## §9. Uniqueness of the Next Extender

In §11 we shall construct an extender sequence $\vec{E}$ such that $L[\vec{E}] \models$ "there is a Woodin cardinal, and every level $\mathcal{J}_\alpha^{\vec{E}}$ of $L[\vec{E}]$ is a 1-small coremouse". The sequence $\vec{E}$ will be defined by recursion. The recursion is substantially more subtle than it is for sequences of measures, but the basic idea is still to define $\vec{E}_\gamma$ by recursion on $\gamma$, by making $\vec{E}_\gamma$ be the least extender which can be added to the sequence $\vec{E} \restriction \gamma$ so that the extender sequence remains good. Part of the strategy will be to pick $\vec{E}_\gamma$ without regard to the initial segment condition and then prove that in fact it does satisfy the initial segment condition as well. We would like to show that there is always only one possible choice of $\vec{E}_\gamma$ for each $\gamma$, so that if $\rho$ is the natural length of $\vec{E}_\gamma \restriction \rho$ and $G$ of length $\gamma'$ is its trivial extension then $G$, being a legal choice for $E_{\gamma'}$, must in fact be $E_{\gamma'}$. Of course this ignores the second alternative in the initial segment condition, but more important we are unable to prove this uniqueness: so far as we know there could be one choice of types I or III and a second of type II. In this section we will prove uniqueness for types I or III, and in section 11 this will be used for the case when $\rho$ is a cardinal in $L[\vec{E}]$. In section 10 we will prove a related result which will apply in the cases when $\rho$ is not a cardinal in $L[\vec{E}]$.

The standard method for showing uniqueness of the next extender on the sequence involves *Doddages* and comparison of a Doddage with itself. The method originates in Mitchell's [M74R], see also [D]. We need only a simple sort of Doddage, dubbed by Jensen a bicephalus. A bicephalus is like an active premouse, except that it has two predicates corresponding to two candidates for a last extender. By comparing bicephali with themselves we show that in sufficiently iterable bicephali, these candidates are not distinct.

Unfortunately, when we want to form an ultrapower of a bicephalus whose last extenders differ in type, we have a problem. We may want to squash for the sake of one extender, but if we do so it is not clear how to carry along the other. This is the reason we will also need the alternative technique from section 10.

The first problem in dealing with bicephali will be to verify that when we form the ultrapower of a bicephalus both of whose last extenders are of type III, the squashing procedures in the two cases are consistent with one another. We shall verify this now, in Lemma 9.1.

If $M$ is an active ppm then $\nu^{\mathcal{M}}$ is just the the natural length of the extender coded by $\dot{F}^{\mathcal{M}}$, that is if $\mathcal{M}$ is of type II or III then $\nu^{M}$ is the strict sup of its generators, while if $\mathcal{M}$ is type I, then $\nu^{\mathcal{M}} = (\kappa^+)^{\mathcal{M}}$.)

**Lemma 9.1.** *Let $\mathcal{M}$ be a type III ppm, and $G$ an extender over $M$ with* $\operatorname{crit} G = \kappa < \nu^{\mathcal{M}}$. *Let $\mathcal{P}$ be the ultrapower of $\mathcal{M}$ via $G$, where functions in $|\mathcal{M}|$ are used, and let $i : \mathcal{M} \to \mathcal{P}$ be the canonical embedding. Assume $\mathcal{P}$ is wellfounded. Let $\nu^* = sup\ i''\nu^{\mathcal{M}}$. Then*

(a) *$\nu^* =$ sup of generators of $\dot{F}^{\mathcal{P}}$.*

(b) *Let $\gamma = (\nu^*)^{+^{\mathcal{P}}}$, or $\gamma = OR^{\mathcal{P}}$ if $\mathcal{P} \models (\nu^*)^+$ doesn't exist. Let*

$$Q = (J_\gamma^{\dot{E}^{\mathcal{P}}}, \in, \dot{E}^{\mathcal{P}} \restriction \gamma, \dot{F}^{\mathcal{P}} \restriction \gamma).$$

*Then $Q$ is a type* III *ppm. and $Q^{sq} = \mathrm{Ult}_0(\mathcal{M}^{sq}, G)$.*

*Remarks.* (1) $\mathcal{P}$ is defined more carefully at the beginning of §4. It is to be constructed without squashing.

(2) According to (b), the structure $Q$ is equal to the structure $\mathrm{Ult}_0(\mathcal{M}, G)$.

(3) If $\nu^* = i(\nu^{\mathcal{M}})$, then $\gamma = \mathrm{OR}^{\mathcal{P}}$, so that $\mathcal{P} = Q$.

(4) If $\nu^* < i(\nu^{\mathcal{M}})$, then $\gamma < \mathrm{OR}^{\mathcal{P}}$ since $i(\nu^{\mathcal{M}})$ is a cardinal of $\mathcal{P}$. Part (a) of 9.1 then tells us that $\mathcal{P}$ is not a ppm, as it violates the bounded generators clause of "good at $\mathrm{OR}^{\mathcal{P}}$". (It also violates the initial segment clause.) Roughly speaking, its last extender $\dot{F}^{\mathcal{P}}$ was added too late; it should have been added at $\gamma$. Replacing $\mathcal{P}$ by $Q$, which is the net effect of squashing, amounts to adding $\dot{F}^{\mathcal{P}}$ at $\gamma$.

Proof of 9.1. (a) Notice that for $\xi < \nu^{\mathcal{M}}$, $\xi$ is a generator of $\dot{F}^{\mathcal{M}}$ iff $i(\xi)$ is a generator of $\dot{F}^{\mathcal{P}}$. The reason is that $\dot{F}^{\mathcal{M}} \restriction (\xi + 1) \in |\mathcal{M}|$, and the fact that $\xi$ is a generator is a $\Sigma_0$ fact about $\dot{F}^{\mathcal{M}} \restriction (\xi + 1)$. Thus $\nu^*$ is a sup of generators of $\dot{F}^{\mathcal{P}}$, and we need only show that no $\eta \geq \nu^*$ is a generator of $\dot{F}^{\mathcal{P}}$.

So let $\nu^* \leq \eta < \mathrm{OR}^{\mathcal{P}}$. We want to find $a \subseteq \nu^*$, $a$ finite, and $h \in |\mathcal{P}|$ such that $\eta = [a, h]^{\mathcal{P}}_{\dot{F}^{\mathcal{P}}}$.

Let $\eta = [b, f]^{\mathcal{M}}_G$, where $b$ is a size $n$ set of coordinates of $G$ and $f \in |\mathcal{M}|$ and $\mathrm{dom}\, f = [\kappa]^n$. It will be enough to find maps $\bar{u} \mapsto a_{\bar{u}}$ and $\bar{u} \mapsto h_{\bar{u}}$, both in $|\mathcal{M}|$, such that

(i) for $G_b$ a.e. $\bar{u}$, $f(\bar{u}) = [a_{\bar{u}}, h_{\bar{u}}]^{\mathcal{M}}_{\dot{F}^{\mathcal{M}}}$

and

(ii) $\bigcup\{ a_{\bar{u}} : \bar{u} \in [\kappa]^n \}$ is bounded in $\nu^{\mathcal{M}}$.

(We can then take $a = [b, \lambda\bar{u} \cdot a_{\bar{u}}]^{\mathcal{M}}_G$ and $h = [b, \lambda\bar{u} \cdot h_{\bar{u}}]^{\mathcal{M}}_G$. Because $\dot{F}^{\mathcal{M}}$ is weakly amenable over $\mathcal{M}$ we have enough of Los' theorem to show this works. We omit the details.)

Now as $f \in |\mathcal{M}|$, the coherence condition on $\dot{F}^{\mathcal{M}}$ implies that $f \in \mathrm{Ult}(\mathcal{M}, \dot{F}^{\mathcal{M}})$. Let

$$f = [c, g]^{\mathcal{M}}_{\dot{F}^{\mathcal{M}}}$$

where $c \subseteq \nu^{\mathcal{M}}$ is finite and $g \in |\mathcal{M}|$. For any $\bar{u} = \{u_1 \cdots u_n\} \in [\kappa]^n$, set

$$a_{\bar{u}} = c \cup \{u_1 \cdots u_n\}.$$

So the map $\bar{u} \mapsto a_{\bar{u}}$ is in $\mathcal{M}$, and condition (ii) above holds.

Let $a_{\bar{u}} = \{\alpha_1 \cdots \alpha_{\mathcal{I}}\}$ in increasing order. Let $c = \{\alpha_{k_0} \cdots \alpha_{k_e}\}$ and $\bar{u} = \{\alpha_{m_0} \cdots \alpha_{m_t}\}$ in increasing order. If $\bar{v} = \{v_1 \cdots v_{\mathcal{I}}\}$ is any sequence, given in increasing order, then we will write $\bar{v}^0_{\bar{u}} = \{v_{k_0} \cdots v_{k_e}\}$ and $\bar{v}^1_{\bar{u}} = \{v_{m_0} \cdots v_{m_t}\}$. Note that the map $(\bar{v}, \bar{u}) \mapsto (\bar{v}^0_{\bar{u}}, \bar{v}^1_{\bar{u}})$ is in $M$. Now set, for $\bar{u} \in [\kappa]^n$ and $\bar{v}$ in the space of $\dot{F}^{\mathcal{M}}_{a_{\bar{u}}}$

$$h_{\bar{u}}(\bar{v}) = g(\bar{v}^0_{\bar{u}})(\bar{v}^1_{\bar{u}}).$$

(Take $h_{\bar{u}}(\bar{v}) = 0$ if this doesn't make sense; it makes sense $\dot{F}^{\mathcal{M}}_{a_{\bar{u}}}$ a.e.) Clearly the map $\bar{u} \mapsto h\bar{u}$ is in $|\mathcal{M}|$. We need only verify condition (i).

Let $j : \mathcal{M} \to \mathrm{Ult}(\mathcal{M}, \dot{F}^{\mathcal{M}})$ be the canonical embedding. Then

$$\begin{aligned}
[a_{\bar{u}}, h_{\bar{u}}]^{\mathcal{M}}_{\dot{F}^{\mathcal{M}}} &= j(h_{\bar{u}})(a_{\bar{u}}) \\
&= j(g)((a_{\bar{u}})^0_{\bar{u}})((a_{\bar{u}})^1_{\bar{u}}) \\
&= j(g)(c)(\bar{u}) = f(\bar{u}),
\end{aligned}$$

as desired.

(b) Clearly, $\mathrm{Ult}_0(\mathcal{M}^{sq}, G) = (J^{\dot{E}^{\mathcal{P}}}_{\nu^*}, \in, \dot{E}^{\mathcal{P}} \restriction \nu^*, \dot{F}^{\mathcal{P}} \restriction \nu^*)$. This structure is $\mathcal{R}^{sq}$ for some type III ppm $\mathcal{R}$, by results of §3. We decode $\mathcal{R}_{\mathcal{P}}$ from $\mathcal{R}^{sq}$ by taking $\mathrm{Ult}_0(\mathcal{R}^{sq}, \dot{F}^{\mathcal{P}} \restriction \nu^*)$ and cutting off at its $(\nu^*)^+$. By (a), this is the same as taking $\mathrm{Ult}_0(R^{sq}, \dot{F}^{\mathcal{P}})$ and cutting off. As $\dot{F}^{\mathcal{P}}$ coheres with the $\mathcal{P}$ sequence, this gives us $Q$. □

DEFINITION 9.1.1. A *bicephalus* is a structure

$$\mathfrak{B} = (J^{\vec{E}}_{\alpha}, \in, \vec{E} \restriction \alpha, F, G)$$

such that $\mathfrak{B}_0 = (J^{\vec{E}}_{\alpha}, \in, \vec{E} \restriction \alpha, F)$ and $\mathfrak{B}_1 = (J^{\vec{E}}_{\alpha}, \in, \vec{E} \restriction \alpha, G)$ are both premice, and either

(a) both $\mathfrak{B}_0$ and $\mathfrak{B}_1$ are of type II

or

(b) neither $\mathfrak{B}_0$ nor $\mathfrak{B}_1$ is of type II.

*Remark.* As the reader may have noticed, the distinction between types I and II is not very important elsewhere - here it is.

If both $\mathfrak{B}_0$ and $\mathfrak{B}_1$ are of type II we say $\mathcal{L}$ has type II. Otherwise $\mathfrak{B}$ has type III.

We let $\dot{F}^{\mathfrak{B}}_0$ and $\dot{F}^{\mathfrak{B}}_1$ be the two last extenders of $\mathfrak{B}$.

Certain notions appropriate for premice - e.g. $\mathcal{J}^{\mathfrak{B}}_{\gamma}$, agreement below $\gamma$- extend to bicephali in an obvious way.

Suppose $\mathfrak{B}$ and $\mathcal{A}$ are bicephali, and $G$ is an extender from the $\mathcal{A}$-sequence such that crit $G = \kappa$. Suppose $P(\kappa) \cap |\mathfrak{B}| = P(\kappa) \cap |\mathcal{A}|$. Suppose also that if $\mathfrak{B}$ is type III, then $\kappa < \nu^{\mathfrak{B}_0}$. (Notice that $\nu^{\mathfrak{B}_0} = \nu^{\mathfrak{B}_1}$ = largest cardinal of $\mathfrak{B}$, in the case $\mathfrak{B}$ is of type III.) Suppose $\mathrm{Ult}_0(\mathfrak{B}_0, G)$ and $\mathrm{Ult}_0(\mathfrak{B}_1, G)$ are wellfounded, and hence premice of the same type as $\mathfrak{B}_0$ and $\mathfrak{B}_1$. We claim there is a unique bicephalus $\mathcal{C}$ such that $\mathcal{C}_0 = \mathrm{Ult}_0(\mathfrak{B}_0, G)$ and $\mathcal{C}_1 = \mathrm{Ult}_0(\mathcal{B}_1, G)$. If $\mathfrak{B}$ is of type II this is obvious, so suppose $\mathfrak{B}$ is of type III.

Suppose one of $\mathfrak{B}_0$ and $\mathfrak{B}_1$ in of type I. If both are type I, there is no problem. Suppose e.g. $\mathfrak{B}_0$ is type I while $\mathfrak{B}_1$ is type III. Then $\nu^{\mathfrak{B}_1} = (\kappa^+)^{\mathfrak{B}}$, where $\kappa$ is the critical point of the last extender of $\mathfrak{B}_0$, i.e. of $\dot{F}_0^{\mathfrak{B}}$. (For $\nu^{\mathfrak{B}_1}$ is the largest cardinal of $\mathfrak{B}_1$ in the type III case, and $(\kappa^+)^{\mathfrak{B}_0}$ is the largest cardinal of $\mathfrak{B}_0$ in the type I case.) But then if $i$ is the canonical embedding from the full ultrapower of $\mathfrak{B}_1$ by $G$, using all functions in $|\mathfrak{B}_1|$, then $i$ is continuous at $\nu^{\mathfrak{B}_1}$. So by Lemma 9.1, the squashed and unsquashed ultrapowers of $\mathfrak{B}_1$ coincide. This gives us the desired $\mathfrak{C}_1$ at once.

Now suppose both $\mathfrak{B}_0$ and $\mathfrak{B}_1$ are of type III. Recall that if $\mathcal{M}$ is a type III premouse, then $\mathrm{Ult}_0(\mathcal{M}, G)$ is the unique $Q$ such that $Q^{sq} = \mathrm{Ult}_0(\mathcal{M}^{sq}, G)$. It will be enough to show that $\mathrm{OR} \cap \mathrm{Ult}_0(\mathcal{B}_0, G) = \mathrm{OR} \cap \mathrm{Ult}_0(\mathcal{B}_1, G)$, and that $\mathrm{Ult}_0(\mathfrak{B}_0, G)$ agrees with $\mathrm{Ult}_0(\mathfrak{B}_1, G)$ below $\mathrm{OR} \cap \mathrm{Ult}_0(\mathcal{B}_0, G)$. But now let $\mathcal{D}$ be the full ultrapower via $G$ of $\mathcal{B}$ formed using all functions in $|\mathcal{B}|$, and $i : \mathcal{B} \to \mathcal{D}$ the canonical embedding. By Lemma 9.1 we see $\mathrm{OR} \cap \mathrm{Ult}_0(\mathcal{B}_0, G) = (\sup i''\nu^{\mathcal{B}_0})^{+\mathcal{D}} = \mathrm{OR} \cap \mathrm{Ult}_0(\mathcal{B}_1, G)$, and that the necessary agreement holds.

So we may define

DEFINITION 9.1.2. In the situation described above, $\mathrm{Ult}_0(\mathfrak{B}, G)$ is the unique bicephalus $\mathfrak{C}$ such that $\mathfrak{C}_0 = \mathrm{Ult}_0(\mathfrak{B}_0, G)$ and $\mathfrak{C}_1 = \mathrm{Ult}_0(\mathfrak{B}_1, G)$.

Notice that if $\mathfrak{B}$ is type II, we have a canonical $i : \mathfrak{B} \to \mathrm{Ult}_0(\mathfrak{B}, G)$ which is $r\Sigma_1$ elementary (in the obvious sense.) If $\mathfrak{B}$ is type III we get an embedding $i : \mathfrak{B}^{sq} \to \mathrm{Ult}_0(\mathfrak{B}, G)^{sq}$ - which is $q\Sigma_1$ elementary. We get an embedding $i : \mathfrak{B} \to \mathrm{Ult}_0(\mathfrak{B}, G)$ in the case $\mathrm{Ult}_0(\mathfrak{B}, G)$ happens to be the full ultrapower of $\mathfrak{B}$ by $G$, using all functions in $|\mathfrak{B}|$. This happens when the canonical embedding of $\mathfrak{B}$ into the full ultrapower, call it $i$, is continuous at $\nu^{\mathfrak{B}_0}$. That is, this happens when $\mathfrak{B} \models \mathrm{cof}\,(\nu^{\mathfrak{B}_0}) \neq \kappa$, where $\kappa = \mathrm{crit}\ G$. Notice in this regard that if $\mathfrak{B} \models \mathrm{cof}\,(\nu^{\mathfrak{B}_0}) = \kappa$, and $\mathfrak{C} = \mathrm{Ult}_0(\mathfrak{B}, G)$, then $\mathfrak{C} \models \mathrm{cof}\,(\nu^{\mathfrak{C}_0}) = \kappa$, since $\nu^{\mathfrak{C}_0} = \sup i''\, \nu^{\mathfrak{B}_0}$. This implies that along any branch of an iteration tree on $\mathfrak{B}$, the natural embeddings map $\mathfrak{B}$ into $\mathrm{Ult}_0(\mathfrak{B}, G)$ in all but at most one instance. This is because we can hit a given $\kappa$ at most once along any branch.

The notion of a 0-maximal iteration tree generalizes in an obvious way to trees on bicephali, so we shall just mention a few points. Let $\mathfrak{B}$ be a bicephalus; a 0-maximal iteration tree on $\mathfrak{B}$ is a system

$$\mathcal{T} = \langle T, \deg, D, \langle E_\alpha, \mathfrak{B}^*_{\alpha+1} \mid \alpha + 1 < \theta \rangle \rangle$$

together with associated models $\mathfrak{B}_\alpha$ and embeddings $i_{\alpha\beta} : \mathfrak{B}_\alpha \to \mathfrak{B}_\beta$ defined whenever $\alpha T \beta$ and $[\alpha, \beta]_T \cap D = \varnothing$. We have $\mathfrak{B}_0 = \mathfrak{B}$. One can build $\mathcal{T}$ freely at successor steps except for the following restrictions:

Let $T$-pred$(\alpha + 1) = \beta$, and crit $E_\alpha = \kappa$. Then we must have $\operatorname{lh} E_\alpha > \operatorname{lh} E_\eta$, all $\eta < \alpha$, and

$$\beta = \text{least } \xi \leq \alpha \text{ such that } \kappa < \text{sup of generators of } E_\xi \,.$$

Let

$$\gamma = \text{largest } \eta \text{ such that } \mathcal{J}_\eta^{\mathfrak{B}_\beta} \text{ exists and}$$
$$P(\kappa) \cap |\mathcal{J}_\eta^{\mathfrak{B}_\beta}| \subseteq |\mathfrak{B}_\alpha|\,;$$

then $\alpha + 1 \in D \Leftrightarrow \mathcal{J}_\gamma^{\mathfrak{B}_\beta}$ is a proper initial segment of $\mathfrak{B}_\beta$, and

$$\deg(\alpha+1) = \begin{cases} 0 & \text{if } [0, \alpha+1]_T \cap D = \varnothing\,, \\ \text{largest } k \text{ s.t. } \kappa < \rho_k^{\mathcal{N}}, & \text{for } \mathcal{N} = \mathcal{J}_\gamma^{\mathfrak{B}_\beta}\,, \text{otherwise} \end{cases}$$

and

$$\mathfrak{B}^*_{\alpha+1} = \mathcal{J}_\gamma^{\mathfrak{B}_\beta}\,.$$

Finally, if $n = \deg(\alpha + 1)$, then

$$\mathfrak{B}_{\alpha+1} = \mathrm{Ult}_n(\mathfrak{B}^*_{\alpha+1}, E_\alpha)$$

and if $\alpha + 1 \notin D$ we have a canonical embedding $i_{\beta,\alpha+1} : \mathfrak{B}_\beta \to \mathfrak{B}_{\alpha+1}$.

This last statement is not true in the case when $\mathfrak{B}_\beta$ is of type III and $\nu^{\mathfrak{B}_\beta}$ has cofinality $\kappa$ in $\mathfrak{B}_\beta$. In this case we let $i_{\beta,\alpha+1}$ be the canonical embedding of $\mathfrak{B}_\beta^{sq}$ into $\mathfrak{B}^{sq}_{\alpha+1}$.]

We also have an embedding $i^*_{\alpha+1} \to \mathfrak{B}_{\alpha+1}$, again with a possible exception in the type III case, when we may only have $i^*_{\alpha+1} : (\mathfrak{B}^*_{\alpha+1})^{sq} \to (\mathfrak{B}_{\alpha+1})^{sq}$.

If $\lambda < \theta$ is a limit, then $D \cap [0, \lambda]_T$ must be finite. Moreover the special case referred to above will only occur finitely often, so that $\operatorname{dom} i^*_{\alpha+1} = |\mathfrak{B}^*_{\alpha+1}|$ for all but finitely many $\alpha + 1 \in [0, \lambda)_T$. Thus the direct limit of the models $\mathfrak{B}_\beta$ under the maps $i_{\beta,\gamma}$ for $\beta, \gamma$ in $[\beta_0, \lambda)$ for some $\beta_0 \in [0, \lambda)$ exists; and we require that $\mathfrak{B}_\lambda$ be this direct limit.

*Remarks.*

1. If $[0, \alpha+1]_T \cap D \neq \varnothing$, then $\mathfrak{B}_{\alpha+1}$ is a premouse rather than a bicephalus. Moreover, one can see by an easy induction that $\mathfrak{B}^*_{\alpha+1}$ is $\deg(\alpha + 1)$ sound, whenever $[0, \alpha+1]_T \cap D \neq \varnothing$. Also, if $\gamma + 1 T \alpha + 1$ and$D \cap (\gamma+1, \alpha+1]_T = \varnothing$, then $\deg(\gamma + 1) \geq \deg(\alpha + 1)$.

2. By coherence and the fact that $\operatorname{lh} E_\alpha$ increases with $\alpha$, we get the counterpart of Lemma 5.1; $\mathfrak{B}_\beta$ agrees with $\mathfrak{B}_\alpha$ below $\operatorname{lh} E_\alpha$, for all $\beta \geq \alpha$, and $\operatorname{lh} E_\alpha$ is a cardinal of $\mathfrak{B}_\beta$ for all $\beta > \alpha$.

The notion of a *simple* iteration tree generalizes in an obvious way to trees on bicephali. We can then define *k-iterability* for bicephali just as we did for premice.

The notion of 1-*smallness* generalizes in the obvious way: a bicephalus $\mathfrak{B}$ is 1-small iff both $\mathfrak{B}_0$ and $\mathfrak{B}_1$ are 1-small. The ***uniqueness theorem*** 6.1 generalizes in an obvious way. (We have no analog of 6.2, strong uniqueness. In general, we don't care about the Levy hierarchy over bicephali.)

The main theorem about bicephali is that there aren't any interesting ones.

**Theorem 9.2.** *Let $\mathfrak{B}$ be a 1-small, 1-iterable bicephalus. Then $\dot{F}_0^{\mathfrak{B}} = \dot{F}_1^{\mathfrak{B}}$.*

PROOF SKETCH. We compare $\mathfrak{B}$ with itself in such a way that the comparison process can only terminate if $\dot{F}_0^{\mathfrak{B}} = \dot{F}_1^{\mathfrak{B}}$. But iterability implies that the process terminates, so $\dot{F}_0^{\mathfrak{B}} = \dot{F}_1^{\mathfrak{B}}$.

Let $\mathcal{T}$ and $\mathcal{U}$ be the 0-maximal iteration trees on $\mathfrak{B}$, with models $\mathfrak{C}_\alpha$ and $\mathcal{D}_\alpha$, built by the method of "iterating the least disagreement". Notice that if $\dot{F}_0^{\mathfrak{B}} \neq \dot{F}_1^{\mathfrak{B}}$ then it is guaranteed that there will be such a disagreement, since even if the bicephalus $\mathfrak{C}_\alpha = (J_\gamma^{\vec{E}}, \in, \vec{E}, F, G)$ is matched to $\mathcal{D}_\alpha$ except for the final extenders $F$ and $G$ then each of $F$ and $G$ would have to agree with whatever is on the $\mathcal{D}_\alpha$ sequence at $\gamma$ (or possibly to both of the final extenders of $\mathcal{D}_\alpha$), and if $F$ and $G$ are different then they cannot both agree with the same extender.

At limit steps $\lambda$, we stop the construction unless $\mathcal{T} \restriction \lambda$ and $\mathcal{U} \restriction \lambda$ are simple. In the latter case, we let $[0,\lambda)_T$ and $[0,\lambda)_U$ be the unique cofinal wellfounded branches of their respective trees, and continue.

Suppose the construction never stops because we reach a $\lambda$ such that one of $\mathcal{T} \restriction \lambda$ and $\mathcal{U} \restriction \lambda$ is not simple. Then the proof of 7.1 shows that we must reach a $\theta$ such that $\mathfrak{C}_\theta$ is an initial segment of $\mathcal{D}_\theta$, or vice-versa. (So the construction stops at $\theta$.) Say $\mathfrak{C}_\theta$ is an initial segment of $\mathcal{D}_\theta$. By the proof of 7.1, there's no dropping on $[0,\theta]_T$. [Otherwise $\mathfrak{C}_\theta$ is unsound, so $\mathfrak{C}_\theta = \mathcal{D}_\theta$. So $\mathfrak{C}_\omega(\mathfrak{C}_\theta)$, the core of $\mathfrak{C}_\theta$, is $\mathfrak{C}^*_{\alpha+1}$ for some $\alpha+1 \in [0,\theta)_{\mathcal{U}}$. This is too much agreement at an earlier stage.] Thus $\mathfrak{C}_\theta$ is a bicephalus. If $\dot{F}_0^{\mathfrak{C}_\theta} \neq \dot{F}_1^{\mathfrak{C}_\theta}$, then $\mathfrak{C}_\theta$ cannot be on initial segment of $\mathcal{D}_\theta$; at worst, one of $\dot{F}_0^{\mathfrak{C}_\theta}$ and $\dot{F}_1^{\mathfrak{C}_\theta}$ will participate in a disagreement with $\mathcal{D}_\theta$. So $\dot{F}_0^{\mathfrak{C}_\theta} = \dot{F}_1^{\mathfrak{C}_\theta}$, so $\dot{F}_0^{\mathfrak{B}} = \dot{F}_1^{\mathfrak{B}}$.

Suppose we reach a $\lambda$ s.t. e.g. $\mathcal{T} \restriction \lambda$ is not simple. Let $b$ and $c$ be distinct cofinal wellfounded branches of $\mathcal{T} \restriction \lambda$. Suppose e.g. $\mathrm{OR}^{\mathfrak{C}_b} \leq \mathrm{OR}^{\mathfrak{C}_c}$. Let $\delta = \delta(\mathcal{T} \restriction \lambda) = \sup\,\{\mathrm{lh}\, E_\alpha^{\mathcal{T}} \mid \alpha \in \lambda\}$. The proof of Claim 1 in the proof of 6.2 shows $\mathrm{lh}\, F < \delta$ for all extenders $F$ from the $\mathfrak{C}_b$ sequence. Clearly, then, $\delta = \sup\,\{\mathrm{lh}\, F : F \text{ from the } \mathfrak{C}_b \text{ sequence}\}$. As $\mathfrak{C}_0 = \mathfrak{B}$ has a maximum length realized by extenders on its sequence, $D^{\mathcal{T}} \cap b \neq \varnothing$. Thus $\mathfrak{C}_b$ is unsound. On the other hand, the proof of Claim 2 in/a the proof of 6.2 shows $\mathfrak{C}_b$ is an initial segment of $\mathfrak{C}_c$. Thus $\mathfrak{C}_b = \mathfrak{C}_c$, and $D^{\mathcal{T}} \cap c \neq \varnothing$. But then the proof of Claim 4 in the proof of 6.2 yields a contradiction.

Thus we can never reach a $\lambda$ such that one of $\mathcal{T} \restriction \lambda$ and $\mathcal{U} \restriction \lambda$ is not simple. This completes the proof. $\square$