# Kurt Gödel and the constructive Mathematics of A.A. Markov

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## 1.

I would like to dedicate this article to the memory of Dr. Oswald Demuth (12.9.1936 – 15.9.1988). Oswald was an excellent Mathematician and a dear friend of mine. I miss him so badly.

## 2.

The Russian School of constructive Mathematics was founded by A.A. Markov, Jr. (1903-1979) in the late 40-ies - early 50-ies [1]. In private conversations Markov used to state that he nurtured a type of constructive convictions for a very long time, long before the Second World War. This is an interesting fact if one considers that this was the time when Markov worked very actively in various areas of classical Mathematics and achieved first-rate results. Perhaps it is worth mentioning that Markov was a scientist with a very wide area of interest. In his freshman years he published works in Chemistry and he graduated from Leningrad University (1924) with a major in Physics. Besides Mathematics, he published works in theoretical Physics, Celestial Mechanics, Theory of Plasticity (cf., e.g., Markov and Nagorny [1988: introduction by Nagorny]; this monograph (originally in Russian, 1984) was completed and published by N.M. Nagorny after Markov's death). It is almost inevitable that a scientist of such universality arrives to philosophical and foundational issues. I believe that the explicitly "constructive" period of Markov's activities began with his work on Thue's Problem which had stood open since 1914. Thue's Problem was solved independently by A.A. Markov (Jr.) and E. Post in 1947. Markov began to develop his concept of so called *normal algorithms* as a tool to present his results on Thue Problem. Markov's publications on normal algorithms appeared as early as 1951 (Markov [1960; an English translation]). In 1954 Markov published his famous monograph

[1] This article is written for the Gödel '96 Proceedings. It was not and will not be published anywehere else in any form.

Theory of Algorithms (Markov [1961; an English Translation]). This monograph can be considered as probably the first systematic presentation of the general theory of algorithms together with related semiotical problems. In this monograph, in particular, one can find a mathematical theory of *words* as special types of sign complexes. As far as I know, such a theory was developed here for the first time. On the other hand, scrupulous proofs that such and such normal algorithm works on given words in a certain way can be considered as the first examples of program correctness verification. All in all, it seems that Markov's monograph is still underestimated by experts in Computer Science. At the same time Markov began to develop a mathematical worldview, and Mathematics in the framework of the above worldview, that was later to be known as "Markov's (or Russian) constructive Mathematics". In my opinion, Markov's constructive Mathematics (MCM; see, e.g.,Markov [1971], Kushner [1984], Kushner [1990], Kushner [1993 a,b])) is one of the three most important and coherent constructivist trends of our Century. The other two are Brouwer's Intuitionism (see, e.g., Heyting [1956], Troelstra [1977, 1990], Troelstra and van Dalen [1988], Kleene and Vesley [1965], Dragalin [1988], Beeson [1985]) and Bishop's constructive Mathematics (BCM) (see, e.g., Bishop [1967, 1970,1984], Bishop and Bridges [1985]) Chronologically, Markov stands between Brouwer and Bishop.

## 3.

Markov's constructive Mathematics (MCM) can be characterized by the following main features (cf., e.g., Markov [1971], Kushner [1984])

1. The objects of study are constructive processes and constructive objects arising as the results of these processes. The concept of constructive object is primitive. The main feature of constructive objects is that they are constructed according to definite rules from certain elementary objects, which are indecomposable in the process of these constructions. Hence we deal with objects of a completely combinatorial and finite nature. Practically, for developing MCM it is enough to consider a special type of constructive objects, namely words in one or another alphabet.

2. A special constructive logic is allowed to be used. This logic takes into account the specific nature of constructive objects and processes. In particular, tertium non datur principle and principle of double negation are not accepted as universal logical principles.

3. The abstraction of potential realizabality is accepted, but the abstraction of actual infinity is completely rejected.

4. The intuitive concept of effectiveness, computability etc is identified with one of the precise concept of algorithm (historically, with Markov's normal algorithms). This means that a version of Church's Thesis is accepted.

MCA differs, roughly speaking, with Intuitionism in features 1, 4 and with BCM, in feature 4. As is well-known, one of the principal achievements of Brouwer was his non-pointwise, Aristotelian-style theory of the real continuum. A very specific tool of choice sequences was developed and used to reach this goal. Choice sequences can be considered as developing, incomplete mathematical objects and their theory was, probably, the first sample of Mathematics of incomplete objects. In any case, they are not constructive objects in Markov's sense and therefore they are outside of MCM. On the other hand, Bishop refused to identify intuitive constructivity, effectiveness etc with ,say, recursiveness (see,e.g., Bishop [1967, 1984], Bishop and Bridges [1985]). It is worth noting that the main part of Brouwer's work on intuitionistic Mathematics was done before the concept of recursive function appeared on mathematical scene. Heyting once noted (Heyting [1962], van Dalen [1995]) that had it been the other way, Brouwer probably would not have introduced choice sequences and it would have been a pity. I can only agree with the last part of the statement - it would be a pity not to have today this marvelous concept. On the other hand, I am not sure about the strength of this "probably" above; in fact, Brouwer rejected the pointwise concept of continuum because of very deep philosophical reasons, hence, recursive functions would not have satisfied him in his task of creating a non-Cantorian theory of the continuum, anyway. The problem was not so much to grasp the volume of the intuitive notion of computability (that is all that recursive functions are about) but rather to present mathematically the Aristotelian idea of developing continuum. And this brings in mathematical objects that are incomplete in principle. It seems that Brouwer did not express in any written form his position with respect of Church's Thesis and, as far as I know, there no other evidences of his point of view in this respect. Nevertheless, it seems highly unlikely that Brouwer accepted Church's Thesis and, anyway, it was not used in the body of Intuitionistic Mathematics developed by him and his disciples. Be that as it may, both Brouwer and Bishop did not join the Church's Thesis Club, though Bishop considered the Thesis practically plausible. It is worth noting that Church's Thesis was taken for long time by the mathematical community almost for granted. In reality, this fundamental principle is not so evident, in particular philosophically. The attitude of such outstanding mathematical thinkers, as Brouwer and Bishop speaks for itself. And, as is known, Gödel was unconvinced by Church's Thesis (very impressive accounts of the early years of the theory of computability can be found in Feferman [1984] and Davis [1982] where the further bibliography can be found, as well) since Church failed to present conceptual analysis of the notion of finite algorithmic procedure. It was only after Turing's work with its analysis of the concept of mechanical procedure that Gödel was ready to accept the identification of intuitive and precise concepts of algorithm. E. Mendelson published recently an interesting work [Mendelson, 1990] on the subject. Along with Turing's conceptual analysis he considers the analysis of the nature of finitary processes that was undertaken by Kolmogorov and Uspensky (Kolmogorov-Uspensky [1958]), as another strong argument for accepting Church's Thesis. Mendelson goes so far as consider it as a legitimate proof of Church's Thesis.

An interesting feature of MCM is its pure syntactic mathematical universe. It is true that the same can be said with some reason about BCM. But Markov placed a special accent on this feature of his system. Consructive objects and constructive processes (algorithms) are the main (and essentially the only) Dramatis Personae on the scene. Thus constructive objects are considered to be initial data for algorithms and this point of view gives a very special touch to MCM which may be of interest for Computer Science. Let as consider an example. A classical real number can be defined as follows. Let $\alpha$ be a Cauchy sequence of rational numbers. This means that

$$\forall n \exists m \forall ij (i,j > m \supset |\alpha(i) - \alpha(j)| < 2^{-n}) \tag{3.1}$$

Clasically, real numbers are classes of equivalent Cauchy sequences. In MCM the arbitrary sequence above is to be replaced by an algorithm of the type $\mathbb{N} \to \mathbb{Q}$, where $\mathbb{N}$ is the set of natural numbers (and natural numbers are words of type $0, 0|, 0||...$) in the alphabet $\{0, |\}$ and $\mathbb{Q}$ is the set of rational numbers (which are words of a special type, as well). Of course, one can speak here about a recursive function with rational values. As for 3.1, the strictest constructive reading of it will be as follows. There is an algorithm $\beta$ of type $\mathbb{N} \to \mathbb{N}$ such that

$$\forall nij (i,j > \beta(n) \supset |\alpha(i) - \alpha(j)| < 2^{-n}) \tag{3.2}$$

Such algorithm $\beta$ we call a Cauchy modulus for $\alpha$. The schemes of algorithms can be coded in some natural way by words (or by natural numbers). The code of $\alpha$ we denote by $\alpha^c$. A constructive number is defined as a couple $\alpha^c * \beta^c$ where $\beta$ is a Cauchy modulus for $\alpha$. Therefore constructive real numbers are *words* in an alphabet. The set of constructive real numbers we denote by $\mathbb{D}$. $\mathbb{D}$ is an adequate continuum for MCM and usual Calculus can be readily developed over $\mathbb{D}$ using Markov's definition of constructive real functions (constructive function for the sake of shortness). A constructive function is an algorithm $f$ of type $\mathbb{D} \to \mathbb{D}$ such that $f(x_1) = f(x_2)$ as soon as $x_1 = x_2$. (Equality of constructive real numbers can be introduced in an obvious way, see, e.g., Kushner [1984]). It should be noted that, as natural as it is, the concept of constructive real number is not completely evident. E.g., there is a temptation to consider computable systematic expansions. And the first definition of computable number published by Turing (Turing [1936/37]) was exactly of this type. A correction (Turing [1937]) followed immediately. Deficiencies of computable expansions are discussed in Kushner [1984]. The essence of those deficiencies was known already to Brouwer (see, e.g. Brouwer [1921]). It is enough to mention that, e.g., there is no algorithm for addition of computable systematic expansions.

A constructive real number as a syntactic object holds in itself information sufficient to find in an effective way rational approximations to the number with every desirable accuracy. Nevertheless, some interesting variations of this concept are possible. First of all it is possible to read 3.1 in a more liberal way, say as

$$\forall n \neg\neg \exists m \forall ij (i,j > m \supset |\alpha(i) - \alpha(j)| < 2^{-n}) \tag{3.3}$$

Formula 3.3 roughly speaking represents classical Cauchy property. We can consider now a new type of computable real numbers. The word $\alpha^c$ where $\alpha$ satisfies 3.3 we call a pseudonumber. The set of all pseudonumbers will be denoted by $\mathbb{P}$. $\mathbb{P}$ presents another model of constructive continuum. Reals from this continuum are computable in the sense that for every such number there is an algorithmic sequence of rational approximations that converges. But not only we do not have a recursive modulus of convergence included in the number-word, it can happen that such a modulus does not exist at all (this follows from a well-known result of Specker [1949], see, also, Kushner [1984]). Therefore there exist pseudonumbers that are not equal to any constructive real number. It is interesting that we can obtain two other variants of computable numbers in a rather syntactic way, by omitting information about a Cauchy modulus in the definition of constructive real numbers. We will call a word $\alpha^c$ an F-number (quasinumber) if there is (can not fail to be) a Cauchy modulus for $\alpha$. Let $\mathbb{F}$ and $\mathbb{K}$ be the sets of all F-numbers and quasinumbers, respectfully. To compare the four models of constructive continuum let us imagine, for a moment, that they are placed on the classical real line. Every model singles out some computable numbers in the classical continuum. It is evident that $\mathbb{P}$ is wider in this sense than $\mathbb{D}$. But $\mathbb{D}, \mathbb{F}$ and $\mathbb{K}$ are from this point of view the same. They single out exactly the same points on the classical real line. The syntactic difference between constructive real numbers and F-numbers (quasinumbers) is evident and they do look as quite different initial data for algorithms. It is well-know that the information about a Cauchy modulus that is absent in F-numbers (quasinumbers) can not be restored in an effective way. Namely, there is no algorithm that finds for every F-number (quasinumber) $p$ a constructive real number that is equal $p$ (see, e.g., Kushner [1984]).

On the other hand, F-numbers and quasinumbers are words of the same type, every F-number is a quasinumber and there is no quasinumber $p$ such that $p \neq q$ for every F-number $q$. The continua $\mathbb{F}$ and $\mathbb{K}$ are the same for a classical mathematician. One can not tell one from another from the classical point of view. But the difference is quite discernible constructively. There is a sequence of quasinumbers $\gamma$ that is not a sequence of F-numbers. Really, in order to prove that $\gamma$ is a sequence of F-numbers a constructivist should develop an algorithm that would give for every $n$ a (code of) Cauchy modulus of $\gamma(n)$. For the sequence $\gamma$ mentioned above such an algorithm does not exist (see, Kushner [1984]).

The difference between F-numbers and quasinumbers can be illustrated by a Brouwerian counterexample, as well. Consider an algorithm $\alpha$ such that

$$\alpha(n) = \begin{cases} 1, & : \text{ if there is a perfect} \\ & : \text{ number among the numbers } 2i+1 \text{ where } i \leq n \\ 0, & : \text{ otherwise} \end{cases}$$

It is evident that if there is no odd perfect number then

$$\forall n(\alpha(n) = 0)$$

and if $2i+1$ is the least odd perfect number than

$$\alpha(n) = \begin{cases} 0, & \text{if} n < i \\ 1, & \text{if} n \geqslant i \end{cases}$$

It is evident that $\alpha$ can not fail to have a Cauchy modulus, hence $\alpha^c$ is a quasinumber. But nobody could present such a Cauchy modulus $\beta$ today. Indeed, it is clear that an odd perfect integer exists iff $\alpha(\beta(1)) = 1$. Thus one can not state that the quasinumber $\alpha^c$ is an F-number.

All in all, we have four pretenders to bear the title of constructive continuum. It is interesting to notice that the completeness theorem holds for $\mathbb{D}$, $\mathbb{P}$, $\mathbb{F}$, but it does not hold for $\mathbb{K}$ (there is a (constructive) Cauchy sequence of quasinumbers that does not have the limit). Certainly, $\mathbb{D}$ looks like the most attractive constructive continuum. Since $\mathbb{D}$ is a countable set from the classical point of view (as are the other three models), the question arises whether our intuitive perception of continuity is grasped in $\mathbb{D}$. There is no immediate answer to this question. On one hand, $\mathbb{D}$ is not a countable set constructively, as Cantor's diagonal construction can be reproduced. Moreover, a constructive function $f$ (which, as is known, is automatically constructively continuous) such that, say, $f(0) < 0$ and $f(1) > 0$, can not fail to have a root between 0 and 1. On the other hand, S.N. Manukyan [1976] constructed the following amazing counterexample

**Theorem 3.1.** *There are two constructive (and therefore continuous) planar curves $\bar{\varphi}_1$ and $\bar{\varphi}_2$ such that*

- $\bar{\varphi}_1(0) = (0,0), \bar{\varphi}_1(1) = (1,1)$;
- $\bar{\varphi}_2(0) = (0,1), \bar{\varphi}_2(1) = (1,0)$;
- *for every $0 < t < 1$ both $\bar{\varphi}_1(t)$ and $\bar{\varphi}_2(t)$ belong to the open unit square;*
- *$\bar{\varphi}$ and $\bar{\varphi}_2$ do not intersect.*

Thus the above continuous curves connect diametrically opposed vertices of the unit square, they do not leave this square and still they do not intersect!

One can consider this example as an argument that the immediate intuition of continuity is not grasped by $\mathbb{D}$. However, this argument can be met by reference to well-known topological examples, like Peano's curve, etc, that show that this very immediate intuition is not quite a reliable facility and sometimes even mislead us.

The splitting of a classical concept (the concept of Cauchy sequence above) into several constructive concepts is a common thing in any constructive mathematics. But the syntactic splitting (e.g., constructive real numbers versus F-numbers or quasinumbers) is very characteristic for MCM with its syntactic mathematical universe. The same can be said about the subtle difference between quasinumbers and F-numbers.

## 4.

Both MCM and BCM took a lot from Brouwer, especially in their critical approach to the set-theoretical (classical) Mathematics and in their understanding of constructive logical operators. I have tried to describe the mathematical and human relations between Markov and Bishop, between MCM and BCM in my essay [1993a] (there exists a Russian version of this essay Kushner [1992]). It seems in general that Bishop's approach to constructive mathematics was mostly pragmatic and foundational problems did not attract him - at least, they were not supposed to be brought in at the expense of concrete mathematical activities. Nevertheless, Bishop still did not avoid the eternal problem of interpretation of implication, this host of Hamlet's father of any constructive mathematics (see, Bishop [1970]). It is worth mentioning that Markov spent the last years of his life struggling to develop a large semantical system to achieve, above all, a satisfactory theory of implication (see, e.g., Markov [1971,1976]). BCM can be considered as neutral in the sense that its results are acceptable both intuitionistically and in MCM. So the remarkable body of mathematical analysis that was built in BCM by Bishop himself and his disciples contributes to the claims of Intuitionism and MCM.

Very interesting remarks revealing Markov's position with respect of Intuitionism can be found in his Editor's Comments to the Russian translation of Heyting's book [1956] (Markov [1965]). As is known, Heyting's book is written artistically in the form of a discussion between several dramatis personae, like Clas, Int, Form etc. In his remarks Markov introduced a new person Con (a Constructivist). Perhaps, it would be interesting to see an edition of Heyting's books with Markov's remark translated and incorporated into the text.

## 5.

60-ies - 70-ies were the best years for MCM. It had active centers in Moscow (headed by A.A. Markov), in Leningrad (headed by N.A. Shanin), in Erevan (headed by I.D. Zaslavsky) and in Prague (headed by O. Demuth). Numerous impressive results were obtained in constructive Analysis, constructive

Logic, Theory of Algorithms, Theory of Complexity of Algorithms and Calculations, and Philosophy of Mathematics. Each of the above centers had a particular face. E.g., pioneering works on automatic theorem proving were done in Leningrad under Shanin's leadership. A body of work on constructive functional analysis was created in Prague by Oswald Demuth and his disciples (see, e.g., Demuth [1965, 1978, 1980], Kushner [1984], Kushner [1990]). Oswald was Markov's PhD student in Moscow University in the same years approximately that I worked on my Thesis under Markov, too. I remember vividly his talks in seminars, his very specific and charming Russian language and our discussions about his approach to the constructive integral of Lebesgue. In the tragic days of August, 1968 he was in Leningrad. He returned to his Mother-Land immediately and took all the consequences of his dignified rejection of Soviet occupation and the events that followed.

The bold attempt to develop Mathematics in the framework of a coherent constructive mathematical worldoutlook that was undertaken by A.A. Markov (Jr.) will beyond doubt be remembered as an exciting chapter in the History of the Mathematics of our century.

## 6.

'he influence of Gödel's works on Markov's program was not, probably, di-ect but it was an essential one. Exactly as one who listens to the grandiose ymphonies of Shostakovich can feel that this great Master has heard the igantic symphonic works of Mahler, one feels, while reading Markov's fun-amental papers on constructive Mathematics, that he was deeply familiar rith the works and ideas of Hilbert, Brouwer and Gödel. It is very character-stic of Markov that he translated into Russian, edited and published in 1948 ı *Uspekhi Mathematicheskikh Nauk* Gödel's work [1940] on the consistency f the Continuum hypothesis and in the same year published his own short aper about the dependence of axioms in the original Bernays-Gödel's sys-em (Markov [1948]) (evidently, this paper originated in Markov's work on ranslating Gödel). And this happened in the year when Markov embarked n his revolutionary activities on developing his own constructive mathemat-s! I was too young then, but later in the early 60-ies I heard many times 1arkov's declarations like "I do not understand it. It is something classical, o, no...Do not even tell it to me..." I always felt a mephistophelian sarcasm ehind such public statements. I do not know if Markov ever met Gödel. I oubt that such a meeting ever took place. Generally, in the political climate f those years personal contacts between Russian and Western mathemati-ians were very scarce. Anyway, I do not think that two men would get along they met. Markov, like the other two great constructive leaders Brouwer nd Bishop, was a quite outspoken person. He just loved to express opinions f paradoxical nature, to amaze and to shock colleagues at any convenient (or non-convenient) occasion by declarations that amounted to mathematical sacrilege. It was sometimes as though somebody would deny the Bible in a Church. On the other hand, as I got to know from an excellent account of S. Feferman [1984], Gödel was quite a different person. It looks that he preferred to be left alone with his great ideas and shied away of every type of publicity. He exercised an extreme caution and did not express openly his strong Cantorian-like platonistic convictions. And it seems that he formulated his incompleteness results in terms of provability, rather than truth, just to avoid discussions (inevitable in those years) on "what the truth is". It somehow reminds me Gauss's reservation with respect of his discovery of non-Euclid Geometry.

Be that as it may, the presence of the titanic personality of Gödel was always felt in our discussion and seminars.

I believe that MCM was influenced the most by three of Gödel's results and ideas: 1) the definition of recursive functions; 2) the incompleteness theorems; 3) the concept of a computable function of a finite type.

Though Markov had introduced his own precise concept of algorithm (normal algorithms of A.A. Markov) , especially designed for his constructive program, and developed an original and deep theory of normal algorithms, one should admit that this work was inspired and influenced by ideas and techniques of the earlier concepts of recursive functions and Turing/Post machines. As is known, Gödel was one of the pioneers in theory of recursive functions. It was he who formulated the first definition of recursive functions that is known today as the Herbrand-Gödel definition. The dramatic history of the first years, better to say months, of the theory of computability is presented in Davis [1982]. It would be simply impossible even to formulate Markov's constructive program without the pre-war achievements in the theory of computability.

As is well-known, the main idea of Hilbert's foundational program was a justification of classical mathematics by a finite proof of its consistency. Sure enough, neither Brouwer nor Markov nor Bishop would buy such a proof as a convincing argument toward legitimation of classical mathematics, as such. The constructive tendency grew not so much from paradoxes, as from intellectual doubts about the main philosophical concepts of classical mathematics, especially about actual infinity and the universality of the tertium non datur principle. (Incidentally, the identification of mathematical existence and consistency was challenged already by H. Poincaré). The most convincing proof of consistency would not make those concepts more feasible. Constructivism, as, probably, every "ism" is about principles, not paradoxes. Moreover, the well-known opinion that Gödel's incompleteness results dealt a death blow to Hilbert's hopes seems exaggerated. The point is that Hilbert's finitism put extremely strong, evidently too strong restrictions on possible proofs of consistency. This was exactly Hilbert's reaction to Gödel's results (see his preface to the first edition (1934) of Hilbert and Bernays' monograph [1968]). I'll cite from Feferman [1993] an excellent description that Gödel gave to Hilbert's finitism in the lecture "The present situation in the foundations of mathematics" delivered in December 1933 to a joint Meeting of The Mathematical Association of America and American Mathematical Society (Cambridge, Massachusetts):

1. The applications of the notion of "all" or "any" is to be restricted to those infinite totalities for which we can give a finite procedure for generating all their elements [such as integers]...

2. Negation must not be applied to propositions stating that something holds for all elements, because this would give existence propositions...[these] are to have a meaning in our system only in the sense that we have found an example but, for the sake of brevity, do not state it explicitly...

3. And finally we require that we should introduce only such notions as are decidable for any particular element and only such functions as can be calculated for any particular element.

It is evident that constructivists are usually far more liberal in their restrictions. Hence, there could be consistency proofs that would satisfy them. In fact, I think that Gentzen's proof is one of them. Various other proofs in this or that extensions of Hilbert's finitism were published subsequently. I would mention recent work of N.M. Nagorny [1995]: the author, a known constructivist of Markov's school, states that his proof of consistency of classical (formal) arithmetic is neutral, i.e. it would be accepted by both intuitionists and constructivists of Markov's school (I think that the same holds for BCM, as well).

Nevertheless, one should not underestimate the significance of Gödel's incompleteness results for constructive mathematics. One of the philosophical consequences of those results was the understanding that Hilbert's goal can not be reached, at least in full. Therefore the conceptual problems that classical mathematics was faced with were more deep and disturbing than this great mathematician believed. It goes without saying that this understanding created more a favorable psychological climate for constructive mathematics and helped to recruit new champions for it. On the other hand, it is worth noting that the conceptual and technical apparatus developed in the framework of Hilbert's program and Gödel's works turned out to be indispensable in building of Markov's constructive mathematics, which is based on a precise concept of an algorithm.

In his work of 1958 Gödel suggested a new interpretation of intuitionistic arithmetic (so called Dialectica-interpretation) by means of computable functions of finite types (the last concept was introduced in the same paper). Hence, the task of developing of an universe of computable functions of finite types arises. Two way of approaching the problem are evident. One is to enumerate objects of lower types and use such Gödel numbers as initial data for functions of higher types. Another is to use approximations to function-arguments, i.e. some topological structure for functions of lower types. In the first case we would speak of Markov operators, in the second of Kleene operators. An interesting theory of computable functions of finite types was later developed topologically in the frameworks of Markov's constructive mathematics by Chernov [1972 a-c]. It is worth noting that everywhere defined constructive functions (the counterpart to the classical concept of everywhere defined real function) can be considered in the spirit of Gödel's approach as computable functions of type ((0,0),(0,0)) with some restrictions on the domain and closure conditions on the range. As was mentioned above, two main ways to operate constructively with (0,0) objects are known: using approximations (e.g. in Bair space) or Gödel codes (numbers) of (0,0) objects. In the first case we arrive to Kleene's partial-recursive operators, in the second case we deal with Markov's constructive functions. It seems that more information about the argument-function is available for a Markov operator than for an operator of Kleene which uses only "beginnings" of the argument-functions. This effect can be really felt in the case of not everywhere defined operators (Muchnik-Friedberg counterexample, see, e.g., Kushner [1984]). But the Kreisel-Lacombe-Schoenfield-Tseitin Continuity Theorem (see, e.g., Kushner [1984]) states that the two above approaches are equivalent for constructive functions that are everywhere defined on Markov's constructive continuum. On the other hand, some results of the author (Kushner [1982]) show that the "Gödel numbers" approach gives a wider class of computable functions than partial-recursive operators if one considers functions everywhere defined on a more liberal version of Markov's constructive continuum, namely $\mathbb{P}$. These results turn out to be closely related to the problem of the compactification of constructive continuum and to uniform continuity of constructive functions. We mention the following theorem (technical detail can be found in Kushner [1982, 1984]).

**Theorem 6.1.** *1. If a constructive function $f$ is everywhere defined and a Kleene operator that computes $f$ on the closed constructive unit interval is defined for all pseudonumbers of this interval, then $f$ is constructively uniformly continuous on this interval.*

*2. There is an everywhere defined constructive function $g$ such that there is an algorithm $G$ of type $\mathbb{P} \to \mathbb{P}$ that extends $g$ and, nevertheless, $g$ is effectively non-uniformly continous on the closed unit interval.*

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## References

1. M. J. Beeson. *Foundations of Constructive Mathematics.* Springer, Berlin, 1985.
2. E. Bishop. *Foundations of Constructive Analysis.* MvGraw- Hill, New York, 1967.

3. E. Bishop. Mathematics as a numerical language. *Intuitionism and Proof Theory (Proc.Conf, Buffalo, N.Y.)*, North-Holland, Amsterdam, 53-71, 1970.
4. E. Bishop. Schizophrenia in Contemporary Mathematics. *Erret Bishop: Reflections on Him and His Research* (San Diego California, 1983) (M. Rosenblatt, Editor), AMS, *Contemp. Math.*, vol.36, Providence, Rhode Island, 1984, 1-32.
5. E. Bishop and D. Bridges. *Constructive Analysis.* Springer-Verlag, Berlin-heidelberg-New York-Tokyo, 1985.
6. L. E. I. Brouwer. Bezitzt jede reelle Zahl eine Dezimalbruchentwicklung? *Nederl.Akad. Wetensch. Vershlagen* , 29, 803-812. Also in *Math. Ann* 83,201-210, 1921, 1921.
7. V. P. Chernov. Topologicheskie varianty teoremy o nepreryvnosti otobrazheniy i rodstvennye teoremy. Russian, *Zapiski Nauchnykh Seminarov LOMI*, 32, 1972a, 129-139.
8. V. P. Chernov. O konstruktivnykh operatorakh konechnykh tipov. Russian, *Zapiski Nauchnykh Seminarov LOMI* , 32, 1972b, 140-147.
9. V. P. Chernov. Klassifikaciya prostranstv operatorov konechnykh tipov. Russian, *Zapiski Nauchnykh Seminarov LOMI* , 32, 1972c, 148-152.
10. D. van Dalen. Why Constructive Mathematics? *The Foundational Debate. Complexity and Constructivity in Mathematics and Physics*, W. Depauli-Schimanovich at al, Eds. Vienna Circle Yearbook, Kluwer, Dordrecht, 141-158, 1995.
11. M. Davis. Why Gödel Didn't Have Church's Thesis. *Information and Control*, 54, 3-24, 1982.
12. O. Demuth. Lebesque integration in constructive analysis Russian, *Dokl. Akad. Nauk SSSR* (Engl. transl. in *Soviet Math. Dokl.* 6 (1965), 160, 1965, 1239-1241.
13. O. Demuth. Nekotorye voprosy teorii konstruktivnykh funktsiy deystvitel'noy peremennoy. Russian, *Acta Universitatis Carolinae, Math. et Phys.*, 19, No1, 61-69, 1978.
14. O. Demuth. O konstruktivnom integrale Perrona. Russian, *Acta Universitatis Carolinae, Math. et Phys.*, 21, No1, 3-57, 1980.
15. A. G. Dragalin. *Mathematical Intuitionism: Introduction to proof theory.* AMS (Translation from the Russian; Russian original 1979) , Providence, Rhode Island , 1988.
16. S. Feferman. Kurt Gödel:Conviction and Caution. *Philosophia Naturalis*, 21 (2-4), 546-562, 1984.
17. S. Feferman. Gödel's *Dialectica* interpretation and its two-way stretch. *Computational Logic and Proof Theory, Proc. of the Third Kurt Gödel Colloquium*, G. Gottlob et al, Eds, Lecture Notes in Computer Science, 713, 1993, 23-40.
18. Gödel. *The consistency of the Continuum Hypothesis.* Princeton University Press, Princeton, New Jersey, 1940.
19. Gödel. Über eine bisher noch nicht benütze Erweiterung des finititen Standpunktes. *Dialectica*, 12, 240-251, 1958.
20. A. Heyting. *Intuitionism. An Introduction.* North-Holland, Amsterdam, 1956.
21. A. Heyting. After thirty years. *Logic, Methodology and Philosophy of Science, Proceedings of the 1960 International Congress* , E. Nagel et all, Eds, Stanford University Press, Stanford, California , 194-197, 1962.
22. D. Hilbert and P. Bernays. *The Grundlagen der Mathematik, 1.* Springer-Verlag, Berlin-Heidelberg-New York, 1968.
23. S. C. Kleene and R. E. Vesley. *The Foundations of Intuitionistic Mathematics. Especially in Relation to Recursive functions.* North-Holland, Amsterdam, 1965.
24. A. N. Kolmogorov and V. A. Uspensky. K opredeleniyu algoritma. Russian, *Uspekhi Mathematicheskikh Nauk*, 13, vyp. 4(82), 3-28, 1958.

25. B. A. Kushner. Some extensions of Markov's constructive continuum and their applications to the theory of constructive functions. *The L.E.J. Brouwer Centenary Symposium*, A. S. Troelstra and D. van Dalen, Eds, North-Holland Publishing Company, Amsterdam-New York-Oxford, 1982 , 261-273.
26. B. A. Kushner. *Lectures on Constructive Mathematical Analysis.* (Translation from the Russian; Russian original 1973) , AMS, Providence, Rhode Island, 1984.
27. B. A. Kushner. Printsip bar-induktsii i teoriya kontinuuma u Brauera. Russian, *Zakonomernosti razvitiya sovremennoy matematiki* , Nauka, Moscow, 1987, 230-250.
28. B. A. Kushner. Ob odnom predstavlenii Bar-Induktsii. Russian, *Voprosy Matematicheskoy Logiki i Teorii Algoritmov*, Vychislitel'ny Tsentr AN SSSR, Moscow, 1988, 11-18.
29. B. A. Kushner. Markov's Constructive Mathematical Analysis: the Expectations and Results. *Mathematical Logic*, P. Petkov, Ed. , 53-58, 1990, Plenum Press, New York-London.
30. B. A. Kushner. A Version of Bar-Induction. Abstract, 1992, 57,No1 *The Journal of Symbolic Logic.*
31. B. A. Kushner. Markov i Bishop. Russian, *Voprosy Istorii Estestvoznaniya i Tekhniki*, 1, 1992, 70-81.
32. B. A. Kushner. Markov and Bishop. *Golden Years of Moscow Mathematics*, S. Zdravkovska, P. Duren, Eds, 179-197, 1993a , AMS-LMS, Providence, Rhode Island.
33. B. A. Kushner. Konstruktivnaya matematika A.A. Markova: nekotorye razmyshleniya. Russian, *Modern Logic*, 3, No2, 119-144, 1993b.
34. S. N. Manukyan. O nekotorykh topologicheskikh osobennostyakh konstruktivnykh prostykh dug. Russian, *Issledovaniya po teorii algorifmov i matematicheskoy logike*, B. A. Kushner and A. A. Markov, Eds , Vychislitel'ny Tsentr AN SSSR , Moscow, 1976, 122-129 .
35. A. A. Markov. O zavisimosti aksiomy B6 ot drugikh aksiom sistemy Bernaysa-Gedelya. Russian, *Izvestiya Akademii Nauk SSSR, ser. matem.* , 12, 1948, 569-570.
36. A. A. Markov. The theory of algorithms. *Amer. Math. Soc. Transl.*, (2) 15, 1960, AMS (Translation from the Russian, Trudy Instituta im. Steklova 38 (1951), 176-189) , Providence, Rhode Island.
37. A. A. Markov. *The theory of algorithms.* Israel Programm Sci. Transl. (Translation from the Russian, Trudy Instituta im. Steklova 42 (1954)) , Jerusalem, 1961.
38. A. A. Markov. On constructive mathematics. *Amer.Math.Soc. Transl.*, (2) 98, 1971, AMS (Translation from the Russian, Trudy Instituta im. Steklova, 67, 8-14, (1964)) , Providence, Rhode Island.
39. A. A. Markov. Comments of the Editor of the Russian translation of Heyting's book *Intuitionism.* Russian, A. Geyting, *Intuitsionizm.* Russian, Moskva, Mir, 1965, 161-193.
40. A. A. Markov. Essai de construction d'une logique de la mathematique constructive. *Revue intern. de Philosophie*, 1971, 98, Fasc.4.
41. A. A. Markov. Popytka postroeniya logiki konstruktivnoy matematiki. Russian, *Issledovaniya po teorii algorifmov i matematicheskoy logike*, B. A. Kushner and A. A. Markov, Eds , Vychislitel'ny Tsentr AN SSSR , Moscow, 1976, 3-31, A Russian version of Markov 1971.
42. A. A. Markov and N. M. Nagorny *The Theory of Algorithms.* Kluwer Academic Publishers (Translation from the Russian; Russian original 1984), Dordrecht-Boston-London, 1988.

43. E. Mendelson. Second thought about Church's thesis and mathematical proofs. *The Journal of Philosophy*, 87, 225-233 , 1990.
44. N. M. Nagorny. *K voprosu o neprotivorechivosti klassicheskoy formal'noy arifmetiki.* Russian, Computing Center RAN, Moscow , 1995 .
45. H. Rogers, Jr. *Theory of Recursive Functions and Effective Computability.* McGraw-Hill Book Company, New York, 1967.
46. E. Specker. Nicht konstruktiv beweisbare Sätze der Analysis. *The Journal of Symbolic Logic*, 14, 145-158, 1949.
47. A. S. Troelstra. On the early history of intuitionistic logic. *Mathematical Logic*, P. P. Petkov, Ed., Plenum Press , New York-London, 1990, 3-17.
48. A. S. Troelstra. *Choice Sequences. A Chapter of Intuitionistic Mathematics.* Clarendon Press, Oxford, 1977.
49. A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics. An Introduction. Vol.1-2*, North-Holland , Amsterdam-New York-Oxford-Tokyo, 1988.
50. A. M. Turing. On computable numbers, with an application to the Entscheidungsproblem. 1936-37, 42 (2), *Proc. London Math. Soc.*, 230-265.
51. A. M. Turing. Correction. 1937, 43 (2), *Proc. London Math. Soc.*, 544-546.