A NOTE ON THE ORDINAL ANALYSIS OF KPM

W. BUCHHOLZ [1]

This note extends our method from (Buchholz [2]) in such a way that it applies also to the rather strong theory KPM. This theory was introduced and analyzed proof-theoretically in (Rathjen [6]), where Rathjen establishes an upper bound for its proof theoretic ordinal |KPM|. The bound was given in terms of a primitive recursive system $\mathcal{T}(\mathrm{M})$ of ordinal notations based on certain ordinal functions χ , ψ_κ ($\omega < \kappa < \mathrm{M}$, κ regular) [2] that had been introduced and studied in (Rathjen [5]). [3] In section 1 of this note we define and study a slightly different system of functions ψ_κ ($\kappa \leq \mathrm{M}$)—where ψ_M plays the rôle of Rathjen's χ—that is particularly well suited for our purpose of extending [2]. In section 2 we describe how one obtains, by a suitable modification of [2], an upper bound for |KPM| in terms of the ψ_κ's from section 1. We conjecture that this bound is best possible and coincides with the bound given in [6]. In section 3 we prove some additional properties of the functions ψ_κ which are needed to set up a primitive recursive ordinal notation system of ordertype $> \vartheta^\star$, where $\vartheta^\star := \psi_{\Omega_1}\varepsilon_{\mathrm{M}+1}$ is the upper bound for |KPM| determined in section 2.

Remark: Another ordinal analysis of KPM has been obtained independently by T. Arai in *Proof theory for reflecting ordinals II: recursively Mahlo ordinals* (handwritten notes, 1989).

§1. Basic properties of the functions ψ_κ ($\kappa \leq \mathrm{M}$). *Preliminaries.* The letters $\alpha, \beta, \gamma, \delta, \mu, \sigma, \xi, \eta, \zeta$ always denote ordinals. On denotes the class of all ordinals, and Lim the class of all limit numbers. Every ordinal α is identified with the set $\{\xi \in \mathrm{On} : \xi < \alpha\}$ of its predecessors. For $\alpha \leq \beta$ we set $[\alpha, \beta[\; := \{\xi : \alpha \leq \xi < \beta\}$. By $+$ we denote ordinary (noncommutative) ordinal *addition.* An ordinal $\alpha > 0$ which is closed under $+$ is called an *additive principal number.* The class of all additive principal numbers is denoted by AP. The *Veblen function* φ is defined by $\varphi\alpha\beta := \varphi_\alpha(\beta)$, where φ_α is the ordering function of the class $\{\beta \in \mathrm{AP} : \forall\xi < \alpha(\varphi_\xi(\beta) = \beta)\}$. An ordinal $\gamma > 0$ which is closed under φ (and thus also under $+$) is said to be *strongly critical.* The class of all strongly critical ordinals is denoted by SC.

---

[1]The final version of this paper was written while the author was visiting Carnegie Mellon University during the academic year 1990/91. I would like to express my sincere thanks to Wilfried Sieg (who invited me) and all members of the Philosophy Department of CMU for their generous hospitality.

[2]M denotes the first weakly Mahlo cardinal.

[3]The essential new feature of [5] is the function χ, while the ψ_κ's ($\kappa < \mathrm{M}$) are obtained by a straightforward generalization of previous constructions in [1], [3], [4].

Some basic facts:

1. $\mathrm{AP} = \{\omega^\alpha : \alpha \in \mathrm{On}\}$
2. $\varphi 0\beta = \omega^\beta$, $\varphi 1\beta = \varepsilon_\beta$
3. For each $\gamma > 0$ there are uniquely determined $n \in \mathbb{N}$ and additive principal numbers $\gamma_0 \geq \cdots \geq \gamma_n$ such that $\gamma = \gamma_0 + \cdots + \gamma_n$.
4. For each $\gamma \in \mathrm{AP} \setminus \mathrm{SC}$ there are uniquely determined $\xi, \eta < \gamma$ such that $\gamma = \varphi\xi\eta$.
5. Every uncountable cardinal is strongly critical.

Definition of $SC(\gamma)$:

1. $SC(0) := \emptyset$
2. $SC(\gamma) := \{\gamma\}$, if $\gamma \in \mathrm{SC}$
3. $SC(\gamma_0 + \cdots + \gamma_n) := SC(\gamma_0) \cup \cdots \cup SC(\gamma_n)$, if $n \geq 1$ and $\gamma_0 \geq \cdots \geq \gamma_n$ are additive principal numbers.
4. $SC(\varphi\xi\eta) := SC(\xi) \cup SC(\eta)$, if $\xi, \eta < \varphi\xi\eta$.

We assume the existence of a weakly Mahlo cardinal M. So every closed unbounded (club) set $X \subseteq \mathrm{M}$ contains at least one regular cardinal, and M itself is a regular cardinal.

DEFINITION 1.1.

$\mathrm{R} := \{\alpha : \omega < \alpha \leq \mathrm{M} \ \& \ \alpha \text{ regular}\}$

$\mathrm{M}^\Gamma := \min\{\gamma \in \mathrm{SC} : \mathrm{M} < \gamma\} =$ closure of $\mathrm{M} \cup \{\mathrm{M}\}$ under $+, \varphi$

$SC_{\mathrm{M}}(\gamma) := SC(\gamma) \cap \mathrm{M}$

$\Omega_0 := 0$, $\Omega_\sigma := \aleph_\sigma$ for $\sigma > 0$.

$\boldsymbol{\Omega} :=$ the function $\sigma \mapsto \Omega_\sigma$ restricted to $\sigma < \mathrm{M}$

Remark: $\forall \kappa \in \mathrm{R}(\ \kappa = \Omega_\kappa$ or $\kappa \in \{\Omega_{\sigma+1} : \sigma < \mathrm{M}\}\ )$

Convention. *In the following the letters κ, π, τ always denote elements of* R.

DEFINITION 1.2 (The collapsing functions ψ_κ).

By transfinite recursion on α we define ordinals $\psi_\kappa\alpha$ and sets $C(\alpha, \beta) \subseteq \mathrm{On}$ as follows. Under the induction hypothesis that $\psi_\pi\xi$ and $C(\xi, \eta)$ are already defined for all $\xi < \alpha$, $\pi \in \mathrm{R}$, $\eta \in \mathrm{On}$ we set

1. $C(\alpha, \beta) :=$ closure of $\beta \cup \{0, \mathrm{M}\}$ under $+, \varphi, \boldsymbol{\Omega}, \psi|\alpha$,
   where $\psi|\alpha$ denotes the binary function given by
   $$\mathrm{dom}(\psi|\alpha) := \{(\pi, \xi) : \xi < \alpha \ \& \ \pi \in \mathrm{R} \ \& \ \pi, \xi \in C(\xi, \psi_\pi\xi)\}$$
   $$(\psi|\alpha)(\pi, \xi) := \psi_\pi\xi.$$
2. $\psi_\kappa\alpha := \min\{\beta \in \mathcal{D}_\kappa(\alpha) : C(\alpha, \beta) \cap \kappa \subseteq \beta\}$
   $$\text{with} \quad \mathcal{D}_\kappa(\alpha) := \begin{cases} \{\beta \in \mathrm{R} : \alpha \in C(\alpha, \mathrm{M}) \Rightarrow \alpha \in C(\alpha, \beta)\} & \text{if } \kappa = \mathrm{M} \\ \{\beta : \kappa \in C(\alpha, \kappa) \Rightarrow \kappa \in C(\alpha, \beta)\} & \text{if } \kappa < \mathrm{M} \end{cases}$$

Abbreviation: $C_\kappa(\alpha) := C(\alpha, \psi_\kappa\alpha)$

The first two lemmata are immediate consequences of Definition 1.2.

LEMMA 1.1.

a) $\alpha_0 \leq \alpha \ \& \ \beta_0 \leq \beta \Longrightarrow C(\alpha_0, \beta_0) \subseteq C(\alpha, \beta)$

b) $\emptyset \neq X \subseteq \mathrm{On} \ \& \ \beta = \sup(X) \Longrightarrow C(\alpha, \beta) = \bigcup_{\eta \in X} C(\alpha, \eta)$

c) $\beta < \kappa \Longrightarrow \mathsf{card}(C(\alpha, \beta)) < \kappa$

LEMMA 1.2.
$C(\alpha,\beta) = \bigcup_{n<\omega} C^n(\alpha,\beta)$, where $C^n(\alpha,\beta)$ is defined by
(i) $C^0(\alpha,\beta) := \beta \cup \{0, \mathrm{M}\}$,
(ii) $C^{n+1}(\alpha,\beta) := \{\gamma : SC(\gamma) \subseteq C^n(\alpha,\beta)\} \cup \{\Omega_\sigma : \sigma \in C^n(\alpha,\beta)\} \cup$
$\cup \{\psi_\pi\xi : \xi < \alpha \ \& \ \pi, \xi \in C^n(\alpha,\beta) \cap C_\pi(\xi)\}$

LEMMA 1.3.
a) $C_\kappa(\alpha) \cap \kappa = \psi_\kappa\alpha < \kappa$
b) $\kappa < \mathrm{M} \Longrightarrow \psi_\kappa\alpha \notin \mathrm{R}$
c) $\psi_\kappa\alpha \in \mathrm{SC} \setminus \{\Omega_\sigma : \sigma < \Omega_\sigma\}$
d) $\kappa \in C(\alpha,\kappa) \Longleftrightarrow \kappa \in C_\kappa(\alpha)$
e) $C(\alpha,\mathrm{M}) = \mathrm{M}^\Gamma = \{\xi : \xi \in C_\mathrm{M}(\xi)\}$
f) $\gamma \in C_\kappa(\alpha) \implies \gamma \in C_\mathrm{M}(\gamma) \ \& \ SC_\mathrm{M}(\gamma) = SC(\gamma) \setminus \{\mathrm{M}\}$
g) $\gamma < \alpha \ \& \ \gamma \in C(\alpha,\beta) \implies \psi_\mathrm{M}\gamma \in C(\alpha,\beta)$

Proof.
a),b) 1. $C_\kappa(\alpha) \cap \kappa = \psi_\kappa\alpha$ is a trivial consequence of the definition of $\psi_\kappa\alpha$. 2. Let $\kappa = \mathrm{M}$. Obviously there exists a $\delta < \kappa$ such that $\mathrm{R} \cap [\delta,\kappa[\subseteq \mathcal{D}_\kappa(\alpha)$. Therefore in order to get $\psi_\kappa\alpha < \kappa$ it suffices to prove that the set $U := \{\beta \in \kappa : C(\alpha,\beta) \cap \kappa \subseteq \beta\}$ is closed unbounded (club) in κ.
i) *closed*: Let $\emptyset \neq X \subseteq U$ and $\beta := \sup(X) < \kappa$. Then $C(\alpha,\beta) \cap \kappa = \bigcup_{\xi\in X}(C(\alpha,\xi) \cap \kappa) \subseteq \bigcup_{\xi \in X} \xi = \beta$, i.e. $\beta \in U$.
ii) *unbounded*: Let $\beta_0 < \kappa$. We define $\beta_{n+1} := \min\{\eta : C(\alpha,\beta_n) \cap \kappa \subseteq \eta\}$ and $\beta := \sup_{n<\omega}\beta_n$. Using L.1.1c we obtain $\beta_n \leq \beta_{n+1} < \kappa$. Hence $\beta_0 \leq \beta < \kappa$ and $C(\alpha,\beta) \cap \kappa = \bigcup_{n<\omega}(C(\alpha,\beta_n) \cap \kappa) \subseteq \bigcup_{n<\omega}\beta_{n+1} = \beta$, i.e. $\beta_0 \leq \beta \in U$. 3. Let $\kappa < \mathrm{M}$. Starting with $\beta_0 := \min(\mathcal{D}_\kappa(\alpha))$ we define the ordinals β_n and β as in 2.(ii). Then we have $\beta \in \mathcal{D}_\kappa(\alpha) \cap U$ and therefore $\psi_\kappa\alpha \leq \beta < \kappa$. — Now assume that $\psi_\kappa\alpha \in \mathrm{R}$. We prove $\beta_n < \psi_\kappa\alpha$ $(\forall n)$. By definition of β_0 and by L.1.1a we have $\beta_0 \leq \psi_\kappa\alpha \ \& \ \beta_0 \notin \mathrm{Lim}$. Hence $\beta_0 < \psi_\kappa\alpha$. From $\beta_n < \psi_\kappa\alpha \in \mathrm{R}$ it follows that $C(\alpha,\beta_n) \cap \kappa \subseteq \psi_\kappa\alpha$ and $\mathsf{card}(C(\alpha,\beta_n) \cap \kappa) < \psi_\kappa\alpha$, and therefore $\beta_{n+1} < \psi_\kappa\alpha$. From $\forall n(\beta_n < \psi_\kappa\alpha \in \mathrm{R})$ we get $\beta < \psi_\kappa\alpha$. *Contradiction.*
c) 1. Obviously $C_\kappa(\alpha) \cap \kappa$ is closed under φ. Together with a) this implies $\psi_\kappa\alpha \in \mathrm{SC}$. — 2. We have $(\psi_\kappa\alpha = \Omega_\sigma > \sigma \Rightarrow \psi_\kappa\alpha \in C_\kappa(\alpha))$ and (by a)) $\psi_\kappa\alpha \notin C_\kappa(\alpha)$. Hence $\psi_\kappa\alpha \notin \{\Omega_\sigma : \sigma < \Omega_\sigma\}$.
d) follows from L.1.1a, L.1.3a and the definition of $\psi_\kappa\alpha$.
e) By L.1.3a $\forall\pi{\in}\mathrm{R}(\psi_\pi\xi < \mathrm{M})$ and therefore $C(\alpha,\mathrm{M}) = \mathrm{M}^\Gamma$. As in d) one obtains $(\alpha \in C(\alpha,\mathrm{M}) \Leftrightarrow \alpha \in C_\mathrm{M}(\alpha))$.
f) and g) follow from e).

LEMMA 1.4.
a) $\gamma \in C(\alpha,\beta) \Longleftrightarrow SC(\gamma) \subseteq C(\alpha,\beta)$
b) $\Omega_\sigma \in C(\alpha,\beta) \Longleftrightarrow \sigma \in C(\alpha,\beta)$
c) $\kappa = \Omega_{\sigma+1} \implies \Omega_\sigma < \psi_\kappa\alpha < \Omega_{\sigma+1}$
d) $\Omega_\kappa = \kappa \implies \Omega_{\psi_\kappa\alpha} = \psi_\kappa\alpha$
e) $\Omega_{\psi_\mathrm{M}\alpha} = \psi_\mathrm{M}\alpha$
f) $\Omega_\sigma \leq \gamma \leq \Omega_{\sigma+1} \ \& \ \gamma \in C(\alpha,\beta) \implies \sigma \in C(\alpha,\beta)$

Proof. a) and b) follow from L.1.2 and L.1.3c. — e) follows from d), since

$M \in R$ and $\Omega_M = M$. — f) follows from a),b),c),d) and L.1.2.
c) Let $\kappa = \Omega_{\sigma+1}$. Then $\kappa \in C(\alpha,\kappa)$ and thus $\kappa \in C_\kappa(\alpha)$. By a) and b) from $\kappa = \Omega_{\sigma+1} \in C_\kappa(\alpha)$ we get $\Omega_\sigma \in C_\kappa(\alpha) \cap \kappa = \psi_\kappa\alpha$.
d) Take $\sigma \in$ On such that $\Omega_\sigma \le \psi_\kappa\alpha < \Omega_{\sigma+1}$. Then we have $\sigma + 1 < \kappa$ and thus $C_\kappa(\alpha) \cap \kappa = \psi_\kappa\alpha < \Omega_{\sigma+1} < \Omega_\kappa = \kappa$. This implies $\Omega_{\sigma+1} \notin C_\kappa(\alpha)$ and then (by a),b)) $\sigma \notin C_\kappa(\alpha)$. Hence $\psi_\kappa\alpha \le \sigma \le \Omega_\sigma \le \psi_\kappa\alpha$.

LEMMA 1.5.
a) $\alpha_0 < \alpha \;\&\; \alpha_0 \in C_M(\alpha) \implies \psi_M\alpha_0 < \psi_M\alpha$
b) $\psi_M\alpha_0 = \psi_M\alpha_1 \;\&\; \alpha_0, \alpha_1 < M^\Gamma \implies \alpha_0 = \alpha_1$

Proof.
a) From the premise we get $\psi_M\alpha_0 \in C_M(\alpha) \cap M = \psi_M\alpha$ by L.1.3a,g.
b) Assume $\psi_M\alpha_0 = \psi_M\alpha_1 \;\&\; \alpha_0 < \alpha_1 < M^\Gamma$. Then $\alpha_0 \in C_M(\alpha_0) \subseteq C_M(\alpha_1)$ and therefore by a) $\psi_M\alpha_0 < \psi_M\alpha_1$. *Contradiction.*

LEMMA 1.6.
For $\kappa < M$ *the following holds*
a) $\alpha_0 < \alpha \implies \psi_\kappa\alpha_0 \le \psi_\kappa\alpha$
b) $\alpha_0 < \alpha \;\&\; \kappa, \alpha_0 \in C_\kappa(\alpha_0) \implies \psi_\kappa\alpha_0 < \psi_\kappa\alpha$

Proof.
a) From $\alpha_0 < \alpha$ it follows that $C(\alpha_0, \psi_\kappa\alpha) \cap \kappa \subseteq \psi_\kappa\alpha$. By definition of $\psi_\kappa\alpha_0$ it therefore suffices to prove $\psi_\kappa\alpha \in \{\beta : \kappa \in C(\alpha_0,\kappa) \Rightarrow \kappa \in C(\alpha_0,\beta)\}$. So let $\kappa \in C(\alpha_0,\kappa)$. — We have to prove $\kappa \in C(\alpha_0, \psi_\kappa\alpha)$.
CASE 1: $\kappa = \Omega_{\sigma+1}$. By Lemma 1.4c we have $\Omega_\sigma < \psi_\kappa\alpha$ and therefore $\sigma + 1 \in C(\alpha_0, \psi_\kappa\alpha)$ which implies $\kappa \in C(\alpha_0, \psi_\kappa\alpha)$.
CASE 2: $\kappa = \Omega_\kappa$. From $\kappa \in C(\alpha_0,\kappa) \subseteq C(\alpha,\kappa)$ we obtain $\kappa \in C_\kappa(\alpha_0) \cap C_\kappa(\alpha)$. From this by L.1.2, L.1.3b, L.1.5b it follows that $\kappa = \psi_M\xi$ with $\xi < \alpha_0$ and $\xi \in C_\kappa(\alpha)$. Now by L.1.4a, L.1.3a,e we get $SC_M(\xi) \subseteq C_\kappa(\alpha) \cap C_M(\xi) \cap M = C_\kappa(\alpha) \cap \kappa = \psi_\kappa\alpha$, and then $\xi \in C(\alpha_0, \psi_\kappa\alpha)$ (by L.1.3f). From this together with $\xi < \alpha_0$ we obtain $\kappa = \psi_M\xi \in C(\alpha_0, \psi_\kappa\alpha)$ (by L.1.3g).
b) The premise together with a) implies $\alpha_0 < \alpha \;\&\; \kappa, \alpha_0 \in C_\kappa(\alpha) \cap C_\kappa(\alpha_0)$ which gives us $\psi_\kappa\alpha_0 \in C_\kappa(\alpha) \cap \kappa = \psi_\kappa\alpha$.

DEFINITION 1.3.
For each set $X \subseteq$ On we set $\mathcal{H}_\gamma(X) := \bigcap\{C(\alpha,\beta) : X \subseteq C(\alpha,\beta) \;\&\; \gamma < \alpha\}$.

§2. Ordinal analysis of KPM.

In this section we show how one has to modify (and extend) [2] in order to establish that the ordinal $\psi_{\Omega_1}\varepsilon_{M+1}$ is an upper bound for |KPM|. Of course we now assume that the reader is familiar with [2].

The theory KPM is obtained from KPi by adding the following axiom scheme:

(Mahlo) $\forall x \exists y \phi(x,y,\vec{z}) \to \exists w[Ad(w) \land \forall x{\in}w \exists y{\in}w \phi(x,y,\vec{z})]$ $\qquad (\phi \in \Delta_0)$

We extend the infinitary system RS^∞ introduced in Section 3 of [2] by adding the following inference rule:

$$\text{(Mah)} \quad \frac{\Gamma, B(\mathsf{L}_{\mathrm{M}}) : \alpha_0}{\Gamma, \exists w{\in}\mathsf{L}_{\mathrm{M}}(Ad(w) \land B(w)) : \alpha} \quad (\alpha_0 + \mathrm{M} < \alpha)$$

where $B(w)$ *is of the form* $\forall x{\in}w\exists y{\in}wA(x,y)$ *with* $\mathrm{k}(A) \subseteq \mathrm{M}$.

We set $\mathrm{R} := \{\alpha : \omega < \alpha \leq \mathrm{M} \ \& \ \alpha \text{ regular}\}$.

Then all lemmata and theorems of Section **3** [4] are also true for the extended system RS^∞(with almost literally the same proofs)[5], and as an easy consequence from Theorem **3.12** one obtains the

EMBEDDING THEOREM for KPM.
If $\mathrm{M} \in \mathcal{H}$ *and if* $\mathcal{H}$ *is closed under* $\xi \mapsto \xi^{\mathrm{R}}$ *then for each theorem* ϕ *of* KPM *there is an* $n \in \mathbb{N}$ *such that* $\mathcal{H}\vert\!\frac{\omega^{\mathrm{M}+n}}{\mathrm{M}+n}\ \phi^{\mathrm{M}}$.

Some more severe modifications have to be carried out on Section **4**. The first part of this section (down to Lemma **4.5**) has to be replaced by Section 1 of the present paper. Then the sets $C(\alpha,\beta)$ are no longer closed under $(\pi,\xi) \mapsto \psi_\pi\xi$ $(\xi < \alpha)$, but only under $(\psi|\alpha)$ as defined in Definition 1.2 above. Therefore we have to add "$\pi, \xi \in C_\pi(\xi)$" to the premise of Lemma **4.6c**, and accordingly a minor modification as to be made in the proof of Lemma **4.7**($\mathcal{A}$1). But this causes no problems. A little bit problematic is the fact that the function ψ_{M} is not weakly increasing. In order to overcome this difficulty we prove the following lemma.

DEFINITION 2.1.
For $\gamma = \omega^{\gamma_0} + \cdots + \omega^{\gamma_n}$ with $\gamma_0 \geq \cdots \geq \gamma_n$ we set $\mathsf{e}(\gamma) := \omega^{\gamma_n+1}$.
Further we set $\mathsf{e}(0) := \mathrm{On}$.

LEMMA 2.1.
For $\gamma \in C_{\mathrm{M}}(\gamma+1)$ *and* $0 < \alpha < \mathsf{e}(\gamma)$ *the following holds*
a) $\psi_{\mathrm{M}}(\gamma+1) \leq \psi_{\mathrm{M}}(\gamma+\alpha) \ \& \ C_{\mathrm{M}}(\gamma+1) \subseteq C_{\mathrm{M}}(\gamma+\alpha)$
b) $0 < \alpha_0 < \alpha \ \& \ \alpha_0 \in C_{\mathrm{M}}(\gamma+1) \implies \psi_{\mathrm{M}}(\gamma+\alpha_0) < \psi_{\mathrm{M}}(\gamma+\alpha)$

Proof:
a) follows from b).
b) We will prove $(*)$ $\psi_{\mathrm{M}}(\gamma+1) \leq \psi_{\mathrm{M}}(\gamma+\alpha)$. From this we get $\gamma+\alpha_0 \in C_{\mathrm{M}}(\gamma+1) \subseteq C_{\mathrm{M}}(\gamma+\alpha)$ and then by L.1.5a the assertion.
For $\gamma = 0$ $(*)$ is trivial. If $\gamma \neq 0$ then $\gamma+\alpha < \mathrm{M}^\Gamma$ and therefore $\gamma+\alpha \in C_{\mathrm{M}}(\gamma+\alpha)$ which (together with $\alpha < \mathsf{e}(\gamma)$) implies $\gamma+1 \in C_{\mathrm{M}}(\gamma+\alpha)$. Hence $\psi_{\mathrm{M}}(\gamma+1) \leq \psi_{\mathrm{M}}(\gamma+\alpha)$ by L.1.5a.

Now we give a complete list of all modifications which have to be carried out in [2] subsequent to Lemma **4.6** .

[4]We use boldface numerals to indicate reference to [2]

[5]In Theorem **3.8** one has to add the clause which corresponds to the new inference rule (Mah). The last line in the proof of Lemma **3.14** has to be modified to *"... cannot be the main part of a* (Ref)*- or* (Mah)*-inference."*. At the end of the proof of Lemma **3.17** one may add the remark *"Due to the premise* $\alpha \leq \beta < \kappa$ *we have* $\alpha < \mathrm{M}$*, and therefore the given derivation of* Γ, C *does not contain any applications of* (Mah)*."*.

(1) Replace I by M in the definition of $\bar{K}$.
(2) Add "$\eta < \gamma + \mathsf{e}(\gamma)$" to the premise of Lemma **4.7**($\mathcal{A}$2).
(3) Add "$\omega^{\mu+\alpha} < \mathsf{e}(\gamma)$" to the premise of Theorem **4.8**.
(4) Add "$\pi \le \mathsf{e}(\gamma')$" to the premise of ($\Box$) in the proof of Theorem **4.8**.
(5) Insert the following proof of "$\psi_\kappa\alpha^* \le \psi_\kappa\hat{\alpha}$" at the end of the proof of ($\Box$):
"*From* $\gamma', \mu', \alpha' \in \mathcal{H}_{\gamma'}[\Theta]$ *we get* $\alpha^* \in \mathcal{H}_{\gamma'}[\Theta]$. *From* $\mathrm{k}(\Theta) \subseteq C_\kappa(\gamma+1) \subseteq C_\kappa(\hat{\alpha})$ & $\gamma' < \hat{\alpha}$ *it follows that* $\mathcal{H}_{\gamma'}[\Theta] \subseteq C_\kappa(\hat{\alpha})$. *Hence* $\alpha^* \in C_\kappa(\hat{\alpha})$ *and thus* $\psi_\kappa\alpha^* \le \psi_\kappa\hat{\alpha}$, *since* $\alpha^* < \hat{\alpha}$."
(6) Extend the proof of Theorem **4.8** by the following treatment of the case where the last inference in the given derivation of Γ is an application of (Mah):
"*5. Suppose that* $\exists w{\in}\mathsf{L}_\mathrm{M}(Ad(w) \wedge B(w)) \in \Gamma$ *and* $\mathcal{H}_\gamma[\Theta] \,|\!\frac{\alpha_0}{\mu}\, \Gamma, B(\mathsf{L}_\mathrm{M})$ *with* $B(w) \equiv \forall x{\in}w\exists y{\in}wA(x,y)$ & $\alpha_0 + \mathrm{M} < \alpha$ & $\mathrm{k}(A) \subseteq \mathrm{M}$.
Then $\kappa = \mathrm{M}$ *(since* $\Gamma \subseteq \Sigma(\kappa)$ *and* $\kappa \le \mathrm{M}$*).*
For $\iota \in \mathcal{T}_\mathrm{M}$ *we set* $\gamma_\iota := \gamma + \omega^{\mu+\alpha_0+|\iota|}$. *Then* $C_\mathrm{M}(\gamma+1) \subseteq C_\mathrm{M}(\gamma_\iota)$, *and since* $SC(|\iota|) \subseteq SC_\mathrm{M}(\gamma_\iota) \subseteq \psi_\mathrm{M}\gamma_\iota$, *we have* $|\iota| < \psi_\mathrm{M}\gamma_\iota$ *and thus* $\mathrm{k}(\Theta,\iota) \subseteq C_\mathrm{M}(\gamma_\iota)$.
From $\gamma, \mu, \alpha_0 \in \mathcal{H}_\gamma[\Theta]$ *we get* $\gamma_\iota \in \mathcal{H}_\gamma[\Theta,\iota]$. *Consequently* $\mathcal{A}(\Theta,\iota;\gamma_\iota,\mathrm{M},\mu)$, *and the Inversion-Lemma gives us* $\mathcal{H}_\gamma[\Theta][\iota] \,|\!\frac{\alpha_0}{\mu}\, \Gamma, \iota \notin \mathsf{L}_0 \to \exists y \in \mathsf{L}_\mathrm{M}A(\iota,y)$.
Now we apply the I.H. and obtain $\mathcal{H}_{\alpha^*_\iota}[\Theta][\iota] \,|\!\frac{\psi_\mathrm{M}\alpha^*_\iota}{\cdot}\, \Gamma, \iota \notin \mathsf{L}_0 \to \exists y{\in}\mathsf{L}_\mathrm{M}A(\iota,y)$ *with* $\alpha^*_\iota := \gamma_\iota + \omega^{\mu+\alpha_0} < \gamma + \omega^{\mu+\alpha_0+\mathrm{M}} =: \alpha^* < \hat{\alpha}$.
Let $\pi := \psi_\mathrm{M}\alpha^*$ & $\beta_\iota := \psi_\mathrm{M}\alpha^*_\iota$. *Then by L.***4.7** $\pi \in \mathcal{H}_{\hat{\alpha}}[\Theta]$ & $\pi < \psi_\mathrm{M}\hat{\alpha}$.
We also have $\forall\iota{\in}\mathcal{T}_\pi(\alpha^*_\iota \in C_\mathrm{M}(\alpha^*))$ *and thus* $\forall\iota{\in}\mathcal{T}_\pi(\beta_\iota < \pi)$.
The Boundedness-Lemma gives us now
$\forall\iota \in \mathcal{T}_\pi(\, \mathcal{H}_{\hat{\alpha}}[\Theta][\iota] \,|\!\frac{\beta_\iota}{\pi}\, \Gamma, \iota \notin \mathsf{L}_0 \to \exists y{\in}\mathsf{L}_\pi A(\iota,y)\,)$.
From this by an application of ($\bigwedge$) *we obtain* $\mathcal{H}_{\hat{\alpha}}[\Theta] \,|\!\frac{\pi}{\cdot}\, \Gamma, B(\mathsf{L}_\pi)$.
*From L.***2.5h** *and L.***3.10** *we get* $\mathcal{H}_{\hat{\alpha}}[\Theta] \,|\!\frac{\delta}{0}\, \Gamma, Ad(\mathsf{L}_\pi)$ *with* $\delta := \omega^{\pi+5}$. *We also have* $\mathcal{H}_{\hat{\alpha}}[\Theta] \,|\!\frac{0}{\cdot}\, \Gamma, \mathsf{L}_\pi \notin \mathsf{L}_0$. *Hence* $\mathcal{H}_{\hat{\alpha}}[\Theta] \,|\!\frac{\delta+2}{\cdot}\, \Gamma, \mathsf{L}_\pi \notin \mathsf{L}_0 \wedge Ad(\mathsf{L}_\pi) \wedge B(\mathsf{L}_\pi)$. *Now we apply* ($\bigvee$) *and obtain* $\mathcal{H}_{\hat{\alpha}}[\Theta] \,|\!\frac{\psi_\mathrm{M}\hat{\alpha}}{\cdot}\, \Gamma$."
(7) Replace I by M in the Corollary to Theorem **4.8** and in Theorem **4.9**.

This yields the following Theorem.

THEOREM.
Let $\vartheta^* := \psi_{\Omega_1}(\varepsilon_{\mathrm{M}+1})$. *Then for each* Σ_1*-sentence* ϕ *of* $\mathcal{L}$ *we have:*
$\mathrm{KPM} \vdash \forall x(Ad(x) \to \phi^x) \implies L_{\vartheta^*} \models \phi$.

COROLLARY. $|\mathrm{KPM}| \le \psi_{\Omega_1}(\varepsilon_{\mathrm{M}+1})$.

§3. Further properties of the functions ψ_κ.

We prove four theorems which together with L.1.3a,b,c and L.1.4a-e provide a complete basis for the definition of a primitive recursive well-ordering (OT,$\prec$) which is isomorphic to $(C(\mathrm{M}^\Gamma,0),<)$. (The set OT consists of terms built up from the constants $\underline{0}$, $\underline{\mathrm{M}}$ by the function symbols $\underline{+}$, $\underline{\varphi}$, $\underline{\Omega}$, $\underline{\psi}$, such that for each $\gamma \in C(\mathrm{M}^\Gamma,0)$ there is a unique term $t \in \mathrm{OT}$ with $|t| = \gamma$, and for all $s,t \in \mathrm{OT}$ one has $(s \prec t \Leftrightarrow |s| < |t|)$. Here $|t|$ denotes the canonical value of t. For details see [1], [4], [5].)

Now the letters $\alpha, \beta, \gamma, \delta, \mu, \sigma, \xi, \eta, \zeta$ always denote ordinals less than M^Γ. So, for all α we have $\alpha \in C_{\mathrm{M}}(\alpha)$ and $SC(\alpha) \setminus \{\mathrm{M}\} = SC_{\mathrm{M}}(\alpha) \subseteq \psi_{\mathrm{M}}\alpha$.

DEFINITION 3.1.

$$\mathsf{sc}_\kappa(\alpha) := \begin{cases} \max SC_{\mathrm{M}}(\alpha) & \text{if } \kappa = \mathrm{M} \ \& \ SC_{\mathrm{M}}(\alpha) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

LEMMA 3.1.

a) $\mathsf{sc}_\kappa(\alpha) < \psi_\kappa\alpha$

b) $\pi = \mathrm{M} \ \& \ \mathsf{sc}_\pi(\beta) < \psi_\kappa\alpha \implies \beta \in C_\kappa(\alpha)$

Proof. Trivial (cf. L.1.a,e,f and L.1.4a).

LEMMA 3.2.

Let $\kappa \in C_\kappa(\alpha) \ \& \ \pi \in C_\pi(\beta)$. *Then*

$\psi_\pi\beta < \kappa < \pi \ \& \ \mathsf{sc}_\pi(\beta) < \psi_\kappa\alpha \implies \psi_\pi\beta < \psi_\kappa\alpha.$

Proof. By L.1.4c,d it follows that $\Omega_\pi = \pi$ and $\Omega_{\psi_\pi\beta} = \psi_\pi\beta$. Therefore if $\kappa = \Omega_{\sigma+1}$ then $\psi_\pi\beta \leq \Omega_\sigma < \psi_\kappa\alpha$, and we may now assume that $\Omega_\kappa = \kappa$. Then by L.1.2 and L.1.3b we obtain $\kappa = \psi_{\mathrm{M}}\gamma$ with $\gamma < \alpha \ \& \ \gamma \in C_\kappa(\alpha) \cap C_{\mathrm{M}}(\gamma)$. By L.1.4a and L1.3a we get $SC_{\mathrm{M}}(\gamma) \subseteq C_\kappa(\alpha) \cap C_{\mathrm{M}}(\gamma) \cap \mathrm{M} = C_\kappa(\alpha) \cap \kappa = \psi_\kappa\alpha$. From $\psi_\pi\beta < \kappa = \psi_{\mathrm{M}}\gamma < \pi$ it follows that $\psi_{\mathrm{M}}\gamma \notin C_\pi(\beta)$ and thus $\beta \leq \gamma$ or $\psi_\pi\beta \leq \mathsf{sc}_{\mathrm{M}}(\gamma)$. — If $\psi_\pi\beta \leq \mathsf{sc}_{\mathrm{M}}(\gamma)$ then $\psi_\pi\beta < \psi_\kappa\alpha$, since $SC_{\mathrm{M}}(\gamma) \subseteq \psi_\kappa\alpha$. If $\mathsf{sc}_{\mathrm{M}}(\gamma) < \psi_\pi\beta \ \& \ \pi = \mathrm{M}$ then we have $\beta \leq \gamma < \alpha$ and $\beta \in C_\kappa(\alpha)$ (since $\mathsf{sc}_\pi(\beta) < \psi_\kappa\alpha$), from which we get $\psi_\pi\beta \in C_\kappa(\alpha) \cap \kappa = \psi_\kappa\alpha$. — For $\pi = \mathrm{M}$ the proof is now finished. — If $\mathsf{sc}_{\mathrm{M}}(\gamma) < \psi_\pi\beta \ \& \ \pi < \mathrm{M}$ then $\psi_{\mathrm{M}}\gamma < \pi < \mathrm{M} \ \& \ \mathsf{sc}_{\mathrm{M}}(\gamma) < \psi_\pi\beta$ which (according to what we already proved for $\pi = \mathrm{M}$) implies $\kappa = \psi_{\mathrm{M}}\gamma < \psi_\kappa\alpha$. *Contradiction.*

DEFINITION 3.2.

$\mathcal{K}(\pi, \beta, \kappa, \alpha)$ abbreviates the disjunction of $(\mathcal{K}1), \ldots, (\mathcal{K}4)$ below:

$(\mathcal{K}1)$ $\pi \leq \psi_\kappa\alpha$

$(\mathcal{K}2)$ $\psi_\pi\beta \leq \mathsf{sc}_\kappa(\alpha)$

$(\mathcal{K}3)$ $\pi = \kappa \ \& \ \beta < \alpha \ \& \ \mathsf{sc}_\pi(\beta) < \psi_\kappa\alpha$

$(\mathcal{K}4)$ $\psi_\pi\beta < \kappa < \pi \ \& \ \mathsf{sc}_\pi(\beta) < \psi_\kappa\alpha$

LEMMA 3.3.

Let $\kappa \in C_\kappa(\alpha) \ \& \ \pi \in C_\pi(\beta)$.

a) $\neg\mathcal{K}(\pi, \beta, \kappa, \alpha) \ \& \ \neg\mathcal{K}(\kappa, \alpha, \pi, \beta) \implies \kappa = \pi \ \& \ \alpha = \beta$

b) $\mathcal{K}(\pi, \beta, \kappa, \alpha) \implies \psi_\pi\beta \leq \psi_\kappa\alpha$

c) $\mathcal{K}(\pi, \beta, \kappa, \alpha) \ \& \ \beta \in C_\pi(\beta) \implies \psi_\pi\beta < \psi_\kappa\alpha$

Proof. a) is a logical consequence of the linearity of $<$. b) and c) follow immediately from L.1.3a, L.1.5a, L.1.6, L.3.1, L.3.2.

As an immediate consequence from lemma 3.3 we get

THEOREM 3.1.

$\kappa, \alpha \in C_\kappa(\alpha) \ \& \ \pi, \beta \in C_\pi(\beta) \ \& \ \psi_\kappa\alpha = \psi_\pi\beta \implies \kappa = \pi \ \& \ \alpha = \beta.$

THEOREM 3.2.

Let $\kappa \in C_\kappa(\alpha) \ \& \ \pi, \beta \in C_\pi(\beta)$.

a) $\psi_\pi\beta < \psi_\kappa\alpha \iff \mathcal{K}(\pi, \beta, \kappa, \alpha)$

b) $\psi_\pi\beta \in C_\kappa(\alpha) \iff (\psi_\pi\beta < \psi_\kappa\alpha \text{ or } [\beta < \alpha \ \& \ \pi, \beta \in C_\kappa(\alpha)])$

Proof. a) "$\Leftarrow$" follows from L.3.3b. "$\Rightarrow$" follows from L.3.3a,c.
b) The "$\Leftarrow$" part is trivial. So let us assume that $\psi_\kappa\alpha \le \psi_\pi\beta \in C_\kappa(\alpha)$. By L.1.2 and L.1.3c this implies the existence of $\tau, \xi \in C_\kappa(\alpha) \cap C_\tau(\xi)$ with $\xi < \alpha$ and $\psi_\pi\beta = \psi_\tau\xi$. From this by Theorem 3.1 we obtain $\pi = \tau \in C_\kappa(\alpha)$ and $\beta = \xi \in C_\kappa(\alpha) \cap \alpha$.

THEOREM 3.3.
$\kappa \in C_\kappa(\alpha) \iff \kappa \in \{\Omega_{\sigma+1} : \sigma < \mathrm{M}\} \cup \{\psi_{\mathrm{M}}\xi : \xi < \alpha\} \cup \{\mathrm{M}\}$

Proof. 1. "$\Rightarrow$" follows from L.1.2 and L.1.3b. — 2. By L.1.3d we have $(\kappa \in C_\kappa(\alpha) \Leftrightarrow \kappa \in C(\alpha,\kappa))$. — 3. If $\kappa = \Omega_{\sigma+1}$ then $\sigma + 1 < \kappa$ and thus $\kappa \in C(\alpha,\kappa)$. — 4. If $\kappa = \psi_{\mathrm{M}}\xi$ with $\xi < \alpha$ then $\xi \in C_{\mathrm{M}}(\xi) = C(\xi,\kappa) \subseteq C(\alpha,\kappa)$ and thus $\kappa \in C(\alpha,\kappa)$.

THEOREM 3.4.
$\kappa = \Omega_{\sigma+1} \implies C_\kappa(\alpha) = C(\alpha, \Omega_\sigma + 1)$

Proof by induction on α. So let us assume that $C_\kappa(\xi) = C(\xi, \Omega_\sigma + 1)$, for all $\xi < \alpha$. — We have to prove $\psi_\kappa\alpha \subseteq C(\alpha, \Omega_\sigma + 1)$. As we will show below the I.H. implies that $\beta := C(\alpha, \Omega_\sigma + 1) \cap \kappa$ is in fact an ordinal. Obviously $\kappa \in C(\alpha, \beta)$ and $C(\alpha,\beta) \cap \kappa \subseteq C(\alpha, \Omega_\sigma + 1) \cap \kappa = \beta$ and thus $\psi_\kappa\alpha \le \beta$, i.e. $\psi_\kappa\alpha \subseteq C(\alpha, \Omega_\sigma + 1)$.
—CLAIM: $\gamma \in C(\alpha, \Omega_\sigma + 1) \cap \kappa \implies \gamma \subseteq C(\alpha, \Omega_\sigma + 1)$.
Proof. 1. $\Omega_\sigma < \gamma \in \mathrm{SC}$. Then $\gamma = \psi_\pi\xi$ with $\xi < \alpha$ & $\xi \in C_\pi(\xi)$. Since $\Omega_\sigma < \gamma < \kappa = \Omega_{\sigma+1}$, we have $\pi = \kappa$ and therefore by the above I.H. $C_\kappa(\xi) = C(\xi, \Omega_\sigma + 1)$. Hence $\gamma = \psi_\kappa\xi \subseteq C(\xi, \Omega_\sigma + 1) \subseteq C(\alpha, \Omega_\sigma + 1)$.
2. Let γ be arbitrary and $\gamma_0 := \max(\{0\} \cup SC(\gamma))$. Then (by 1. above) $\gamma_0 \cup \{\gamma_0\} \subseteq C(\alpha, \Omega_\sigma + 1)$. From this we get $\gamma \subseteq \gamma^* \subseteq C(\alpha, \Omega_\sigma + 1)$, where $\gamma^* := \min\{\eta \in \mathrm{SC} : \gamma_0 < \eta\}$.

COROLLARY. $\psi_{\Omega_1}\alpha = C(\alpha, 0) \cap \Omega_1$

REFERENCES

[1] BUCHHOLZ, W. *A new system of proof-theoretic ordinal functions.* ***Annals of Pure and Applied Logic***, vol. 32 (1986), pp. 195–207.

[2] BUCHHOLZ, W. *A simplified version of local predicativity.* In: P. Aczel, H. Simmons, S. Wainer (eds.), ***Proof Theory***, Cambridge University Press, 1993.

[3] BUCHHOLZ, W., and SCHÜTTE, K. *Ein Ordinalzahlsystem für die beweistheoretische Abgrenzung der Π^1_2-Separation und Bar-Induktion.* ***Sitzungsberichte der Bayerischen Akademie der Wissenschaften***, Mathematisch-Naturwissenschaftliche Klasse (1983).

[4] JÄGER, G. *ρ-inaccessible ordinals, collapsing functions and a recursive notation system.* ***Archiv für mathematische Logik und Grundlagenforschung***, vol. 24 (1984), pp. 49–62.

[5] RATHJEN, M. *Ordinal notations based on a weakly Mahlo cardinal.* ***Archiv für mathematische Logik und Grundlagenforschung***, vol. 29 (1990), pp. 249–263.

[6] RATHJEN, M. *Proof-theoretic analysis of KPM.* ***Archiv für mathematische Logik und Grundlagenforschung***, vol. 30 (1991), pp. 377–403.

Mathematisches Institut der Universität München
Theresienstr. 39, D-80333 München, Germany