ON ω_1-COMPLETE FILTERS

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Let us start with a definition. For an uncountable cardinal κ set

$$\mu(\kappa) = \min\{|H| \mid H \text{ is a set of } \omega_1\text{-complete uniform filters on } \kappa \text{ and } \forall A \subseteq \kappa\ \exists F \in H(A \in F \text{ or } \kappa - A \in F)\}$$

Clearly, $1 \leq \mu(\kappa) \leq 2^\kappa$.

A classical result of Ulam says that κ must be very large, if $\mu(\kappa) = 1$. On the other hand, by definition we have that $\mu(\kappa) = 1$ if κ is bigger than some strongly compact cardinal. Only recently (see [3]) Gitik has shown that $\mu(\kappa) \leq \omega$ implies that $\mu(\kappa) = 1$.

Can $\mu(\kappa)$ be small for small cardinals κ? Using a huge cardinal, Magidor showed in [4] that $\mu(\omega_3) \leq \omega_3$ is consistent. Shelah constructed a model of $\mu(\omega_1) = \omega_1$ starting with many supercompact cardinals (see [6]). With an almost huge cardinal Woodin produced a model where $\mu(\omega_1) = \omega_1$ is witnessed by normal filters. It seems to be an open problem whether $\mu(\omega_2) = \omega_1$ is consistent.

In this note we treat the following question. Is there always some κ such that $\mu(\kappa) \leq \kappa$? Prikry showed in [5] that $\mu(\omega_1) > \omega_1$ is consistent. Jensen showed later that the appropriate combinatorial principle holds in L which implies that $\mu(\omega_1) > \omega_1$ is true in L. We shall show:

THEOREM 1. *Assume $V = L$. Then $\mu(\kappa) > \kappa$ for all regular $\kappa > \omega$.*

To prove this we reduce the problem to a purely combinatorial question. So let us introduce the following principle. Let $\kappa > \omega$ be regular. Then Q_κ denotes the following property:

> There is some $G \subseteq \{f \mid f : \kappa \to 2\}$ such that $|G| > \kappa$ and for all $G^* \subseteq G$ such that $|G^*| > \kappa$ there is a countable $\overline{G} \subseteq G^*$ such that $\{\alpha < \kappa \mid \forall f, g \in \overline{G}, f(\alpha) = g(\alpha)\}$ is nonstationary.

This principle is closely related to some properties discussed in [7]. So the interested reader might also consult that paper. Now we have:

LEMMA 1. *Let $\kappa > \omega$ be regular and assume that Q_κ holds. Then $\mu(\kappa) > \kappa$.*

Proof. Assume not. Let $\mu(\kappa) \leq \kappa$ be given by H. By a result of Taylor (see [8]) we may assume that all $F \in H$ contain the club filter on κ. Let Q_κ be given by G. For each $f \in G$ choose $F_f \in H$ and $i_f < 2$ such that $\{\alpha < \kappa \mid f(\alpha) = i_f\} \in F_f$. Then there are $G^* \subseteq G$, $i < 2$, $F \in H$ such that

$|G^*| > \kappa$ and $F_f = F$, $i_f = i$ for all $f \in G^*$. Choose some countable $\overline{G} \subseteq G^*$ as in Q_κ. Then $\bigcap\{\alpha < \kappa \mid \forall f \in \overline{G} f(\alpha) = i\} \in F$ by ω_1-completeness and is nonstationary. This is a contradiction. □

So in order to prove Theorem 1 we only need to show:

PROPOSITION 1. *Assume $V = L$. Then Q_κ holds for all regular $\kappa > \omega$.*

Proof. We shall use the natural $(\kappa, 1)$-morass and the natural $\square_\infty$-sequence in L. The reader should look at [1] for the basic definitions. We use the standard notations. So for example $S = \{\nu \mid \nu > \omega,\ \nu \text{ p.r. closed},\ \nu \text{ singular}\}$, $\langle C_\nu \mid \nu \in S\rangle$ is the $\square_\infty$-sequence, $\prec$ is the morass tree, $\pi_{\overline{\nu}\nu}$ are the morass maps. Set $E = \{\nu \in S \cap \kappa^+ \mid C_\nu = \emptyset\}$. So we have

(1) (a) E is stationary in κ^+
 (b) for all singular $\tau, E \cap \tau$ is not stationary in τ
 (c) if $\overline{\nu} \prec \nu, \overline{\nu} \in E$ and $\pi_{\overline{\nu}\nu}$ is cofinal, then $\nu \in E$.

Set $E_0 = \{\nu < \kappa \mid \nu \in S^+ \cap E,\ \nu \text{ is minimal in } \prec\}$. We also need:

(2) There is a sequence $\langle X_\eta \mid \eta \in E_0\rangle$ such that
 (a) $\mathrm{otp}(X_\eta) = \omega$, $X_\eta \subseteq \eta$ is cofinal in η
 (b) for all unbounded $X \subseteq \kappa^+$ there are $\nu \in S_\kappa$ and $\eta \in E_0$ such that $\eta \prec \nu$ and $\pi_{\eta\nu}$"$X_\eta \subseteq X$.

The proof of this is very similar to the argument used in §3 of [1]. So we only give a sketch. We define $\langle X_\eta \mid \eta \in E_0\rangle$ by recursion. Given $\eta \in E_0$ let Z_η be the $<_L$-least unbounded subset of η such that there are no $\nu \in S_{\alpha_\eta}$ and $\tau \in E_0$ such that $\tau \prec \nu$ and $\pi_{\tau\nu}$"$X_\tau \subseteq Z_\eta$. Then choose $X_\eta \subseteq Z_\eta$ such that $\mathrm{otp}(X_\eta) = \omega$ and $\sup X_\eta = \eta$. Note that every element of E_0 has cofinality ω. This will do it.

Now using (1) we easily get:

(3) For $\alpha < \kappa$ and $\mu < \alpha^+$ there is a function $h^\mu_\alpha : \mu \to 2$ such that for all $\nu \in S_\alpha \cap \mu$, $\eta \prec \nu$, $\eta \in E_0$ we have that $h^\mu_\alpha \restriction \pi_{\eta\nu}$"$X_\eta$ is not eventually constant.

Now let $\nu \in S_\kappa$. Set $A_\nu = \{\alpha_\tau \mid \tau \prec \nu\}$. We define a function $f_\nu : A_\nu \to \kappa$ such that $f_\nu(\alpha) < \alpha^+$ as follows. Let $\tau \prec \nu$, $\alpha = \alpha_\tau$ and $\pi = \pi_{\tau\nu}$. Here we regard π as a map from L_τ to L_ν. Set $U = \{X \subseteq \alpha \mid X \in L_\tau,\ \alpha \in \pi(X)\}$. Define a sequence $\langle \tau_i \mid i \leq \gamma\rangle$ as follows. Set $\tau_0 = \alpha + 1$. If $\tau_i > \tau$, then set $\gamma = i$ and stop. If $\tau_i \leq \tau$, then let τ_{i+1} be the least ordinal Θ such that $U \cap L_{\tau_i} \in L_\Theta$. If λ is a limit ordinal, set $\tau_i = \sup\{\tau_i \mid i < \lambda\}$. Because we are in L it is easy to see that $\gamma \leq \omega + 1$. Set $f_\nu(\alpha) = \tau_\gamma$.

We are now ready to define the set of functions G which will give us Q_κ. It suffices that every element of G is defined on a club subset of κ. So let $\nu \in S_\kappa$. We define $g_\nu : A_\nu \to 2$ by $g_\nu(\alpha) = h^\mu_\alpha(\tau)$ where $\mu = f_\nu(\alpha)$ and τ is the unique $\tau \prec \nu$ such that $\alpha = \alpha_\tau$. Then set $G = \{g_\nu \mid \nu \in S_\kappa\}$. Finally, we show that G satisfies Q_κ. So let $X \subseteq S_\kappa$ be unbounded. By (2)(b) choose $\nu_0, \nu_1 \in S_\kappa, \eta_0, \eta_1 \in E_0$ such that $\nu_0 < \nu_1$ and $\eta_i \prec \nu_i, \pi_{\eta_i\nu_i}$"$X_\eta \subseteq X$ for $i < 2$. Set $Y_i = \pi_{\eta_i\nu_i}$"X_{η_i} and $Y = Y_0 \cup Y_1$. It suffices to show that there is a club $C \subseteq \kappa$ such that for all $\alpha \in C$ there are $\tau_0, \tau_1 \in Y$ such that $F_{\tau_0}(\alpha) = 0$ and $f_{\tau_1}(\alpha) = 1$. For this let $\alpha \in A_{\nu_0} \cap A_{\tau_1}$ be sufficiently large. Let $\tau_i \prec \nu_i$ such that $\alpha_{\tau_i} = \alpha$. Set $\pi_i = \pi_{\tau_i \nu_i}$. Then $\pi_0 \subseteq \pi_1$. Looking at the definition of the functions f_ν we see that the sequence $\langle f_\nu(\alpha) \mid \nu \in Y_0 \rangle$ or the sequence $\langle f_\nu(\alpha) \mid \nu \in Y_1 \rangle$ is eventually constant. So (3) gives us what we need. □

We conjecture that $\mu(\kappa) \leq \kappa$ implies that there is an inner model with a measurable cardinal. Let us mention that in Theorem 1 we can replace the assumption $V = L$ by $V = K$, where K denotes the Dodd–Jensen core model. We now indicate a proof of a very special case of our conjecture.

THEOREM 2. *Assume $\mu(\omega_1) = \omega_1$. Then there is an inner model with a measurable cardinal.*

For this we use a result of Taylor (see [8]). He showed that $\mu(\omega_1) > \omega_1$ is true if every ω_1-complete filter on ω_1 containing the club filter possesses an almost disjoint family of sets of positive F-measure of size ω_2. Now let $\langle f_\nu \mid \nu < \omega_2 \rangle$ be the sequence of canonical functions for ω_1. By Taylor's result Theorem 2 follows from the following proposition.

PROPOSITION 2. *Let F be an ω_1-complete filter on ω_1 which contains every club subset of ω_1. Assume that for every $f : \omega_1 \to \omega_1$ there is some $\nu < \omega_2$ such that $\{\alpha < \omega_1 \mid f(\alpha) < f_\nu(\alpha)\} \in F$. Then there is an inner model with a measurable cardinal.*

Proof. This just uses the method applied in the proof of Theorem 2 in [2]. So we build the same system of embeddings as there. It is well known that we may assume that for all $\nu \in E$ and $\alpha \in C_\nu$ that $f_\nu(\alpha) = \nu_\alpha$. So by our assumption on F for all $f : \omega_1 \to \omega_1$ there is some $\nu \in E$ such that $\{\alpha \in C_\nu \mid f(\alpha) < \nu_\alpha\} \in F$. So we can easily construct $X \subseteq E$ such that $\text{otp}(X) = \omega^2$ and $S_{\nu\tau} = \{\alpha \mid [\nu_\alpha, \tau_\alpha] \cap I_\alpha \neq \emptyset\} \in F$ for all $\nu, \tau \in X, \nu < \tau$. Then $S = \bigcap \{S_{\nu\tau} \mid \nu, \tau \in X, \nu < \tau\} \in F$. So S is stationary. Now we argue exactly as in [2]. □

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