders is decidable.*

2. *Over scattered flows of time, any temporal logic using connectives with first order tables is decidable.*

PROOF.

1. Let σ be a monadic second order sentence in the signature $\{=,<\}$, where quantification over elements and subsets is allowed. Let Q be a new unary relation symbol and let σ^Q denote the relativisation of σ to Q. (I.e., the first order quantifiers $\exists x, \forall x$ are replaced by $\exists x \in Q, \forall x \in Q$ respectively, and the second order quantifiers $\exists X, \forall X$ by $\exists X \subseteq Q$ and $\forall X \subseteq Q$ respectively. Later we give a formal definition of relativisation in the first order case.) Let $\xi(Q)$ be the formula
$$\forall R \subseteq Q([\exists xy(R(x) \wedge R(y) \wedge x < y)] \rightarrow$$
$$\exists xy(R(x) \wedge R(y) \wedge x < y \wedge \neg\exists z(x < z < y \wedge R(z))).$$
So $\xi(Q)$ says that the set of points where Q holds is a scattered ordering. Now any countable linear ordering embeds into $(\mathbf{Q}, <)$. So $\mathbf{Q} \models \exists Q(\xi(Q) \wedge \sigma^Q)$ iff σ has a countable scattered model.

   It follows from the celebrated result of Rabin [R] that the monadic second order theory of $\mathbf{Q}$ is decidable: cf. [BG, Theorem 2.6]. Hence there is an algorithm to decide whether $\mathbf{Q} \models \exists Q(\xi(Q) \wedge \sigma^Q)$. This completes the proof.

2. It follows from the downward Löwenheim–Skolem theorem (see [CK]) that if A is a temporal formula with a first order table, then A has a scattered model iff A has a countable scattered model. Let A use atoms $p_1, \ldots, p_n$ and have table $\alpha(x, P_1, \ldots, P_n)$, where the P_i are unary relation symbols corresponding to the atoms. Then A has a scattered model iff the monadic second order sentence
$$\exists P_1 \cdots P_n \exists x \alpha(x, P_1, \ldots, P_n)$$
holds in some countable scattered linear order. By (1) there is an algorithm to decide this question. □

REMARKS.

1. It follows trivially that given any set of connectives with first order tables, there is a recursive axiomatisation of the class K of temporal structures with scattered flow of time. We simply take as axioms $\{A :A$ is valid in every structure in $K\}$; this set is recursive by Proposition 4. The only proof rule required is substitution.

2. In [GH] a finite (not merely recursive) axiomatisation of the temporal logic with Until and Since over the real numbers $\mathbf{R}$ was given. In that proof a certain condensation $\sim_r$ $(r < \omega)$ was defined, and the irreflexivity rule used to show that every $\sim_r$-class was a closed interval of the flow of time. (Reynolds [Re] has since eliminated the use of the IRR rule.) The temporal translation B of $\neg\exists y < x(y \sim_r x)$ was then true exactly

at the left-hand endpoint of each $\sim$-class, so a single axiom could be used to specify properties of the condensation $M/\sim_r$, uniformly in r. In our case the relevant axiom would be $\Diamond(B \wedge FB) \rightarrow \Diamond(B \wedge U(B, \neg B))$ (cf. Proposition 2), but we have not found a formula true exactly once in each $\sim_r$-class (our proof of Proposition 2 uses the axiom of choice). So this method does not appear to be applicable in the scattered case.

3. [BG, Theorem 2.9] proves the decidability of the temporal logic with Until and Since over the real numbers. Their argument is a variant of the 'finite model property' approach to decidability, and goes back to [LL] and [Ra]. Also see [D]. This technique can be used to give another proof of our Proposition 4(2).

§8. Expressive completeness of U, S & Stavi connectives over linear time.

In this section we will prove Theorem 3. That is, we establish expressive completeness of U, S and the Stavi connectives for arbitrary linear flows of time. The formal statement follows after some initial definitions. Our argument was sketched in [GPSS] for the case of U and S over natural numbers time; the generalisation to arbitrary linear time was indicated but not proved.

DEFINITION 8.1.

1. *We fix an arbitrary* finite *set L of propositional atoms. We will consider first order formulas $\varphi(\overline{x})$ in the 'monadic' language with $=, <$ and a unary relation symbol Q for every atom $q \in L$. We also consider temporal formulas. Unless otherwise stated, a temporal formula will be one built from the atoms of L using the Boolean connectives and the binary temporal connectives U, S, U' and S' (standing for* Until *and* Since *and the Stavi connectives).*

2. *A* temporal (L-) structure *is formally a triple $N = (T, <, h)$, where $(T, <)$ is an irreflexive poset (the* flow of time *of N) and $h : L \rightarrow \mathcal{P}(T)$ is the assignment map. We will often abuse notation by identifying N with its flow of time T. Moreover, as every temporal formula A defines a subset of a structure—the set of time points $h(A)$ (cf. Definition 1.1(3)) where A is true—we will regard A as an extra atom and use it in monadic first order formulas as a monadic relation symbol. This simplifies the notation a little. So for example, $N \models \forall x U(A, B)(x)$ iff $U(A, B)$ is true at every point of N.*

3. *We will usually use Roman letters for temporal formulas and Greek for classical first order ones.*

In this setting, Theorem 3 becomes:

For all L-formulas $\varphi(x)$ there is a temporal formula A such that if N is a linear temporal structure (i.e., one with linear flow of time) in which the atoms of

φ have interpretations then for all $t \in N, N \models \varphi(t)$ iff $N \models A(t)$. Moreover, A is effectively obtainable from φ (i.e., by an algorithm).

This says that the temporal logic with Until, Since and the Stavi connectives is *expressively (functionally) complete* over linear time. Our proof here is based on the sketch in [GPSS]; an alternative proof using separation (cf. Theorem 2, and [G2]) will appear in [GHR]. The algorithm resulting from separation is probably more efficient than ours.

We begin with some definitions.

DEFINITION 8.2. (Rank.) *The* rank *of a temporal formula A is defined to be the maximum depth of nesting of temporal connectives in A. Example: if p, q are atoms then rank($p \wedge q$) $= 0$, and rank($\neg U(p, \neg S'(\neg q, q))$) $= 2$. Since L is finite, it is easy to show by induction on r that for each $r < \omega$ there is a finite set of temporal formulas of rank r such that every rank r formula is logically equivalent to one of them.*

DEFINITION 8.3. (Gaps.) *We will use the definition of a* gap *in a linear order discussed in Section 2. We need a few extra notions. Let $M = (M, <, h)$ be any linear temporal structure. If γ is a gap and $S \subseteq M$, we say that γ =sup(S) if for all $t \in M, t \geq s$ for all $s \in S$ iff $t \geq \gamma$. We also say that γ =inf(S) if for all $t \in M, t \leq s$ for all $s \in S$ iff $t \leq \gamma$.*

Let γ be a gap and let D be a temporal formula. We say that γ is definable on the left by *D if D is true at all points of M in some non-empty interval (t, γ) on the left of γ, and not true throughout any non-empty interval (γ, u) on the right. The definition of a gap's being* definable (by *D*) on the right *is made in a similar way. If $r < \omega$, an* r-definable gap *is one that is definable (on the left or right) by a formula D of rank at most r. For $r < \omega$ we let $M_r = M \cup \{r-$definable gaps of $M\}$, with the induced ordering $<$. So in general $M \subset M_0 \subset M_1 \subset \cdots$. For example, if M has no last element then $+\infty$ is a gap of M definable on the left by $\top$, so that $\infty \in M_0 \backslash M$. The situation for $-\infty$ is similar.*

We will refer to the elements of M as *points*.

DEFINITION 8.4. (Relativised connectives.) *There is a natural way of evaluating temporal formulas of the form $\sharp(A, B)$ for $\sharp \in \{U, S, U', S'\}$ at gaps. For example, $U(A, B)$ holds at a gap γ (i.e., $\gamma \in M_n$ for some n) iff there is a* point *$t > \gamma$ (so $t \in M$) where A holds, with B holding at all* points *$u \in (\gamma, t)$. To formalise this we relativise our connectives to points.*

Fix $r < \omega$ and let $\mu \notin L$ be a new propositional atom. We define M_r as a temporal $L \cup \{\mu\}-$structure $(M_r, <, h')$ by:

$$h'(q) = h(q) \subseteq M \text{ for all } q \in L$$
$$h'(\mu) = M.$$

We will relativise U, S, U' and S' to μ.

Let $\varphi(\overline{x})$ be any first order formula in the signature consisting of $=, <$ and a unary relation symbol for each atom of $L \cup \{\mu\}$. We define the relativisation *φ^μ of φ to μ by induction on φ:*

$$\text{if } \varphi \text{ is quantifier free then } \varphi^\mu = \varphi;$$

$$(\neg\varphi)^\mu = \neg(\varphi^\mu);$$
$$(\varphi \wedge \psi)^\mu = \varphi^\mu \wedge \psi^\mu;$$
$$[\exists y \varphi(\overline{x}, y)]^\mu = \exists y(\mu(y) \wedge \varphi^\mu(\overline{x}, y)).$$

We introduce connectives U^μ, S^μ, U'^μ and S'^μ whose tables are the relativisations to μ of the tables of U, S, U' and S' respectively. We can write formulas using these connectives that are meaningful in any $L \cup \{\mu\}$-structure. In particular we can interpret them in M_r. If A is any formula of $USU'S'$, we let A^μ be the formula obtained by replacing each U in A by U^μ, and similarly for S, U' and S'.

REMARK.

1. Let $\alpha(x)$ be the canonical *first order table* of the temporal formula A, as defined in Definition 1.2: in any temporal structure T, $\{t \in T : T \models A(t)\} = \{t \in T : T \models \alpha(t)\}$. Then it is easily seen that the table of A^μ is just α^μ: i.e., for all $t \in M_r$, $M_r \models A^\mu(t)$ iff $M_r \models \alpha^\mu(t)$ (this holds for any $L \cup \{\mu\}$-structure).

2. If $t \in M$ then $M \models A(t)$ iff $M_r \models A^\mu(t)$.

3. Let $A = S'(B, C)$ where C has rank $\leq r$. If $t \in M_r$ and $M_r \models A^\mu(t)$, then the gap that A asserts the existence of actually lies in M_r (as C defines it on the right).

The Stavi connectives can express existence of gaps, but cannot talk directly about what formulas are 'true' at them. So we need to transform properties of a gap into properties of 'real' points. This is done in the following definition and lemma.

DEFINITION 8.5. *Let D be any temporal L-formula. We define a temporal L-formula* left(A, D) *by induction on A:*

- *left$(p, D) = \perp$ for atomic p*
- *left$(\neg A, D) = U'(\top, D) \wedge \neg left(A, D)$*
- *left$(A \wedge B, D) =$ left$(A, D) \wedge$ left(B, D)*
- *left$(U(A, B), D) = U'(B \wedge U(A, B), D)$*
- *left$(U'(A, B), D) = U'(B \wedge U'(A, B), D)$*
- *left$(S(A, B), D) = U(D \wedge B \wedge S(A, B) \wedge U'(\top, B \wedge D) \wedge \neg U'(D, B \wedge D), D)$*
- *left$(S'(A, B), D) = U(D \wedge B \wedge S'(A, B) \wedge U'(\top, B \wedge D) \wedge \neg U'(D, B \wedge D), D)$.*

So rank(left$(A, D)) \leq$ max$(rk(A), rk(D)) + 2$. We define right(A, D) *similarly by swapping each U with S and U' with S' in the definition above.*

The point of this definition is given by the following lemma.

LEMMA 9. *Let A, D be temporal formulas with D of rank at most r. Let $m \in M_r$. Then the following are equivalent:*

1. $M_r \models \mathit{left}(A, D)^{\mu}(m)$;
2. *There is $\gamma \in M_r - (M \cup \{\pm\infty\}), \gamma$ a gap of M defined by D to the left, with (a) $\gamma > m$,(b) D holds in M on (m, γ), and (c) $M_r \models A^{\mu}(\gamma)$.*

PROOF. Clear. A corresponding result holds for right(A, D). □

DEFINITION 8.6. (Games.) *We will need some results on Ehrenfeucht–Fraïssé games. Let Σ be any finite first order signature without function symbols. Let M, N be Σ-structures. If $n < \omega$ we define a game $G^n(M, N)$ between two players, $\forall$ (male) and $\exists$ (female). The game has n rounds. In each round, $\forall$ chooses an element from whichever of M, N he wishes. Then $\exists$ responds by choosing an element of the other structure. After n rounds, two n-tuples $\overline{a}, \overline{b}$ of elements have been chosen from M, N respectively; the order of the elements in each tuple is the order in which they were chosen as the game was played. $\exists$ wins this 'play' $(\overline{a}, \overline{b})$ of the game iff for all quantifier-free formulas $\varphi(\overline{x})$ of Σ, $M \models \varphi(\overline{a})$ iff $N \models \varphi(\overline{b})$. This is slightly stronger than saying that the map $\overline{a} \mapsto \overline{b}$ is a partial isomorphism, since Σ may have constant symbols.*

A strategy *for $\exists$ in a game is a set of rules (not necessarily deterministic) telling her what to do—this can be formalised as a family of functions. The strategy is said to be* winning *if whenever she uses it she wins.*

The following is a well-known result of Ehrenfeucht–Fraïssé game theory.

PROPOSITION 5. *Let Σ be any signature as above. Let M, N be Σ-structures and let $n < \omega$. The following are equivalent:*

1. *$\exists$ has a winning strategy for $G^n(M, N)$*
2. *$M \models \sigma$ iff $N \models \sigma$ for all Σ-sentences σ of quantifier depth of nesting at most n.*

PROOF. See [E]. As is well known, (2) $\rightarrow$ (1) can fail if Σ is assumed infinite or to have function symbols. □

NOTATION. If $x < y$ in M_r we write (x, y) for $\{t \in M : x < t < y\}$, and if $n \leq r, (x, y)_n$ for $\{t \in M_n : x < t < y\}$. We write $[x, y]_n$ for $\{t \in M_n : x \leq t \leq y\}$, etc. We do not require that $x, y \in M_n$.

DEFINITION 8.7. (Special games on temporal structures.) *We now introduce a modified version of the game above. Let M and N be linear temporal structures. The game $G_{n;r}(M, xy; N, x'y')$ for $n, r < \omega, x < y$ in M_r, and $x' < y'$ in N_r, is played as follows. There are only two rounds. $\forall$ begins by choosing n elements $a_1, \ldots, a_n \in [x, y]_r$; $\exists$ responds with elements $a'_1, \ldots, a'_n \in [x', y']_r$. Then $\forall$ chooses one more element $b' \in [x', y']$—so b' must not be a gap—and $\exists$ replies with $b \in [x, y]$.*

$\exists$ wins iff:

1. *the tuples $xy\bar{a}b$ and $x'y'\overline{a'}b'$ have the same order type; and if $t \in xy\bar{a}b$ and t' is the corresponding element of $x'y'\overline{a'}b'$, then:*

2. *t is a gap of M iff t' is a gap of N*

3. *for each temporal L-formula A of rank at most $r, M_r \models A^\mu(t)$ iff $N_r \models A^\mu(t')$.*

LEMMA 10. *Let M, N etc. be as above. Suppose that $\exists$ has a winning strategy σ for $G_{n;r}(M, xy; N, x'y')$ for some $n, r < \omega$. Let $n' \leq n, r' \leq r$. Then σ gives in the natural way a winning strategy for $G_{n';r'}(M, xy; N, x'y')$ provided that $x, y \in M_{r'}$ and $x', y' \in N_{r'}$.*

PROOF. Recall that K^+X abbreviates the formula $\neg U(\top, \neg X)$, and $K^-X = \neg S(\top, \neg X)$.

Suppose in a play of $G_{n';r'}(M, xy; N, x'y')$, $\forall$ chooses $a_1, \ldots, a_{n'} \in [x, y]_{r'}$. Then $\exists$ defines $a_{n'+1}, \ldots, a_n$ to be x, say. So $a_1, \ldots, a_n \in [x, y]_r$. She applies σ to $\bar{a}$ to obtain $\bar{e} \in [x', y']_r$.

We claim that each $e_i \in [x', y']_{r'}$. This is clear if $r' = r$, so assume that $r' < r$. Take i; certainly if $a_i \in M$ then $e_i \in N$. Otherwise a_i is defined by some formula $\neg D$ of rank $\leq r'$. So letting $D' = (K^+D \wedge \neg K^-D) \vee (K^-D \wedge \neg K^+D)$, a formula of rank $\leq r' + 1 \leq r$, we have $M_r \models D'^\mu(a_i)$. As σ is winning, $N_r \models D'^\mu(e_i)$. Hence e_i is also a gap defined by $\neg D$; so $e_i \in N_{r'}$.

If $\forall$ now chooses $a' \in [x', y']$ then $\exists$ simply uses σ to respond with $e' \in [x, y]$. Then $\bar{a}e'$ and $\bar{e}a'$ satisfy the same order relations and rank r temporal formulas, hence also the same temporal formulas of rank r'. Hence $\exists$ has won the play. □

We want to characterise the formulas associated with these games.

DEFINITION 8.8.

1. *Let $r < \omega$ and $t \in M_r$ be given. Define X_t to be the conjunction of all temporal L-formulas X of rank $\leq r$ with $M_r \models X^\mu(t)$. This conjunction is effectively finite, as because L is finite there are up to logical equivalence only finitely many distinct formulas of any rank. Hence X_t can be taken to be a temporal formula of rank r.*

   If $t < u$ in M_r, define $X_{(t,u)}$ to be $\bigvee_{v \in (t,u)} X_v$. Again the disjunction is effectively finite, so that $X_{(t,u)}$ can be taken to be a formula of rank r. Note that only points (non-gaps) contribute to the disjunction.

2. *(This definition is from [GPSS].) An $n;r$-*decomposition *formula is a first order formula of the form:*

   $$\psi(x_1, x_2) = \exists y_1, \ldots, y_n[x_1 < y_1 < \cdots < y_n < x_2 \wedge \forall z \chi(x_1, x_2, \bar{y}, z)],$$

   where χ is a conjunction of formulas of the following kinds:

   (a) *$\theta(t)$, where t is an element of $x_1x_2\bar{y}$ and θ is either $\mu, \neg\mu$, or A^μ for some temporal L-formula A of rank $\leq r$;*

(b) $\mu(z) \wedge a < z < b \to B^{\mu}(z)$, *where* $a < b$ *are adjacent elements of the sequence* $x_1 y_1 \cdots y_n x_2$, *and* B *is a temporal formula of rank* $\leq r$.

LEMMA 11. *Let* $M, N, x, y, x,', y'$ *be as above. Let* $n, r < \omega$. *Then the following are equivalent:*

1. $\exists$ *has a winning strategy for* $G_{n;r}(M, xy; N, x'y')$.
2. *for all* $n; r$*-decomposition formulas* $\varphi(x_1, x_2), M_r \models \varphi(x, y) \Rightarrow N_r \models \varphi(x', y')$.

PROOF. (1) $\Rightarrow$ (2)—clear.
(2) $\Rightarrow$ (1) Let $\forall$ choose $a_1, \ldots, a_n \in [x, y]$ in his first move. Assume without loss that $x < a_1 < \cdots < a_n < y$. Write a_0 for x and a_{n+1} for y. Let $\psi(y_0, y_{n+1}) = \exists y_1 \cdots y_n [y_0 < y_1 < \cdots < y_{n+1} \wedge \forall z (\bigwedge_{a_i \in M} \mu(y_i) \wedge \bigwedge_{a_i \notin M} \neg\mu(y_i) \wedge \bigwedge_{i \leq n+1} X_{a_i}(y_i) \wedge \bigwedge_{i \leq n} (\mu(z) \wedge y_i < z < y_{i+1} \to X_{(a_i, a_{i+1})}(z))]$. Then ψ is an $n; r$-decomposition formula and $M_r \models \psi(x, y)$. Hence by assumption $N_r \models \psi(x', y')$, and so there are $e_i \in (x', y')$ witnessing the $\exists$'s in ψ. If $\exists$ chooses the e_i she can easily win the game. □

The main step in our proof is

THEOREM 6. *Suppose that* M, N *are linear temporal structures. Then* $(*)_n$ *holds for all* $n < \omega$:

$(*)_n$ *For all* $r < \omega$, *if* $x < y$ *in* M_r, $x' < y'$ *in* N_r, *and* $\exists$ *has a winning strategy for*

$$G_{1+3n;r+4n}(M, xy; N, x'y'),$$

then $\exists$ *has a winning strategy for* $G_{n;r}(N, x'y'; M, xy)$.

This says that if $\exists$ possesses winning strategies for enough 'forward' games $G(M, xy; N, x'y')$ then she has a winning strategy for a given 'backward' game $G(N, x'y'; M, xy)$. The proof does not use compactness and is really a syntactic result—we could equally prove that a certain class of formulas is closed under negation up to equivalence, which is what is done (for the case of U and S over $\mathbb{N}$ and without full proof) in [GPSS]. However, the game approach, though still complicated, seems rather simpler to present.

Before we prove Theorem 6 we finish our result on expressive completeness.

PROPOSITION 6. *Let* M, N *be linear temporal structures and let* $x \in M$, $y \in N$. *Suppose* $n, r < \omega$ *and that* x *and* y *satisfy the same temporal formulas of rank* $r + 4n + 1$ *in their respective structures. Then* $\exists$ *has a winning strategy for* $G_{n;r}(M, -\infty x; N, -\infty y)$ *and* $G_{n;r}(M, x\infty; N, y\infty)$.

PROOF. (Sketch.) Suppose for simplicity that $\forall$ chooses n points $x < a_1 < \cdots < a_n$ in M in the future of x. Let $a_0 = x$. Define C_n to be $X_{a_n} \wedge \neg U(\neg X_{(a_n, \infty)}, \top)$, and for $i < n, C_i$ to be $X_{a_i} \wedge U(C_{i+1}, X_{(a_i, a_{i+1})})$. So rank $(C_i) = r + n + 1 - i$. Then $M \models C_0(x)$, so that $N \models C_0(y)$. $\exists$ can use the form of C_0 to choose points $y = e_0 < e_1 < \cdots < e_n$ in N such that $N \models X_{a_i}(e_i)$ and $N \models X_{(a_i,a_{i+1})}(t)$ for all (non-gaps) $t \in (e_i, e_{i+1})$. If $\forall$ now chooses $t \in (e_i, e_{i+1})$ then $N \models X_u(t)$ for some $u \in (a_i, a_{i+1})$. If $\exists$ responds with such a u, she wins the game. The argument for the 'past' game is similar. If some of the a_i are gaps, the idea is the same but the formulas C are more complicated and involve formulas D defining the gaps, together with the formulas left(X_{a_i}, D) or right(X_{a_i}, D)—cf. the proof of Cases III, IV of Theorem 6. In all cases we have rank$(C_0) \leq r+4n+1$. $\square$

DEFINITION 8.9. *Let f, g be any functions on ω satisfying $f(0) = g(0) = 0, f(n+1) \geq (1+3f(n)).(2k_n)+1$, and $g(n+1) \geq g(n)+4f(n)$, where k_n is the number of inequivalent $(1+3f(n)); (g(n)+4f(n))$-decomposition formulas.*

PROPOSITION 7. *For all $n < \omega$ the following holds. Let M, N be linear temporal structures and let $x_1 < \cdots < x_m, y_1 < \cdots < y_m$ be increasing m-tuples of elements of M, N respectively, for arbitrary $m < \omega$. Define $x_0 = -\infty$ and $x_{m+1} = \infty$ in M_0. Define y_0, y_{m+1} similarly.*

Suppose that $\exists$ has winning strategies for

$$G_{f(n);g(n)}(M, x_i, x_{i+1}; N, y_i, y_{i+1})$$

and

$$G_{f(n);g(n)}(N, y_i, y_{i+1}; M, x_i, x_{i+1})$$

for all $0 \leq i \leq m$. Then $\exists$ has a winning strategy for the Ehrenfeucht–Fraïssé game $G^n((M,\overline{x}),(N,\overline{y}))$.

PROOF. By induction on n. If $n = 0$ the result is trivial. Assume it true for n, let $r = g(n)+4f(n) \leq g(n+1)$, and suppose that $\exists$ has winning strategies for the games $G_{f(n+1);r}(M, x_i, x_{i+1}; N, y_i, y_{i+1})$ and $G_{f(n+1);r}(N, y_i, y_{i+1}; M, x_i, x_{i+1})$.

Let $\forall$ begin $G^{n+1}((M,\overline{x}),(N,\overline{y}))$ by choosing without loss $a \in M$. (If $\forall$ chooses in N the proof is the same as we have complete symmetry.) If $a \in \{x_1, \ldots, x_m\}$ then $\exists$ chooses the corresponding element of $\overline{y}$, and the result then follows using the induction hypothesis and Lemma 10. So let $i \leq m$ be such that $x_i < a < x_{i+1}$. List as $\varphi_1, \ldots, \varphi_j$ the $[1+3f(n)]$;r-decomposition formulas $\varphi(u,v)$ such that $M_r \models \varphi(x_i, a)$, and as $\psi_1, \ldots, \psi_k$, the $[1+3f(n)]$;r-decomposition formulas $\psi(u,v)$ with $M_r \models \psi(a, x_{i+1})$.

Let $\exists$ choose witnesses for the existential quantifiers of each φ, ψ, together with a, making at most $n' = (1+3f(n)).(j+k)+1 \leq f(n+1)$ elements of $(x_i, x_{i+1})_r$ in all. She now applies her winning strategy for $G_{f(n+1);r}(M, x_i x_{i+1}; N, y_i y_{i+1})$. Let e be the point she chooses corresponding to a. Clearly (cf. Lemma 11) we have $N_r \models \varphi_s(y_i, e)$ for all $s \leq j$ and $N_r \models \psi_s(e, y_{i+1})$ for $s \leq k$. By Lemma 11, $\exists$ has a winning strategy for $G_{1+3f(n);r}(M, x_i a; N, y_i e)$ and for $G_{1+3f(n);r}(M, a x_{i+1}; N, e y_{i+1})$. Crucially, by Theorem 6, she also has winning strategies for

$$G_{f(n);g(n)}(N, y_i e; M, x_i a)$$

and

$$G_{f(n);g(n)}(N, e y_{i+1}; M, a x_{i+1}).$$

By the induction hypothesis, $\exists$ has a winning strategy σ for

$$G^n((M,\overline{x}a),(N,\overline{y}e)).$$

So in $G^{n+1}(M,\overline{x}),(N,\overline{y}))$, $\exists$ can choose e in response to $\forall$'s choice of a and then follow σ. This strategy wins the game for her. $\square$

COROLLARY 5. *Let M,N be linear temporal structures and let $x \in M, y \in N$. Suppose that x and y satisfy the same temporal formulas of rank $g(n+1)+1$ in their respective structures. Then for all monadic first order formulas φ (of L) of quantifier depth $\leq n, M \models \varphi(x)$ iff $N \models \varphi(y)$.*

PROOF. By Propositions 5, 6, 7. $\square$

Expressive completeness now follows easily. For given $\varphi(x)$ of quantifier depth n, we may choose a finite L with atoms corresponding to the monadic predicates of φ. Now take a finite set Ψ of temporal formulas of rank $1+g(n+1)$ such that (1) if $A,B \in \Psi$ and $A \wedge B$ is consistent then $A=B$; (2) each temporal formula C of rank $1+g(n+1)$ is equivalent to a disjunction of formulas in Ψ. Let $\Psi' = \{B \in \Psi :$ for some linear M and $t \in M, M \models B(t)$ and $M \models \varphi(t)\}$. Then by Corollary 5, φ is equivalent over linear time to the rank $1+g(n+1)$-formula $\bigvee \Psi'$.

Note that by a result of Gurevich [BG, 2.7(a)], the universal monadic second order theory of linear order is decidable. Hence Ψ' is computable by an algorithm, so that the translation of first order formulas into temporal ones is effective.

PROOF (of Theorem 6). We must prove

$(*)_n$ For all $r<\omega$, if $x<y$ in $M_r, x'<y'$ in N_r, and $\exists$ has a winning strategy for $G_{1+3n;r+4n}(M,xy;N,x'y')$, then $\exists$ has a winning strategy for $G_{n;r}(N,x'y';M,xy)$.

We prove $(*)_n$ by induction on n. For the case $n=0$ (r is arbitrary) assume that $\exists$ has a winning strategy σ for $G_{1;r}(M,xy;N,x'y')$ and that $\forall$ chooses $a \in (x,y)$ in the second round of $G_{0;r}(N,x'y';M,xy)$ (as $n=0$ the first round is 'empty'). a is not a gap. $\exists$ simply applies σ to choose a response $e \in (x',y')$. Clearly $\exists$ has won.

Assume $(*)_n$ for $n<\omega$; we prove $(*)_{n+1}$. Fix $r<\omega, x<y$ in M_r and $x'<y'$ in N_r. Assume that $\exists$ has a winning strategy for

$$G_{4+3n;r+4(n+1)}(M,xy;N,x'y').$$

We will construct a winning strategy for $\exists$ in $G_{n+1;r}(N,x'y';M,xy)$.

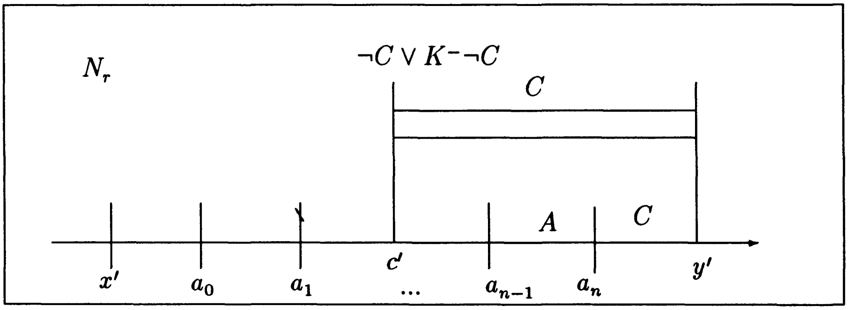
Suppose $\forall$ chooses $n+1$ points $x'<a_0<\cdots<a_n<y'$ in N_r (we may assume that they are all distinct, for otherwise the result follows by the inductive hypothesis and Lemma 10). Define the following rank r temporal formulas:

$$A = X_{(a_{n-1},a_n)}$$
$$C = X_{(a_n,y')}$$

where if $n=0$ we take a_{n-1} in A to be x'. Clearly in N, A holds on (a_{n-1},a_n) and C on (a_n,y'). Let

- $c = \inf\{t \in [x,y] : M \models C(u) \text{ for all } u \in (t,y)\}$.

If $c \notin M$ then either $c = x \in M_r$ already, or c is a gap definable on the right by C. Hence $c \in M_r$. Define $c \in N_r$ similarly.



Claim 1.
Consider a play of the game $G_{m;r'}(M, xy; N, x'y')$ for arbitrary $r' > r, m \geq 1$ in which $\exists$ uses a winning strategy. Let $\forall$ begin by choosing c plus $m-1$ other points, and let $\exists$'s response to c be d (plus $m-1$ other points). Then $d = c'$.

Proof of Claim.
As the strategy is winning, any rank r' temporal formula satisfied by one of $\forall$'s choices must also be satisfied by the corresponding choice of $\exists$. Now the rank $r+1$ formula $C' = \neg C \vee K^- \neg C$ satisfies $M_r \models C'^{\mu}(c)$. Hence also $N_r \models C'^{\mu}(d)$, so $d \leq c'$.

If $d < c'$ then $\forall$ can choose $d' \in (d, y')$ with $N \models \neg C(d')$. $\exists$ now has no winning response, a contradiction. Hence $d = c'$. This proves the claim.

Claim 2.
$\exists$ has a winning strategy for

$$G_{1+3n;r+4(n+1)}(M, xc; N, x'c')$$

and for

$$G_{1+3n;r+4(n+1)}(M, cy; N, c'y').$$

Proof of Claim.
Let $r' = r+4(n+1)$. Suppose that $\forall$ chooses $1+3n$ elements in the interval $[x, c]_{r'}$. By assumption $\exists$ has a winning strategy σ for the game $G_{4+3n;r'}(M, xy; N, x'y')$. $\exists$ adds c to $\forall$'s choices and applies σ (cf. Lemma 10). As the order of $\exists$'s element choices from σ matches the order of $\forall$'s, Claim 1 ensures that her responses to $\forall$'s choices all lie in $[x', c']_{r'}$. If $\forall$ then chooses in $[x', c']$ then again $\exists$'s strategy will yield an answer in $[x, c]$. The strategy is clearly winning. To sum up, the restriction of σ to games in which $\forall$ always chooses in $[x, c]_{r'}$ and then in $[x', c']$ can yield a winning strategy for $G_{1+3n;r+4(n+1)}(M, xc; N, x'c')$. Similarly for the intervals $[c, y], [c', y']$. This establishes the claim. We will use this argument repeatedly.

Hence by inductive hypothesis $(*)_n$, $\exists$ has winning strategies σ, τ for the backward games $G_{n;r+4}(N, x'c'; M, xc)$ and $G_{n;r+4}(N, c'y'; M, cy)$.

Now clearly $c' \leq a_n$, so $(x', c')_r$ contains at most n points from $\{a_0, \ldots, a_n\}$. The proof will divide into cases, mainly according to whether a_n is a point of N, a left- or a right-definable gap.

Case I: $a_0 \le c'$.

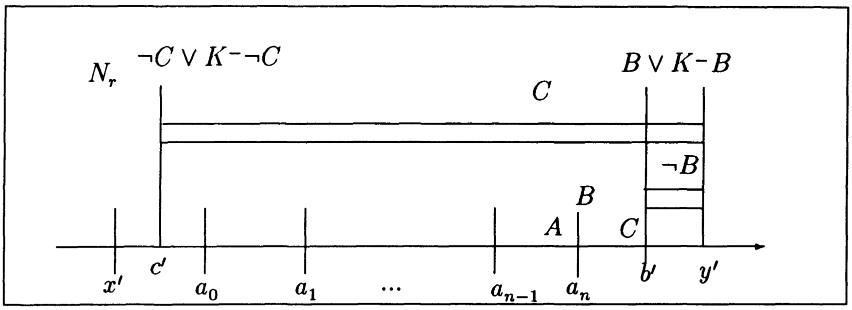
Then $(c', y')_r$ also contains at most n points from $\{a_0, \ldots, a_n\}$. So as $\exists$ is trying to win

$$G_{n+1;r}(N, x'y'; Mxy),$$

she can use σ and τ to choose points $e_0, \ldots, e_n \in M_r$. She applies σ to those a_i in $(x', c')_r$ and τ to the rest using the method of Lemma 10; if an a_i happens to be c' it can be dealt with by either strategy. If $\forall$ then responds in $[x, c)$ she uses σ, and if in $[c, y], \tau$. If she does this then by Lemma 10 she will win the game.

Case II: All the points $a_0, \ldots, a_n$ lie in (c', y'), and $a_n \in N$ is not a gap.

Recall that $\exists$ is trying to win $G_{n+1;r}(N, x'y'; M, xy)$—i.e., to preserve all rank r formulas. Define $B = X_{a_n}$, and $b = \sup\{t \in (x, y) : M \models B(t)\}$. As before, either $b \in M, b = y$ or b is an r-definable gap, defined on the right by $\neg B$, so that $b \in M_r$. Define $b' \in N_r$ similarly. Then clearly $b' \ge a_n$.



As in Claim 1, in any play of $G_{4+3n;r+4(n+1)}(M, xy; N, x'y')$ in which $\exists$ is using her winning strategy and $\forall$ chooses b, c amongst other points, $\exists$ will respond with b', c' amongst others. Hence again $\exists$ has a winning strategy for $G_{1+3n;r+4(n+1)}(M, cb; N, c'b')$. So by the induction hypothesis $(*)_n$ she has a winning strategy τ for $G_{n;r+4}(N, c'b'; M, cb)$. She already has a winning strategy σ for $G_{n;r+4}(N, x'c'; M, xc)$.

Let her first use τ in response to $a_0, \ldots, a_{n-1}$. It delivers n points $e_0, \ldots, e_{n-1} \in (c, b)_r$ (cf. Lemma 10). Now clearly $N_r \models U(B, A)^\mu(a_{n-1}) : a_n$ is a witness to this. (This holds even if a_{n-1} is a gap; if $n = 0$ we take $a_{-1}$to be c' and (see below) e_{-1} to be c.) $U(B, A)$ has rank $r + 1$, so as τ preserves formulas up to rank $r + 4, M_r \models U(B, A)^\mu(e_{n-1})$. Hence there is $z > e_{n-1}$ in M with $M \models B(z)$ and $M \models A(t)$ for all $t \in (e_{n-1}, z)$. But $e_{n-1} < b$. Hence we can assume that $z \le b$. $\exists$ defines e_n to be such a z, completing her move. Clearly e_n and a_n satisfy the same temporal formulas of rank r, as they both satisfy B.

Suppose that $\forall$ continues by choosing $t \in [x, y]$. Recall that by the game rules, t is not a gap. If $t < c$ then $\exists$ uses σ to respond, and if $c \le t \le e_{n-1}$ she uses τ. If $t \in (e_{n-1}, e_n)$ then $M \models A(t)$. By definition of A there is $t' \in (a_{n-1}, a_n)$ with $M \models X_{t'}(t)$. $\exists$ can then choose any such t' as her response. It follows that t and t' agree on all rank r temporal formulas, as required. If $t = e_n$ then $\exists$ responds with

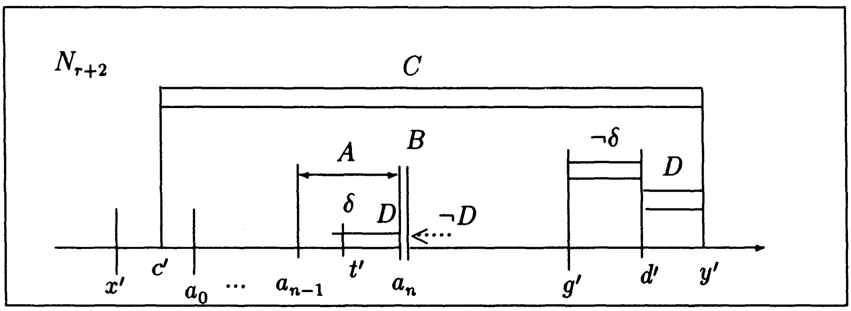
a_n. Finally, if $y > t > e_n$ then certainly $t > c$, so $M \models C(t)$. By definition of C there is $t' > a_n$ with $M \models X_{t'}(t)$, and $\exists$ can choose such a t' in response to t. If $\exists$ follows these directions she will win.

The remaining cases are similar to Case II, which gave a response e_n to a_n by letting B describe a_n and making $U(B, A)$ true at e_{n-1}. But a_n will now be a gap, so we must use the Stavi U'—and $U'(B, A)$ does not say that B^μ is true at the gap. So we use the formulas left(-,-) and right(-,-) instead.

Case III: All the points $a_0, \ldots, a_n$ lie in $(c', y')_r$, and a_n is a gap defined on the left by some formula D of rank $\leq r$. Clearly a_n is also defined by $A \wedge D$, so we can assume that $D \vdash A$.

Write B for X_{a_n}, and δ for $A \wedge$ left(B, D). δ is a formula of rank $\leq r+2$, and $N_r \models U(\delta, A)^\mu(a_{n-1})$ (again we set a_{n-1} to be c' if $n = 0$). Define d', g' by:

- $d' = \sup\{t \in (x', y') : N \models \neg D(t)\}$
- $g' = \sup\{t \in (x', d') : N \models \delta(t)\}$.



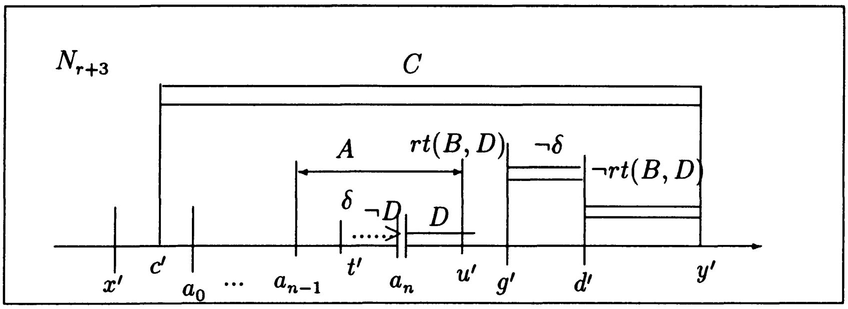
Define d, g similarly. Note that as before, all these points lie in M_{r+2}, N_{r+2}. Clearly, $a_n < d'$ and the fact that $N_r \models U(\delta, A)^\mu(a_{n-1})$ is witnessed at a point $t' \in N$ where δ holds, with $t' < g'$.

Now if $\exists$ uses a winning strategy for $G_{4+3n;r+4(n+1)}(M, xy; N, x'y')$ and adds c, g and d to $\forall$'s choices, then as before, her strategy delivers *inter alia* c', g' and d' in response. So again, $\exists$ has a winning strategy for $G_{1+3n;r+4(n+1)}(M, cg; N, c'g')$ for all m, r'. By $(*)_n$, $\exists$ has a winning strategy for $G_{n;r+4}(N, c'g'; M, cg)$. Let her use it to choose $e_0, \ldots, e_{n-1}$ in response to $a_0, \ldots, a_{n-1}$. Then as in Lemma 10, $e_0, \ldots, e_{n-1} \in (c, g)_r$, and as rank $r+4$ formulas are preserved, $M_r \models U(\delta, A)^\mu(e_{n-1})$. As $e_{n-1} < g$ we can choose $t \leq g$ in M with $M \models \delta(t)$ and such that A holds at all $u \in (e_{n-1}, t]$.

By definition of δ and Lemma 9, there is a gap $e_n \in (t, d)_r$ defined by D on the left, and such that A holds between t and e_n. Moreover, any rank r formula holds at e_n iff it holds at a_n, as they both satisfy B. $\exists$ chooses e_n in response to a_n, so completing her move. The same argument as in Case I allows $\exists$ to complete the remainder of the game, winning it.

Case IV: $a_0, \ldots, a_n \in (c', y'), a_n \in N_r - N$, and a_n is not definable on the left by any formula of rank $\leq r$.

It follows from the case assumption that A holds throughout some interval containing a_n. Choose D of rank $\leq r$ defining a_n on the right. Define $B = X_{a_n}$ and $\delta = A \wedge \neg D \wedge U(\text{right}(B, D), A)$ (rank $r + 3$). Let $d' = \sup\{t \in (x', y') : N \vDash \text{right}(B, D)(t)\}$, and then $g' = \sup\{t \in (x', d') : N \vDash \delta(t)\}$. Define $d, g \in M_{r+3}$ similarly.



Clearly there are $a_{n-1} < t' < a_n < u' < y'$, with $t', u' \in N$, $N \vDash \delta(t')$, $N \vDash \text{right}(B, D)(u')$, and A holding on (t', u') (if $n = 0$ we take a_{n-1} to be c' as usual). Hence $t' \leq g'$ and $u' \leq d'$. As usual, if $\exists$ uses a winning strategy for $G_{4+3n;r+4(n+1)}(M, xy; N, x'y')$ and adds c, g and d to $\forall$'s choices she can derive a winning strategy for $G_{1+3n;r+4(n+1)}(M, cg; N, c'g')$. So by $(*)_n$ she has a winning strategy for $G_{n,r+4}(N, c'g'; M, cg)$. Let her use it to respond to $a_0, \ldots, a_{n-1}$ with $e_0, \ldots, e_{n-1}$. So as $U(\delta, A)$ has rank $\leq r + 4$, $M_r \vDash U(\delta, A)^{\mu}(e_{n-1})$. We can choose $e_{n-1} < t \leq g$ with $t \in M, M \vDash \delta(t)$, and A holding on (e_{n-1}, t). Then we can choose $u \in M$ with $t < u \leq d, M \vDash \text{right}(B, D)(u)$ and such that A holds in (e_{n-1}, u).

By Lemma 9 there is a gap $e_n \in (t, u)$ defined by D and at which the same relativised rank r formulas hold as at a_n in N_r. (We have $e_n > t$ because $M \vDash \neg D(t)$.) Then $\exists$ adds e_n to her choices to complete the move. The remainder of the game is as before.

This ends the proof of the theorem. □

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