§6. Uniqueness of Wellfounded Branches

We shall show that, roughly speaking, all iteration trees which are important for the comparison of 1-small mice are simple.

Let $\mathcal{T} = \langle T, \deg, D, \langle E_\alpha, \mathcal{M}^*_{\alpha+1} \mid \alpha + 1 < \theta\rangle\rangle$ be an iteration tree of length θ. We set

$$\vec{E}(\mathcal{T}) = \bigcup_{\alpha<\theta} (\dot{E}^{\mathcal{M}_\alpha} \restriction \operatorname{lh} E_\alpha)$$

$$\delta(\mathcal{T}) = \bigcup_{\alpha<\theta} \operatorname{lh} E_\alpha$$

By 5.1, $\dot{E}^{\mathcal{M}_\alpha} \restriction \operatorname{lh} E_\alpha = \dot{E}^{\mathcal{M}_\beta} \restriction \operatorname{lh} E_\alpha$ for all $\beta > \alpha$, so that $\vec{E}(\mathcal{T})$ is a good extender sequence with domain included in $\delta(\mathcal{T})$. Notice that if b is a cofinal wellfounded branch of $\mathcal{T}$, then $\vec{E}(\mathcal{T}) = \dot{E}^{\mathcal{M}_b} \restriction \delta(\mathcal{T})$.

Theorem 6.1 (Uniqueness Theorem). *Let $\mathcal{T}$ be an iteration tree of limit length θ, and b and c be distinct cofinal wellfounded branches of $\mathcal{T}$. Let $\alpha = OR^{\mathcal{M}_b} \cap OR^{\mathcal{M}_c}$, so that $\alpha \geq \delta(\mathcal{T})$, and suppose that $\alpha > \delta(\mathcal{T})$. Then*

$$J_\alpha^{\vec{E}(\mathcal{T})} \models \delta(\mathcal{T}) \quad \textit{is Woodin}\,.$$

Proof. Just as in [MS]. Here is a slightly cleaner presentation of that argument, adapted to our context.

Let $\delta = \delta(\mathcal{T})$, $\vec{E} = \vec{E}(\mathcal{T})$, and let $f : \delta \to \delta$ with $f \in J_\alpha^{\vec{E}}$. Let $\beta < \theta$ be large enough that

$$D \cap (b \cup c) \subseteq \beta$$

and

$$b \cap \beta \neq c \cap \beta$$

and

$$\gamma \in b - \beta \Rightarrow f, \vec{E}, \delta \in \operatorname{ran} i_{\gamma b}\,,$$

$$\gamma \in c - \beta \Rightarrow f, \vec{E}, \delta \in \operatorname{ran} i_{\gamma c}\,,$$

and $\alpha \in \operatorname{ran} i_{\gamma b}$ if $\alpha \neq \mathrm{OR}^{\mathcal{M}_b}$, and $\alpha \in \operatorname{ran} i_{jc}$ if $\alpha \neq \mathrm{OR}^{\mathcal{M}_c}$.

Claim 1. If $\gamma \in b - \beta$ and $\eta \in c - \beta$, then

$$(\operatorname{ran} i_{\gamma b} \cap \operatorname{ran} i_{\eta c} \cap J_\alpha^{\vec{E}}) \prec_{\Sigma_1} J_\alpha^{\vec{E}}\,.$$

Proof. Straightforward. The restriction to Σ_1 is due to the limited elementarity of the maps $i_{\gamma b}, i_{\eta c}$.

CLAIM 2. Let $\gamma + 1 \in b$ with T-pred$(\gamma + 1) = \xi \geq \beta$, and let η be a member of c such that $\beta < c < \gamma + 1$ such that if $c < \xi$ then η is the largest member of c such that $\eta < \gamma + 1$. Then

$$\operatorname{ran} i_{\xi b} \cap \operatorname{ran} i_{\eta c} \cap \delta = \inf\{\operatorname{crit} i_{\xi b},\ \operatorname{crit} i_{\eta c}\}.$$

PROOF. $\supseteq$ is obvious. Let us define

$$\begin{aligned} \gamma_0 &= \gamma + 1 \\ \eta_n &= \text{least ordinal in } c - \gamma_n \\ \gamma_{n+1} &= \text{least ordinal in } b - \eta_n \end{aligned}$$

for all $n < \omega$. The γ_n's and η_n's are all successor ordinals. Also we have $\sup_{n<\omega}\gamma_n = \sup_{n<\omega}\eta_n$, so the common sup is θ. Notice also that T-pred$(\eta_n) < \gamma_n$ and T-pred$(\gamma_{n+1}) < \eta_n$ by the minimality of our choices. Also T-pred$(\eta_0) = \eta$ (unless $\eta \geq \xi$ in which case this may fail), and T-pred$(\gamma_0) = \xi$.

Now suppose $\mu \in \operatorname{ran} i_{\xi b} \cap \operatorname{ran} i_{\eta c} \cap \delta$. As $\mu < \delta$, we have an $n < \omega$ such that

$$\mu < \operatorname{lh} E_{\gamma_{n+1}-1}.$$

Since $\mu \in \operatorname{ran} i_{\xi b}$ and $\xi T \gamma_{n+1}$,

$$\mu < \operatorname{crit}\ E_{\gamma_{n+1}}.$$

By clauses (3) and (4) on iteration trees,

$$\mu < \operatorname{lh} E_{T\text{-pred}(\gamma_{n+1})} \leq \operatorname{lh} E_{\eta_n - 1}.$$

Since $\mu \in \operatorname{ran} i_{\eta c}$ and $\eta T \eta_n$,

$$\mu < \operatorname{crit}\ E_{\eta_n - 1}.$$

By clauses (3) and (4) on iteration trees

$$\mu < \operatorname{lh} E_{T\text{-pred}(\eta_n)} \leq \operatorname{lh} E_{\gamma_n - 1}.$$

So we may repeat the cycle until we get $\mu < \operatorname{lh} E_{\gamma_0 - 1}$. Then applying the argument again we get

$$\mu < \operatorname{crit} E_{\gamma_0 - 1} < \operatorname{lh} E_\xi.$$

So if $\nu + 1 \in b - (\xi + 1)$ or $\nu + 1 \in c - (\eta + 1)$ then $\nu \geq \xi$ (under either hypothesis on η) so that $\mu < \operatorname{lh} E_\nu$, so $\mu < \operatorname{crit}\ E_\nu$. Thus $\mu < \operatorname{crit}\ i_{\eta c}$ and $\mu < \operatorname{crit}\ i_{\xi b}$.

CLAIM 3. Claim 2 holds with the roles of b and c reversed.

PROOF. The proof is the same as that of claim 2.

Now fix $\beta' > \beta$ such that $b \cap (\beta' - \beta) \neq \varnothing$ and $c \cap (\beta' - \beta) \neq \varnothing$. Let

$$\kappa = \text{ least } \nu \text{ such that } \nu = \text{crit } E_\gamma \text{ for some } \gamma + 1 \in (b \cup c) - \beta' .$$

Let γ be largest such that $\kappa = \text{crit } E_\gamma$ and $\gamma + 1 \in (b \cup c) - \beta'$, and suppose without loss of generality that $\gamma + 1 \in b$. Let η be the largest element of c which is $< \gamma + 1$. Notice crit $i_{\eta c} = \text{crit } E_\nu$ for some $\nu + 1 \in c$ such that $\gamma + 1 < \nu + 1$; thus crit $i_{\eta c} > \kappa$. So

$$\kappa = \text{ran } i_{\eta c} \cap \text{ran } i_{\xi b} \cap \delta$$

where $\xi = T\text{-pred}(\gamma + 1)$, and it follows by Claim 1 that κ is closed under f. Now let $\nu = \inf\{\text{crit } i_{\eta c}, \text{crit } i_{\gamma+1,b}\}$ Claim 3 implies that

$$\nu = \text{ran } i_{\eta c} \cap \text{ran } i_{\gamma+1,b} \cap \delta$$

so that ν is closed under f. Note also that $\kappa < \nu$.

We claim that $\nu < \rho_\gamma$. (Recall that ρ_γ is the sup of the generators for E_γ.) Let $\tau \in c$ and $T\text{-pred}(\tau) = \eta$. Then $\nu \leq \text{crit } i_{\eta c} \leq \text{crit } E_{\tau-1} < \rho_\eta$. So if $\eta = \gamma$ we're done. Otherwise $\eta < \gamma$, so $\text{lh } E_\eta$ is a cardinal of $\mathcal{M}_\gamma$, and as $\text{lh } E_\eta < \text{lh } E_\gamma$, $\text{lh } E_\eta \leq \rho_\gamma$. As $\nu < \rho_\eta$, $\nu < \rho_\gamma$.

Our initial segment condition on good extender sequences implies that $E_\gamma \restriction \nu$ is an initial segment of some extender F which is on the sequence of $\mathcal{M}_\gamma$ before E_γ. By coherence we see that F is one of the extenders on $\vec{E} = \vec{E}(\mathcal{T})$. So $E_\gamma \restriction \nu \in J_\alpha^{\vec{E}}$.

We leave it to the reader to check that ν is an inaccessible cardinal of $J_\alpha^{\vec{E}}$. By strong acceptability and the fact that F coheres with $\vec{E}$,

$$J_\alpha^{\vec{E}} \models \text{“}V_\nu \in \text{Ult}(V, E_\gamma \restriction \nu)\text{”} .$$

Finally, suppose $i_{\xi b}(\bar{f}) = f$. Then $\bar{f} \restriction \kappa = f \restriction \kappa$, and

$$i_{\xi,\gamma+1}(\bar{f}) \restriction \nu = f \restriction \nu$$

so

$$i_{\xi,\gamma+1}(f \restriction \kappa)(\kappa) < \nu .$$

But

$$i_{\xi,\gamma+1}(f \restriction \kappa) \restriction \nu = i_{E_\gamma \restriction \nu}(f \restriction \kappa) \restriction \nu$$

as computed in $J_\alpha^{\vec{E}}$. Thus $E_\gamma \restriction \nu$ witnesses that δ is Woodin with respect to f in $J_\alpha^{\vec{E}}$. □

For the purpose of comparison we are only interested in iteration trees in which each E_α is applied to the earliest model to which it can be.

DEFINITION 6.1.1. $\mathcal{T} = \langle T, \deg, D, \langle E_\alpha, \mathcal{M}^*_{\alpha+1} \mid \alpha+1 < \theta\rangle\rangle$ is *non-overlapping* iff whenever T-pred$(\gamma+1) = \beta$, then $\rho_\eta \le \text{crit } E_\gamma$ for all $\eta < \beta$.

Here ρ_η is the sup of the generators for E_η, so that crit $E_\gamma < \rho_\beta$. Clearly, generators are not moved along the branches of a nonoverlapping tree, and in fact not moving generators is equivalent to being non-overlapping.

We want also to restrict ourselves to trees in which $\mathcal{M}^*_{\gamma+1}$ and $\deg(\gamma+1)$ are as large as possible, subject perhaps to an n-boundedness requirement.

DEFINITION 6.1.2. Let $\mathcal{T} = \langle T, \deg, D, \langle E_\alpha, \mathcal{M}^*_{\alpha+1} \mid \alpha+1 < \theta\rangle\rangle$ be an iteration tree, and $n \le \omega$. We say $\mathcal{T}$ is *n-maximal* iff $\mathcal{T}$ is non-overlapping, and whenever T-pred$(\gamma+1) = \beta$, $E_\gamma = \dot{F}^{\mathcal{N}}$ where $\mathcal{N}$ is an initial segment of $\mathcal{M}_\gamma$, and $\kappa =$ crit E_γ, then

(a) $\mathcal{M}^*_{\gamma+1}$ is the longest initial segment $\mathcal{P}$ of $\mathcal{M}_\beta$ such that $P(\kappa) \cap |\mathcal{P}| = P(\kappa) \cap |\mathcal{N}|$, and
(b) if $D \cap [0, \gamma+1]_T = \varnothing$ then $\deg(\gamma+1)$ is the largest integer $k \le n$ such that $\kappa < \rho_k^{\mathcal{M}^*_{\gamma+1}}$, and
(c) if $D \cap [0, \gamma+1]_T \neq \varnothing$, then $\deg(\gamma+1)$ is the largest $k \in \omega$ such that $\kappa < \rho_k^{\mathcal{M}^*_{\gamma+1}}$.

Notice that in (a) of the definition $\mathcal{P}$ is the longest initial segment Q of $\mathcal{M}_\beta$ such that

$$P(\kappa) \cap J^{\mathcal{M}_\beta}_{\text{lh } E_\beta} = P(\kappa) \cap Q\,.$$

Since $J^{\mathcal{M}_\beta}_{\text{lh } E_\beta} = J^{\mathcal{M}_\gamma}_{\text{lh } E_\beta}$ it follows that if $\beta \neq \gamma$ then $\mathcal{P}$ is the longest initial segment Q of $\mathcal{M}_\beta$ such that $P(\kappa) \cap Q = P(\kappa) \cap |\mathcal{M}_\gamma|$.

The iteration trees for which we have any practical use are all n-maximal for some $n \le \omega$. One important elementary property of such trees is the following.

Lemma 6.1.5. *Let $\mathcal{T} = \langle T, \deg, D, \langle E_\alpha, \mathcal{M}^*_{\alpha+1} \mid \alpha+1 < \theta\rangle\rangle$ be an n-maximal iteration tree, where $n \le \omega$; then for any $\alpha+1 < \theta$, E_α is close to $\mathcal{M}^*_{\alpha+1}$.*

PROOF. By induction on α. Let $\beta = T$-pred$(\alpha+1)$. We may assume $\beta < \alpha$; otherwise E_α is on the $\mathcal{M}_\beta$ sequence, and so by the restrictions on how far $\mathcal{M}^*_{\alpha+1}$ can drop in $\mathcal{M}_\beta$, on the $\mathcal{M}^*_{\alpha+1}$ sequence. Thus E_α is close indeed to $\mathcal{M}^*_{\alpha+1}$.

Let $a \subseteq \text{lh } E_\alpha$ be finite. We wish to verify the two conditions in closeness to $\mathcal{M}^*_{\alpha+1}$ for $(E_\alpha)_a$. We begin with the second.

Let $\kappa =$ crit E_α and $\tau = \text{lh } E_\beta$. As $\beta = T$-pred$(\alpha+1)$, $\kappa < \tau$, and as τ is a cardinal of $\mathcal{M}_\alpha$, $(\kappa^+)^{\mathcal{M}_\alpha} \le \tau$. Let $A \subseteq P([\kappa]^{\text{card } a})$, $A \in |\mathcal{M}^*_{\alpha+1}|$, be such that $\mathcal{M}^*_{\alpha+1} \models \text{card}(A) \le \kappa$. We want to see that $(E_\alpha)_a \cap A \in |\mathcal{M}^*_{\alpha+1}|$. Now $P(\kappa) \cap |\mathcal{M}_\alpha| = P(\kappa) \cap |\mathcal{M}^*_{\alpha+1}|$, so A has cardinality $\le \kappa$ in $\mathcal{M}_\alpha$. But then $(E_\alpha)_a \cap A$ is in $\mathcal{M}_\alpha$ and has cardinality $\le \kappa$ there, by weak amenability. But then $(E_\alpha)_a \cap A \in |\mathcal{M}^*_{\alpha+1}|$, as desired.

It remains to show $(E_\alpha)_a$ is Σ_1 over $\mathcal{M}^*_{\alpha+1}$. The following claim is useful; notice that $\mathcal{J}_\tau^{\mathcal{M}_\beta}$ is an initial segment of $\mathcal{M}^*_{\alpha+1}$.

CLAIM 1. If $A \subseteq \tau$ and $A \in |\mathcal{M}_\gamma|$ for some $\gamma > \beta$, then A is Σ_1 over $\mathcal{J}_\tau^{\mathcal{M}_\beta}$.

PROOF. By 5.1, $A \in |\mathcal{M}_{\beta+1}|$. Let $A = [a, f]^Q_{E_\beta}$, where $Q = \mathcal{M}^*_{\alpha+1}$. Since $A \subseteq \tau$, we can take f to map $[\mu]^{\text{card}\, a}$ into J^Q_μ, where $\mu = \text{crit } E_\beta$. We can therefore assume $f \in |Q|$, as $\mu < \rho^Q_m$ where $\mathcal{M}_{\beta+1} = \text{Ult}_m(Q, E_\beta)$. But also, $\mathcal{M}_\beta$ agrees with Q below τ, and $f \in J_\tau^{\mathcal{M}_\beta} = \mathcal{P}$. Moreover, $A = [a, f]^Q_{E_\beta} = [a, f]^{\mathcal{P}}_{E_\beta}$. It is easy, then, to define A in a Σ_1 way over $\mathcal{P}$ from the parameters a and f. □

It follows that if $(E_\alpha)_a \in |\mathcal{M}_\alpha|$, then since $(E_\alpha)_a$ is coded by a subset of τ, $(E_\alpha)_a$ is Σ_1 over $\mathcal{J}_\tau^{\mathcal{M}_\beta}$, hence Σ_1 over $\mathcal{M}^*_{\alpha+1}$, as required. Thus we may assume that $(E_\alpha)_a \notin |\mathcal{M}_\alpha|$, and hence E_α is on the $\mathcal{M}_\alpha$ sequence, $\mathcal{M}_\alpha$ is active and $E_\alpha = \dot{F}^{\mathcal{M}_\alpha}$.

CLAIM 2. Let $\gamma \in [0, \alpha]_T$ be such that $\gamma \geq \beta$ and $D \cap (\gamma, \alpha]_T = \varnothing$. Then $\text{crit}(i_{\gamma\alpha}) > \kappa$, and $(E_\alpha)_a$ is Σ_1 over $\mathcal{M}_\gamma$. If, in addition, $\gamma > \beta$ and γ is a successor ordinal, then $\text{crit}(i_{\gamma,\alpha} \circ i^*_\gamma) > \kappa$ and $(E_\alpha)_a$ is Σ_1 over $\mathcal{M}^*_\gamma$.

PROOF. Since $\kappa = \text{crit } E_\alpha$ and $E_\alpha = \dot{F}^{\mathcal{M}_\alpha}$, $\kappa \in \text{ran } i_{\gamma\alpha}$. On the other hand, every extender used in $i_{\gamma\alpha}$ has length at least $\text{lh}\, E_\beta$, since $\gamma \geq \beta$. It follows that $\kappa < \text{crit}(i_{\gamma\alpha})$.

By our induction hypothesis, E_η is close to $\mathcal{M}^*_{\eta+1}$ for all $\eta < \alpha$. Thus the preservation facts recorded in 4.5, 4.6, and 4.7 hold for the embeddings of $\mathcal{T} \restriction (\alpha + 1)$. Now $\rho_1^{\mathcal{M}_\alpha} \leq \tau = (\kappa^+)^{\mathcal{M}_\alpha}$ since $(E_\alpha)_a \notin |\mathcal{M}_\alpha|$, and $\tau \leq \text{crit } i_{\gamma\alpha}$, so $\deg(\eta) = 0$ for all $\eta \in (\gamma, \alpha]_T$. The proofs of 4.5 and 4.7 (see especially 4.5) show that every $\Sigma_1^{\mathcal{M}_\alpha}$ subset of $\text{crit}(i_{\gamma\alpha})$ is $\Sigma_1^{\mathcal{M}_\gamma}$. Thus $(E_\alpha)_a$ is $\Sigma_1^{\mathcal{M}_\gamma}$, as desired.

Suppose finally that $\gamma > \beta$ and γ is a successor ordinal. The extenders used in $i_{\gamma\alpha} \circ i^*_\gamma$ are just those used in $i_{\gamma\alpha}$ together with $E_{\gamma-1}$. Since $\gamma - 1 \geq \beta$, all these have length at least $\text{lh}\, E_\beta$, hence $> \kappa$. The argument of the previous paragraph now shows $\text{crit}(i_{\gamma\alpha} \circ i^*_\gamma) > \kappa$ and $(E_\alpha)_a$ is Σ_1 over $\mathcal{M}^*_\gamma$. □

Now let $\eta \in [0, \alpha]_T$ be least such that $\beta \leq \eta$. Suppose first that $D \cap (\eta, \alpha]_T \neq \varnothing$. Let γ be largest in $D \cap (\eta, \alpha]_T$, and $\xi = T\text{-pred}(\gamma)$. Since $\gamma > \beta$, Claim 2 implies that $(E_\alpha)_a$ is Σ_1 over $\mathcal{M}^*_\gamma$. Since $\gamma \in D$, $\mathcal{M}^*_\gamma \in |\mathcal{M}_\xi|$, so $(E_\alpha)_a \in |\mathcal{M}_\xi|$. Since $\xi \geq \beta$, Claim 1 implies that $(E_\alpha)_a$ is Σ_1 over $\mathcal{M}^*_{\alpha+1}$, as desired.

So we may assume $D \cap (\eta, \alpha]_T = \varnothing$. We claim that $\eta = \beta$. For if $\eta > \beta$, then the leastness of η implies that η is not a limit, so let $\delta = T\text{-pred}(\eta)$. Since η is least, $\delta < \beta$. By Claim 2 with $\gamma = \eta$, $\text{crit}(i^*_\eta) = \text{crit}(E_{\eta-1}) > \kappa$. But $\text{crit}(E_{\eta-1}) < \rho_\delta$, so $\kappa < \rho_\delta$. But the rules for non-overlapping trees then require that $T\text{-pred}(\alpha + 1) \leq \delta$, a contradiction.

So $\eta = \beta$. Also, by Claim 2, $\text{crit } i_{\beta\alpha} > \kappa$, and $(E_\alpha)_a$ is Σ_1 over $\mathcal{M}_\beta$. But then

$P(\kappa) \cap |\mathcal{M}_\beta| = P(\kappa) \cap |\mathcal{M}_\alpha|$, and since $\mathcal{T}$ is n-maximal, $\mathcal{M}_\beta = \mathcal{M}^*_{\alpha+1}$. Thus $(E_\alpha)_a$ is Σ_1 over $\mathcal{M}^*_{\alpha+1}$, as desired. □

Lemma 6.1.5 has the important consequence that the preservation facts listed in 4.5, 4.6, and 4.7 apply to the embeddings along the branches of an n-maximal tree. We shall use this repeatedly and without explicit mention in the future.

The following is a crucial strengthening of the uniqueness theorem (6.1). It will imply that only simple iteration trees arise in our proof that 1-small, k-iterable premice are k-solid for all k. This is important because our proof of that fact uses heavily the Dodd-Jensen lemma, which requires a simplicity hypothesis.

If $\mathcal{M}$ is a ppm, an "extender from the $\mathcal{M}$-sequence" is an extender E such that $E = \dot{F}^{\mathcal{M}}$ or E is on the sequence $\dot{E}^{\mathcal{M}}$.

Theorem 6.2 (Strong uniqueness). *Let $\mathcal{M}$ be an n-sound, 1-small n-iterable premouse and $\rho^{\mathcal{M}}_{n+1} \leq \operatorname{lh} E$ for some extender E from the $\mathcal{M}$-sequence and some integer n. Let $\mathcal{T}$ be an n-maximal iteration tree on $\mathcal{M}$. Then $\mathcal{T}$ is simple.*

PROOF. Assume toward a contradiction that b and c are distinct cofinal well-founded branches of $\mathcal{T}$ with $\mathrm{OR}^{\mathcal{M}_b} \leq \mathrm{OR}^{\mathcal{M}_c}$. Let $\delta = \delta(\mathcal{T})$.

CLAIM 1. $\operatorname{lh} F < \delta$ for all extenders F from the $\mathcal{M}_b$ sequence.

PROOF. Let F be the first extender on the $\mathcal{M}_b$ sequence such that $\operatorname{lh} F \geq \delta$. Notice δ is a limit of $\mathcal{M}_b$ cardinals, as crit $i_{\alpha b}$ is an $\mathcal{M}_b$ cardinal whenever $i_{\alpha b}$ is defined. Thus $\operatorname{lh} F > \delta$, as $\exists \nu < \operatorname{lh} F \, \forall \gamma < \operatorname{lh} F \, (\mathcal{M}_b \models \operatorname{card} \gamma \leq \nu)$. Let $\gamma = \operatorname{lh} F$. By Theorem 6.1,

$$J^{\vec{E}(\mathcal{T})}_\gamma \models \delta \text{ is Woodin}$$

so

$$\mathcal{J}^{\mathcal{M}_b}_\gamma = (J^{\vec{E}(\mathcal{T})}_\gamma, \in, \vec{E}(\mathcal{T}), \tilde{F}) \models \delta \text{ is Woodin}.$$

Now let $\mathcal{N} = \mathrm{Ult}_0(\mathcal{J}^{\mathcal{M}_b}_\gamma, F)$. As F is a pre-extender over $\mathcal{J}^{\mathcal{M}_b}_\gamma$, $\gamma \in \mathrm{wfp}(\mathcal{N})$. By coherence and strong acceptability and the fact that γ is a cardinal of $\mathcal{N}$,

$$\mathcal{N} \models \delta \quad \text{is Woodin.}$$

But then $\mathcal{N}$ is not 1-small, so that $\mathcal{M}_b$ is not 1-small and hence $\mathcal{M}$ is not 1-small, which is a contradiction. □

CLAIM 2. $\mathcal{M}_b$ is an initial segment of $\mathcal{M}_c$.

PROOF. Otherwise $\mathcal{M}_c$ is not 1-small. For let F be the first extender from the $\mathcal{M}_c$ sequence with $\operatorname{lh} F \geq \delta$; if none exists Claim 2 is obvious from Lemma 5.1. So $\operatorname{lh} F > \delta$ as in Claim 1. If $\mathcal{M}_b$ is not an initial segment of $\mathcal{M}_c$, $\operatorname{lh} F \leq \mathrm{OR}^{\mathcal{M}_b}$. But now we can show $\mathcal{M}_c$ is not 1-small as in Claim 1. □

CLAIM 3. If $\mathrm{OR}^{\mathcal{M}_b} < \mathrm{OR}^{\mathcal{M}_c}$, then there is no dropping of any kind along b; that is, $D^{\mathcal{T}} \cap b = \varnothing$ and $\deg^{\mathcal{T}}(\alpha+1) = n$ for all $\alpha + 1 \in b$.

PROOF. If $\mathrm{OR}^{\mathcal{M}_b} < \mathrm{OR}^{\mathcal{M}_c}$, then $\mathcal{M}_b$ is a proper initial segment of $\mathcal{M}_c$, and hence $\mathcal{M}_b$ is ω-sound since $\mathcal{M}_c$ is a premouse. But now suppose the last drop of any kind along b occurs at $\alpha + 1$. Then $\alpha + 1 \in b$, and $k = \deg(\alpha+1) = \deg(\gamma)$ for all $\gamma \in b - (\alpha + 1)$. Also, $\mathcal{M}^*_{\alpha+1}$ is $k+1$ sound and $\mathrm{crit}(i_{\alpha+1,b} \circ i^*_{\alpha+1}) = \mathrm{crit}(i^*_{\alpha+1}) \geq \rho_{k+1}^{\mathcal{M}^*_{\alpha+1}}$. From Lemma 4.7 it follows that $\mathcal{M}_b$ is not $k+1$-sound, a contradiction. □

CLAIM 4. If $\mathrm{OR}^{\mathcal{M}_b} = \mathrm{OR}^{\mathcal{M}_c}$, then on one of b and c there's no dropping of any kind.

PROOF. Suppose the last drop along b occurs at $\eta + 1$, and the last drop along c at $\gamma + 1$. Since $\mathcal{M}_b = \mathcal{M}_c$, $\deg(\eta+1) = \deg(\gamma+1) = k$, where $k < \omega$ is least such that $\mathcal{M}_b = \mathcal{M}_c$ is not $k+1$-sound. But then

$$\mathcal{M}^*_{\eta+1} = \mathfrak{C}_{k+1}(\mathcal{M}_b) = \mathfrak{C}_{k+1}(\mathcal{M}_c) = \mathcal{M}^*_{\gamma+1} .$$

This implies that T-pred$(\eta+1) = T$-pred$(\gamma+1)$. For let $\beta = T$-pred$(\eta+1)$; then E_β is on the $\mathcal{M}^*_{\eta+1}$ sequence, so E_β is on the $\mathcal{M}^*_{\gamma+1}$ sequence, so E_β is on the $\mathcal{M}_\xi$-sequence where $\xi = T$-pred$(\gamma+1)$. Thus $\xi \leq \beta$ by remark (a) following 5.1. That $\beta \leq \xi$ is proved symmetrically.

Now then

$$i_{\eta+1,b} \circ i^*_{\eta+1} = i_{\gamma+1,c} \circ i^*_{\gamma+1} ,$$

since by lemma 4.7 each side is the natural embedding from $\mathfrak{C}_{k+1}(\mathcal{M}_b)$ to $\mathfrak{C}_k(\mathcal{M}_b) = \mathcal{M}_b$ inverting the collapse.

Since $\mathcal{T}$ is non-overlapping, crit $i_{\eta+1,b} \geq \rho_\eta$ and crit $i_{\gamma+1,b} \geq \rho_\gamma$. So letting $\nu = \inf(\rho_\eta, \rho_\gamma)$, we have crit $E_\eta =$ crit $E_\gamma < \nu$ and $E_\eta \restriction \nu = E_\gamma \restriction \nu$. By remark (a) following 5.1 we see that $\eta = \gamma$.

Now let β be largest in $b \cap c$; from the above we know that there's no dropping after β on b or c, that is, $\eta + 1 = \gamma + 1 \in b \cap c$. Let

$$\rho = \sup\{\mathrm{lh}\, E_\xi \mid \xi + 1 \in b \cap c\} ;$$

then

$$\mathcal{M}_\beta = \mathcal{H}_{k+1}^{\mathcal{M}_\beta}(\rho \cup \{q_\beta\})$$

where for any $\xi \in b \cup c$ such that $\xi \geq \eta + 1$

$$q_\xi = i_{\eta+1,\xi} \circ i^*_{\eta+1}(p_{k+1}(\mathcal{M}^*_{\eta+1})) .$$

But then

$$i_{\beta,b} = i_{\beta,c} ,$$

as $i_{\beta,b} \restriction \rho = i_{\beta,c} \restriction \rho = \mathrm{id}$, and $i_{\beta,b}(q_\beta) = i_{\beta,c}(q_\beta) = \langle r, u\rangle$, where r is the $k+1$st standard parameter of $(\mathcal{M}_b, u)$ and u is as in the definition of $p_{k+1}(\mathcal{M}_b)$ (cf. Lemma 4.7). Let $\sigma+1 \in b$, $\tau+1 \in c$, and $T\text{-pred}(\sigma+1) = T\text{-pred}(\tau+1) = \beta$. As $i_{\beta,c} = i_{\beta,b}$, we see that crit E_σ = crit E_τ, and $E_\sigma \restriction \nu = E_\tau \restriction \nu$, where $\nu = \inf(\rho_\sigma, \rho_\tau)$. This implies $\sigma = \tau$, a contradiction. □

In view of Claims 3 and 4, we may assume there's no dropping of any kind along b (perhaps by exchanging b for c). The proof of the following claim will take several pages and will nearly finish the proof of theorem 6.2.

CLAIM 5. $\rho_{n+1}^{\mathcal{M}_b} < \delta$.

PROOF. We show by induction on $\eta \in b$, that if $\alpha T \eta$, or if $\eta = b$ and $\alpha \in b$, then

$$(*) \qquad \rho_{n+1}^{\mathcal{M}_\eta} \leq i_{\alpha,\eta}(\rho_{n+1}^{\mathcal{M}_\alpha}),$$

and

$$(**) \quad \text{If } \rho_{n+1}^{\mathcal{M}_\eta} = i_{\alpha\eta}(\rho_{n+1}^{\mathcal{M}_\alpha}) \text{ and } \mathrm{Th}_{n+1}^{\mathcal{M}_\alpha}(\rho_{n+1}^{\mathcal{M}_\alpha} \cup \{q\} \notin \mathcal{M}_\alpha)$$
$$\text{then } \mathrm{Th}_{n+1}^{\mathcal{M}_\eta}(\rho_{n+1}^{\mathcal{M}_\eta} \cup \{i_{\alpha\eta}(q)\}) \notin \mathcal{M}_\eta.$$

By (*) for $\eta = b$ and $\alpha = 0$ we have $\rho_{n+1}^{\mathcal{M}_b} \leq i_{0b}(\rho_{n+1}^{\mathcal{M}_0}) \leq \mathrm{lh}\, E$ for some extender E from the $\mathcal{M}_b$ sequence, so that $\rho_{n+1}^{\mathcal{M}_b} < \delta$, as desired.

Consider first the case η is a limit or $\eta = b$. Let $\alpha T \eta$ be the least ordinal such that $i_{\alpha\gamma}(\rho_{n+1}^{\mathcal{M}_\alpha}) = \rho_{n+1}^{\mathcal{M}_\gamma}$ whenever $\alpha T \gamma T \eta$. Such an ordinal α exists by (*). It will be enough to show that whenever $\gamma \in [\alpha, \eta)_T$ and $\mathrm{Th}_{n+1}^{\mathcal{M}_\gamma}(\rho_{n+1}^{\mathcal{M}_\gamma} \cup \{q\})$ is not a member of $\mathcal{M}_\gamma$, then

$$\mathrm{Th}_{n+1}^{\mathcal{M}_\eta}(i_{\gamma\eta}(\rho_{n+1}^{\mathcal{M}_\gamma}) \cup \{i_{\gamma\eta}(q)\}) \notin |\mathcal{M}_\eta|.$$

For this, suppose $\mathrm{Th}_{n+1}^{\mathcal{M}_\eta}(i_{\gamma\eta}(\rho_{n+1}^{\mathcal{M}_\gamma}) \cup \{i_{\gamma\eta}(q)\}) = i_{\xi\eta}(x)$, where we may assume $\gamma T \xi T \eta$. As $i_{\xi\eta}$ is generalized $r\Sigma_{n+1}$ elementary, we see $x = \mathrm{Th}_{n+1}^{\mathcal{M}_\xi}(i_{\gamma\xi}(\rho_{n+1}^{\mathcal{M}_\gamma} \cup \{i_{\gamma\xi}(q)\})$. This contradicts (**) at ξ.

Now let $\eta = \xi + 1$ and set $\beta = T\text{-pred}(\eta)$. If (*) or (**) fails at η we must have $q \in |\mathcal{M}_\beta|$ such that

$$\mathrm{Th}_{n+1}^{\mathcal{M}_\beta}(\rho_{n+1}^{\mathcal{M}_\beta} \cup \{q\}) \notin |\mathcal{M}_\beta|$$

but

$$\mathrm{Th}_{n+1}^{\mathcal{M}_\eta}(i_{\beta\eta}(\rho_{n+1}^{\mathcal{M}_\beta}) \cup \{i_{\beta\eta}(q)\}) = [a, f]_{E_\xi}^{\mathcal{M}_\beta} \in |\mathcal{M}_\eta|.$$

Fix such a q. Let $\rho = \rho_{n+1}^{\mathcal{M}_\beta}$, $i = i_{\beta\eta}$, $E = E_\xi$.

We may assume $f(\bar{u}) \subseteq \rho$ for all $\bar{u} \in \mathrm{dom}\, f$. Also $\rho < \rho_n^{\mathcal{M}_\beta}$ by (*) and the fact that $\rho_{n+1}^{\mathcal{M}_0} < \rho_n^{\mathcal{M}_0}$. If we let $A = \{(\bar{u}, \nu) \mid \nu \in f(\bar{u})\}$, then A is (generalized) $r\Sigma_n$, so $A \in |\mathcal{M}_\beta|$. Thus $f \in |\mathcal{M}_\beta|$.

Now

$$(\dagger) \qquad x \in \mathrm{Th}_{n+1}^{\mathcal{M}_\beta}(\rho \cup \{q\}) \Leftrightarrow i(x) \in [a, f]_E^{\mathcal{M}_\beta}$$

since i is generalized $r\Sigma_{n+1}$ elementary. This gives an $r\Delta_1^{\mathcal{M}_\beta}$ definition of $\mathrm{Th}_{n+1}^{\mathcal{M}_\beta}(\rho \cup \{q\})$ since E_a is $r\Sigma_1^{\mathcal{M}_\beta}$. This is a contradiction if $n > 0$, so we now assume $n = 0$.

Let $\kappa = \mathrm{crit}\ E$. We have $\kappa < \rho$ by Lemma 4.5. On the other hand, $E_a \notin |\mathcal{M}_\beta|$, as otherwise ($\dagger$) would imply $\mathrm{Th}_1^{\mathcal{M}_\beta}(\rho \cup \{q\}) \in |\mathcal{M}_\beta|$. Thus $\rho = \rho_1^{\mathcal{M}_\beta} = (\kappa^+)^{\mathcal{M}_\beta}$.

We will now complete the proof of claim 5 by showing that there is a $r\Sigma_1^{\mathcal{M}_\beta}$ function $t : \kappa \to \rho$ such that $\mathrm{ran}(t)$ is cofinal in ρ. To see that this proves claim 5, we let S be the set of triples (α, γ, ν) such that $\gamma \prec_{t(\alpha)} \nu$, where $\prec_{t(\alpha)}$ is the first well ordering of κ in the natural order of $\mathcal{M}_\beta$ which has order type $t(\alpha)$. Then $S \subseteq \kappa$ and S is $r\Sigma_1^{\mathcal{M}_\beta}$, so that $S \in |\mathcal{M}_\beta|$ and hence $\rho < (\kappa^+)^{\mathcal{M}_\beta}$, contradiction.

For any $\mathcal{N}$ and $X \subseteq |\mathcal{N}|$, let

$$\overline{\mathrm{Th}}_1^{\mathcal{N}}(X) = \mathrm{Th}_1^{\mathcal{N}}(X) \cap \{(\varphi, \bar{a}) \mid \varphi \text{ is pure } r\Sigma_1\}.$$

Using the proof of Lemma 2.10 we see that $\mathrm{Th}_1^{\mathcal{M}_\beta}(\rho \cup \{q\}) \notin |\mathcal{M}_\beta|$ implies that $\overline{\mathrm{Th}}_1^{\mathcal{M}_\beta}(\rho \cup \{q\}) \notin |\mathcal{M}_\beta|$, so we can use $\overline{\mathrm{Th}}_1^{\mathcal{M}_\beta}(\rho \cup \{q\})$ instead of $\mathrm{Th}_1^{\mathcal{M}_\beta}(\rho \cup \{q\})$. Let f be the function representing $\overline{\mathrm{Th}}_1^{\mathcal{M}_\eta}(i(\rho) \cup \{i(q)\})$. We need to consider two cases:

Case 1. There is a total, continuous, order-preserving, $r\Sigma_1^{\mathcal{M}_\beta}$ function $g : \kappa \to \mathrm{OR}^{\mathcal{M}_\beta}$ such that $g''\kappa$ is cofinal in $\mathrm{OR}^{\mathcal{M}_\beta}$.

In this case, we set for $\bar{u} \in \mathrm{dom}(f)$

$$h(\bar{u}) = \overline{\mathrm{Th}}_1^{J^{\mathcal{M}_\beta}_{g(u_0)}}(\rho \cup \{q\}),$$

so that h is $r\Sigma_1^{\mathcal{M}_\beta}$. Notice that if $A \in E_a$, then $\exists \bar{u} \in A\ h(\bar{u}) \neq f(\bar{u})$, as otherwise $h \restriction A \in |\mathcal{M}_\beta|$, so that $\overline{\mathrm{Th}}_1^{\mathcal{M}_\beta}(\rho \cup \{q\}) \in |\mathcal{M}_\beta|$, a contradiction.

Now set, for all $\bar{u} \in \mathrm{dom}(f)$

$$t(\bar{u}) = \begin{cases} \text{least } \alpha & \text{such that } (f(\bar{u}) \triangle h(\bar{u})) \cap \ (\omega \times (\alpha \cup \{q\})^{<\omega}) \neq \emptyset \\ 0 & \text{if no such } \alpha \text{ exists}. \end{cases}$$

So t is total and $r\Sigma_1^{\mathcal{M}_\beta}$. It is enough to see $\mathrm{ran}(t)$ is unbounded in ρ. Fix any ordinal $\theta < \rho$. We will complete the proof of case 1 by finding a $\bar{u}$ such that $t(\bar{u}) > \theta$. Define a function k by

$$k(\bar{v}) = h(\bar{v}) \cap (\omega \times (\theta \cup \{q\})^{<\omega}).$$

Then $k \in |\mathcal{M}_\beta|$ since it can be computed from $\mathrm{Th}_1^{\mathcal{M}_\beta}(\theta \cup \{q, r\})$, where r is a parameter chosen so that the function g is $\Sigma_1^{\mathcal{M}_\beta}(\{r\})$. Moreover

$$(\dagger\dagger) \qquad [a, k]_E^{\mathcal{M}_\beta} = \overline{\mathrm{Th}}_1^{\mathcal{M}_\eta}(i(\theta) \cup \{i(q)\}).$$

One direction, $\supseteq$, of equation $(\dagger\dagger)$ is easy. To prove $\subseteq$, let $[b, \mathcal{I}]_E^{\mathcal{M}_\beta} \in [a, k]_E^{\mathcal{M}_\beta}$, where we may assume $a \subseteq b$. We may assume that for all $\bar{v} \in \operatorname{dom} \mathcal{I}$

$$\mathcal{I}(\bar{v}) \in k(\bar{v}^*) = \overline{\mathrm{Th}}_1^{J_{g(v_0)}^{\mathcal{M}_\beta}}(\theta \cup \{q\})$$

where $\bar{v}^*$ is the appropriate subsequence of $\bar{v}$. For $\bar{v} \in \operatorname{dom} \mathcal{I}$ such that v_0 is a limit, let

$$s(\bar{v}) = \text{least } \alpha < v_0 \text{ such that } \mathcal{I}(\bar{v}) \in \overline{\mathrm{Th}}_1^{J_{g(\alpha)}^{\mathcal{M}_\beta}}(\theta \cup \{q\}).$$

Then s is a $r\Sigma_1^{\mathcal{M}_\beta}$ map from κ^n to κ, so $s \in |\mathcal{M}_\beta|$. By normality, fix α_0 such that $s(\bar{v}) = \alpha_0$ for E_b a.e. $\bar{v}$, and let $\xi = g(\alpha_0)$. Then

$$\begin{aligned} [a, \mathcal{I}]_E^{\mathcal{M}_\beta} &\in i(\overline{\mathrm{Th}}_1^{J_\xi^{\mathcal{M}_\beta}}(\theta \cup \{q\})) \\ &= \overline{\mathrm{Th}}_1^{J_{i(\xi)}^{\mathcal{M}_\eta}}(i(\theta) \cup \{i(q)\}) \subseteq \overline{\mathrm{Th}}_1^{\mathcal{M}_\eta}(i(\theta) \cup \{i(q)\}), \end{aligned}$$

as desired. This completes the proof of equation $(\dagger\dagger)$.

It follows that there is an $A \in E_a$ such that for all $\bar{u} \in A$,

$$f(\bar{u}) \cap (\omega \times (\theta \cup \{q\})^{<\omega}) = h(\bar{u}) \cap (\omega \times (\theta \cup \{q\})^{<\omega}).$$

Let $\bar{u} \in A$ be such that $h(\bar{u}) \neq f(\bar{u})$; then $t(\bar{u}) > \theta$. This completes the proof of case 1 of claim 5.

Case 2. There is no function g as in case 1.

In this case, define the function $t(\bar{u})$, where $\bar{u} \in \operatorname{dom}(f)$, by

$$t(\bar{u}) = \text{least } \alpha \text{ such that } \big(f(\bar{u}) \triangle \overline{\mathrm{Th}}_1^{\mathcal{M}_\beta}(\rho \cup \{q\})\big) \cap (\omega \times (\alpha \cup \{q\})^{<\omega}) \neq \varnothing.$$

Thus t is total $r\Sigma_1^{\mathcal{M}_\beta}$. To see that $\operatorname{ran} t$ is unbounded in ρ, note that for $\theta < \rho$

$$\overline{\mathrm{Th}}_1^{\mathcal{M}_\eta}(i(\theta) \cup \{i(q)\}) = i(\overline{\mathrm{Th}}_1^{\mathcal{M}_\beta}(\theta \cup \{q\}))$$

as

$$\overline{\mathrm{Th}}_1^{\mathcal{M}_\beta}(\theta \cup \{q\}) = \overline{\mathrm{Th}}_1^{J_\xi^{\mathcal{M}_\beta}}(\theta \cup \{q\})$$

for some $\xi < \mathrm{OR}^{\mathcal{M}_\beta}$ by case hypothesis.

This completes the proof of case 2, and hence of Claim 5. □

Fix now $p \in |\mathcal{M}_\beta|$ and $\rho < \delta$ such that $\mathrm{Th}_{n+1}^{\mathcal{M}_b}(\rho \cup \{p\}) \notin |\mathcal{M}_b|$. We obtain a contradiction via an easy generalization of the proof of 6.1.

Fix $\beta <$ length of $\mathcal{T}$ so large that

(1) $b \cap \beta \neq c \cap \beta$, and there's no dropping on $b \cup c$ above β.

(2) $\gamma \in b - \beta \Rightarrow$ crit $i_{\gamma b} > \rho$ and $p \in$ ran $i_{\gamma b}$ and ($\delta < \mathrm{OR}^{\mathcal{M}_b} \Rightarrow \delta \in$ ran $i_{\gamma b}$).

(3) $\gamma \in c - \beta \Rightarrow$ crit $i_{\gamma c} > \rho$ and $p \in$ ran $i_{\gamma c}$ and ($\delta < \mathrm{OR}^{\mathcal{M}_b} \Rightarrow \delta \in$ ran $i_{\gamma c}$) and ($\mathrm{OR}^{\mathcal{M}_b} < \mathrm{OR}^{\mathcal{M}_c} \Rightarrow \mathrm{OR}^{\mathcal{M}_b} \in$ ran $i_{\gamma c}$).

As in Claim 2 of the proof of 6.1, we can find $\gamma \in b - \beta$ and $\eta \in c - \beta$ such that

$$\mathrm{ran}\ i_{\gamma b} \cap \mathrm{ran}\ i_{\eta c} \cap \delta = \kappa$$

where $\rho < \kappa < \delta$. Let

$$\pi : |\mathcal{N}| \cong X \subseteq |\mathcal{M}_b|$$

where $X =$ ran $i_{\gamma b} \cap$ ran $i_{\eta c}$ and π is the inverse of the collapse. Then π is generalized $r\Sigma_{n+1}$ elementary. This follows from the fact that both $i_{\gamma b}$ and $i_{\gamma c}$ are generalized $r\Sigma_{n+1}$ elementary. To see that $i_{\gamma c}$ is generalized $r\Sigma_{n+1}$ elementary, note that if $\mathcal{M}_b = \mathcal{M}_c$, then $\deg(\xi + 1) \geq n$ for all sufficiently large $\xi + 1 \in c$, so $i_{\eta c}$ is generalized $r\Sigma_{n+1}$ elementary. If $\mathcal{M}_b$ is a proper initial segment of $\mathcal{M}_c$, then $i_{\eta c} \restriction i_{\eta c}^{-1}(\mathcal{M}_b)$ is in fact fully elementary.

Notice that crit $\pi = \kappa$, and $\mathcal{N} = \mathcal{J}_\alpha^{\vec{E}(\mathcal{T})\restriction\kappa}$ for some $\alpha \geq \kappa$. Also $\mathrm{Th}_{n+1}^{\mathcal{M}_b}(\rho \cup \{p\})$ is definable over $\mathcal{N}$, and hence is a member of $L[\vec{E}(\mathcal{T}) \restriction \kappa]$. As $\vec{E}(\mathcal{T}) \restriction \kappa \in |\mathcal{M}_b|$ and $\mathcal{M}_b$ has an internally iterable extender on its sequence with critical point greater than κ, we get $\mathrm{Th}_{n+1}^{\mathcal{M}_b}(\rho \cup \{p\}) \in |\mathcal{M}_b|$, a contradiction. This completes the proof of theorem 6.2. □