1. PRELIMINARIES

In this chapter we introduce the basic notation and terminology which will be used throughout this book. We also state a number of basic Facts. These Facts will not be proved; some of them are rather obvious (and easy to believe), others are substantial and well–known theorems; further Facts will be stated when they are needed. These Facts are sufficient for most of the proofs in this book; the chief exceptions are the proofs of Theorems 3.5 and 6.4. Finally, we prove the very important fixed point lemma (Lemma 1) and apply it to prove that Robinson's Arithmetic Q is essentially undecidable (Theorem 2) and that in extensions of Q there are no truth–definitions (Theorem 3).

The language L_A of elementary arithmetic can be described as follows. The alphabet consists of:

the propositional connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$,
the quantifiers: $\exists$, $\forall$,
the equality symbol: $=$,
symbols used to form (individual) variables: v, $'$,
parentheses: (,),
the arithmetical constants: 0, S, +, $\times$.

(The intended interpretation of S is the successor function.) Thus, the alphabet is finite. The *variables* of L_A are the expressions v, v$'$, v$''$, etc. We write v_n for v followed by n occurrences of $'$. In most contexts x, y, z, u, v, w, possibly with subscripts etc., will be used for variables. The terms, formulas, and sentences of L_A are defined as usual. Among the terms we distinguish the *numerals* 0, S0, SS0, SSS0, ... These will be written 0, 1, 2, ... Thus, we shall omit bars and other devices ordinarily used to indicate numerals (or Gödel numbers) and use the same symbols for natural numbers and for formal numerals. In most cases this will cause no trouble as long as the symbols for formal variables are kept strictly apart from the symbols for natural numbers (and numerals). For the latter we use k, m, n, p, q, r, s, possibly with subscripts etc. and symbols for formulas (see below). N is the set of natural numbers.

For sentences and formulas of L_A we use lower case Greek letters. Sentences will be written as φ, ψ, θ, χ, etc. and formulas as $\alpha(x)$, $\beta(x,y)$, $\gamma(x)$, $\xi(x)$, $\eta(x_1,...,x_n)$, $\rho(x,x')$, $\tau(x)$, ξ, γ, etc. The variables displayed are almost always exactly the free variables of the formula. $\xi(y)$ is obtained from $\xi(x)$ by replacing x by y, assumed not to be free in $\xi(x)$, and, possibly renaming bound variables in the usual way. $\xi(k)$ is obtained from $\xi(x)$ by replacing x by the numeral k (or, if you prefer, by the numeral for the number k). This generalizes in the obvious way to substitutions involving more than one variable. We use := to denote equality between formulas.

By a *theory* T we understand a set of sentences (to be thought of as the (nonlogical) axioms of T). (It would be inconvenient to identify a theory T with the set of its theorems, since quite often we need to know that there is a formula binumerating (defined below) (the set of axioms of) T.) Note that, although we shall mainly be interested in theories that are reasonable from an arithmetical point of view, such reasonableness is not part of the concept *theory*. T + φ is the theory obtained from T by adding φ as a (new) axiom. T + X, where X is a set of sentences, is understood similarly. We assume given a fixed complete deductive calculus PL for first order logic. Referring to PL certain (finite) formal objects (sequences of sentences) are *proofs* (in T). A proof is a *proof of* its last sentence. The sentence φ is *provable in* T, $T\vdash \varphi$, if there is a proof of φ in T. $T\vdash \xi(x_1,\ldots,x_n)$, where $x_1,\ldots,x_n$ are all the free variables of $\xi(x_1,\ldots,x_n)$, is short for $T\vdash \forall x_1\ldots x_n\xi(x_1,\ldots,x_n)$. Th(T) is the set of *theorems* of, i.e. sentences provable in, T. If X is a set of sentences, we write $T\vdash X$ or $X\dashv T$ to mean that $T\vdash \varphi$ for every $\varphi\in X$. Thus, $S\dashv T$ means that S is a *subtheory* of T (T is an *extension* of S). We write $\vdash \varphi$ for $\varnothing\vdash \varphi$, where $\varnothing$ is the empty set. Thus, $\vdash \varphi$ means that φ is provable in logic (PL).

N = (N, +, ×, S, 0) is the standard model of arithmetic. A sentence φ is *true* if it is true in **N**. A theory is *true* if all its axioms (and therefore, all its theorems) are true.

There are two (true) theories PA (Peano Arithmetic) and Q (Robinson's Arithmetic) that will play a prominent role in what follows. Q is a finite theory; its axioms are (we omit the initial universal quantifiers):

Q1 $Sx = Sy \to x = y$,

Q2 $\neg 0 = Sx$,

Q3 $\neg 0 = x \to \exists y(x = Sy)$,

Q4 $x + 0 = x$,

Q5 $x + Sy = S(x + y)$,

Q6 $x \times 0 = 0$,

Q7 $x \times Sy = (x \times y) + x$.

We introduce the two–place predicates $\leq$ and $<$ by means of the definitions:

$$x \leq y =_{df} \exists z(z + x = y),$$

$$x < y =_{df} x \leq y \wedge \neg x = y.$$

With our present simplified notation certain (harmless) ambiguities arise. For example, 2 + 3 can be read as a numeral but also as an expression containing the symbol +. In Fact 1 below we have indicated that the latter is the intended reading by underscoring the relevant function symbol. But, of course, terms such as Sy, x + y, x + 3, 4 × z, etc. are unambiguous. Also, with very few exceptions, it is, in view of Fact 1, not important which way, say, 2 + 3 is understood.

Fact 1. The following formulas are provable in Q for all k, m, n,

(i) $\neg\, k = m$ for $k \neq m$,

(ii) $k \,\underline{+}\, m = k + m$,

(iii) $k \,\underline{\times}\, m = k \times m$,

(iv) $x \leq m \to x = 0 \vee x = 1 \vee \ldots \vee x = m$,

(v) $x \leq m \vee m \leq x$.

Q is a very weak theory. For example, the sentences $\forall x(0 + x = x)$, $\forall x \neg(x = Sx)$, $\forall x(x \leq x)$, cannot be proved in Q.

The axioms of PA consist of the axioms of Q plus the (universal closures) of formulas of the form

$$\alpha(0) \wedge \forall x(\alpha(x) \rightarrow \alpha(Sx)) \rightarrow \forall x\alpha(x).$$

(Here $\alpha(x)$ may contain free variables other than x.) This is the *induction scheme* and is as close as we can get to the full (second order) induction axiom in first order arithmetic.

From the induction scheme we can derive the *least number principle*: for every formula $\alpha(x)$ as above,

$$\mathrm{PA} \vdash \exists x\alpha(x) \rightarrow \exists x(\alpha(x) \wedge \forall y(y < x \rightarrow \neg\alpha(y))).$$

Obviously, $\mathrm{Q} \dashv \mathrm{PA}$. In PA axiom Q3 is redundant. $\forall xy(x \leq y \vee y \leq x)$ is provable in PA, but not in Q. In fact, this is sometimes the sole reason for writing "$\mathrm{PA}\vdash$" rather than "$\mathrm{Q}\vdash$".

Gödel proved that every primitive recursive function is definable in first order arithmetic. Formalizing this proof, he proved that:

Fact 2. For each primitive recursive function $f(k_0,\ldots,k_n)$ there is a formula $\delta_f(x_0,\ldots,x_n,y)$ such that for all $k_0,\ldots,k_n$,

(i) $\mathrm{Q} \vdash \delta_f(k_0,\ldots,k_n,y) \leftrightarrow y = f(k_0,\ldots,k_n)$.

(ii) $\mathrm{PA} \vdash \delta_f(x_0,\ldots,x_n,y) \wedge \delta_f(x_0,\ldots,x_n,z) \rightarrow y = z$,

(iii) $\mathrm{PA} \vdash \exists y\delta_f(x_0,\ldots,x_n,y)$.

In Fact 2 (i) $f(k_0,\ldots,k_n)$ is, of course, a numeral, i.e. does not contain the symbol f.

A formula $\delta(x_0,\ldots,x_n,y)$ such that for all $k_0,\ldots,k_n$,

$$\mathrm{T} \vdash \delta(k_0,\ldots,k_n,y) \leftrightarrow y = f(k_0,\ldots,k_n)$$

will be said to *define* f *in* T.

For (general) recursive functions we have the following weaker Fact (for Fact 3 (b), see below):

Fact 3. (a) For every (total) recursive function $f(k_0,\ldots,k_n)$, there is a formula $\delta_f(x_0,\ldots,x_n,y)$ defining f in Q.

The formula $\rho(x_0,\ldots,x_n)$ *numerates* the relation $R(k_0,\ldots,k_n)$ *in* the theory S if for all $k_0,\ldots, k_n$ (as usual "iff" is short for "if and only if"),

$$R(k_0,\ldots,k_n) \text{ iff } \mathrm{S} \vdash \rho(k_0,\ldots,k_n).$$

Thus, $\xi(x)$ numerates X in S if for every k,

$$k \in X \text{ iff } \mathrm{S} \vdash \xi(k).$$

$\rho(x_0,\ldots,x_n)$ *binumerates* the relation $R(k_0,\ldots,k_n)$ *in* S if for all $k_0,\ldots,k_n$,

$$R(k_0,\ldots,k_n) \text{ iff } \mathrm{S} \vdash \rho(k_0,\ldots,k_n),$$

$$\text{not } R(k_0,\ldots,k_n) \text{ iff } \mathrm{S} \vdash \neg\rho(k_0,\ldots,k_n).$$

In particular, $\xi(x)$ binumerates X in S if for every k,

$k \in X$ iff $S \vdash \xi(k)$,
$k \notin X$ iff $S \vdash \neg\xi(k)$.

If a formula binumerates X (R) in S, it binumerates X (R) in every consistent extension of S.

If S is recursively enumerable (r.e.), any set (relation) numerated by some formula in S is r.e. and any set (relation) binumerated by some formula in S is recursive.

Fact 3 (a) has the following:

Corollary 1. (a) A set X (relation R) is recursive iff there is a formula binumerating X (R) in Q.

(b) A set X (relation R) is r.e. iff there is a formula numerating X (R) in Q.

This corollary and most of those below in this chapter are easy consequences of the relevant Facts; their proofs are, therefore, left to the reader.

Note that, in view of Corollary 1, we have the remarkable fact that any set X (relation R) which is (bi)numerated by some formula in some r.e. theory, is (bi)numerated by a (possibly different) formula already in Q.

We write $\exists x{\le}y\beta(x)$ for $\exists x(x \le y \wedge \beta(x))$ and $\forall x{\le}y\beta(x)$ for $\forall x(x \le y \rightarrow \beta(x))$. $\exists x{<}y\beta(x)$ and $\forall x{<}y\beta(x)$ are defined in a similar way. The initial quantifiers of these formulas are *bounded*.

A formula is *primitive recursive in the strict sense* (*SPR*) if it is of the form $\delta_f(x_0,\dots,x_n,0)$, where f is primitive recursive and $\delta_f(x_0,\dots,x_n,y)$ is as in Fact 2. We define the *primitive recursive* (*PR*) formulas to be the members of the least set F of formulas containing the SPR formulas such that F is closed under propositional connectives, bounded quantification, replacing variables by numerals, and if ξ is a member of F and $\delta_f(x_0,\dots,x_n,y)$ is as in Fact 2 with f primitive recursive, then $\exists z(\delta_f(x_0,\dots,x_n,z) \wedge \xi)$ and $\forall z(\delta_f(x_0,\dots,x_n,z) \rightarrow \xi)$ are members of F. (Every PR formula is, provably in PA, equivalent to an SPR formula.)

Exactly which formulas turn out to be PR will depend on the details of the proof of Fact 2. However, regardless of those details we have the following consequence of Fact 2. A formula $\eta(x)$ is *decidable* in T if for every k, $T \vdash \eta(k)$ or $T \vdash \neg\eta(k)$; and similarly for formulas with more than one free varaible; a sentence φ is *decidable* in T if either $T \vdash \varphi$ or $T \vdash \neg\varphi$.

Corollary 2. If $\rho(x_1,\dots,x_n)$ is PR, then $Q \vdash \rho(k_1,\dots,k_n)$ iff $\rho(k_1,\dots,k_n)$ is true. It follows that

(i) every PR formula is decidable in Q,

(ii) a set X (relation R) is primitive recursive iff there is a PR formula binumerating X (R) in Q.

A PR formula binumerating X (R) in Q will be called a PR *binumeration of* X (R). A *numeration of* X is a formula numerating X in PA.

In what follows Corollary 2 will be applied without further mention.

Suppose PA$\dashv$ T. Then, by Fact 2, if $f(k_0,\dots,k_n)$ is a primitive recursive function and $\delta_f(x_0,\dots,x_n,y)$ is the corresponding formula, we can add the function symbol f to the language of T and add

$$f(x_0,\dots,x_n) = y \leftrightarrow \delta_f(x_0,\dots,x_n,y)$$

as a new axiom. (Thus, we shall be using the same function–symbol in the object language as in the metalanguage.) The resulting theory S is then a conservative extension of T in the sense that every sentence in the language of T provable in S is provable already in T. Thus, we may assume that f is a symbol in the language of T. Occasionally, the choice of the defining formula $\delta_f(x_0,\dots,x_n,y)$ is essential, e.g. in the proof of Theorem 3.5, but most of the time it is not.

In particular, we shall use the function symbols $<x,y>$ and $(x)_y$ for the primitive recursive functions $<k,m>$ and $(k)_m$ defined by:

$$<k,m> = 2^k \times 3^m,$$

$$(k)_m = \text{the number n such that } p_m^n \text{ divides k, but } p_m^{n+1} \text{ doesn't if } k > 0,$$
$$= 0 \text{ if } k = 0.$$

(Here p_m is the m^{th} prime number: $p_0 = 2$, $p_1 = 3$, etc.) The function $(k)_m$ will be used to code finite sequences of natural numbers; namely, for each finite sequence $n_0,\dots, n_k$ of natural numbers, there is a number n such that $(n)_i = n_i$ for $i \le k$.

The function $(k)_m$ can be used to transform an inductive definition into an explicit definition in PA in the following way. Suppose, for example, f(k) is defined by:

$$f(0) = 0,$$
$$f(n+1) = g(f(n),n) \text{ if } n \in X,$$
$$= h(f(n)) \text{ if } n \notin X.$$

Suppose g(k,m), h(k), and X are formally represented by g, h, and $\xi(x)$. Let $\delta(x,y) :=$

$$\exists z\big((z)_0 = 0 \wedge \forall u{<}x(\xi((z)_u) \rightarrow (z)_{u+1} = g((z)_u,u)) \wedge$$
$$\forall u{<}x(\neg\xi((z)_u) \rightarrow (z)_{u+1} = h((z)_u)) \wedge (z)_x = y\big).$$

Then $\delta(x,y)$ defines f in PA and

$$PA \vdash \forall x \exists y \forall z(\delta(x,z) \leftrightarrow z = y).$$

Thus, we may introduce a function symbol f by means of the definition:

$$f(x) = y \leftrightarrow \delta(x,y).$$

It is then easy to see that the formalizations of the clauses of the definition of f(k) become provable in PA:

$$PA \vdash f(0) = 0,$$
$$PA \vdash \xi(x) \rightarrow f(x+1) = g(f(x),x),$$
$$PA \vdash \neg\xi(x) \rightarrow f(x+1) = h(f(x)).$$

We assume given a Gödel numbering of the formal objects of L_A (extensions of L_A obtained by adding symbols for certain functions) among them all proofs. Since there is really no reason to distinguish between a formal object and its Gödel number, we shall "identify" the two. (We do not really care exactly what the formulas, proofs etc. of a theory are; the only thing that matters is how they are related to one another.) Thus, on the pages of this book you will find no formulas of L_A, only "formulas" referring to such formulas. (But the reader may, of course, still think of formal objects as strings of symbols.)

Formulas and sentences being numbers, it follows that symbols for formulas and sentences are (symbols for) numerals. Thus, for example, $\xi(\eta(y))$ makes perfectly good sense; it is the result of replacing x in $\xi(x)$ by (the numeral for) $\eta(y)$. (Note that y is not free in $\xi(\eta(y))$.) $\xi(\eta(k))$ is obtained by first replacing y by k in $\eta(y)$, giving $\eta(k)$, and then replacing x by $\eta(k)$ in $\xi(x)$.

The Gödel numbering can be defined in such a way that everything, that should be primitive recursive, is. In particular, the following is true (see also Fact 4 (d) below):

Fact 4. (a) The function corresponding to concatenation is primitive recursive.

(b) The function corresponding to substitution of numerals for variables is primitive recursive.

(c) The sets of formulas and sentences are primitive recursive.

By Fact 4 (a), $\neg\varphi$, $\varphi \rightarrow \psi$, etc. are primitive recursive functions of φ and ψ and so we may, and shall, use $\neg$, $\rightarrow$, etc. as formal symbols for these functions and write $\neg x$, $x \rightarrow y$, etc. for $\neg(x)$, $\rightarrow(x,y)$, etc.

As has already been mentioned, in many cases our (simplified) notation is not unambiguous. For example, $x = \varphi \rightarrow \psi$ can be read in three different ways. One of these is eliminated by writing $x = (\varphi \rightarrow \psi)$. But this formula is still ambiguous: does it contain the function symbol $\rightarrow$, or doesn't it? The answer to this and similar questions will always be clear from the context, when it matters. For example, we are allowed to add the symbol f for an (arbitrary) primitive recursive function f to the vocabulary of T only if we have assumed that PA$\dashv$ T, and then it doesn't really matter which way f(2+3), say, or $\varphi \rightarrow \psi$, occurring as a term, is understood. On the other hand, terms such as f(x) or $y \rightarrow \varphi$ are, of course, unambiguous.

In this book we shall be interested in r.e. theories only. In most contexts it is not necessary to distinguish between deductively equivalent theories. Thus, we may take advantage of the following result known as Craig's theorem.

Theorem 1. For any r.e. set X, there is a primitive recursive set Y such that Y$\dashv\vdash$ X.

Proof. If $X = \varnothing$, this is trivial. Suppose $X \neq \varnothing$. There is a primitive recursive function f such that $X = \{f(k)\colon k \in N\}$. For any sentence φ, let $\varphi^{(0)} := \varphi$ and $\varphi^{(n+1)} := \varphi^{(n)} \wedge \varphi$. Let $Y = \{f(k)^{(k)}\colon k \in N\}$. ■

In view of Craig's theorem we may adopt the first of the following three conventions; the other two are introduced to avoid needless repetition:

Convention 1. All theories denoted by single (decorated) letters, S, S_0, T, T′, A, B etc. are primitive recursive.

Convention 2. All theories denoted by single (decorated) letters, S, S_0, T. T′, A, B, etc. are consistent.

Convention 3. From now on until Chapter 8, T is an extension of Q, $Q \dashv T$.

If a theory is written as S, S′, S_0, etc. this is meant to indicate that, unless the contrary is explicitly assumed, the fact that this theory is formalized in L_A is really irrelevant.

We now define the *arithmetical hierarchy* of formulas (sentences) of L_A, in other words, the sets Σ_n and Π_n in the following way. Σ_n and Π_n are the least sets containing PR, closed under $\wedge$, $\vee$, and bounded quantification and such that (i) $\Sigma_n \cup \Pi_n \subseteq \Sigma_{n+1} \cap \Pi_{n+1}$, (ii) if ξ is Σ_n (Π_n), then $\neg\xi$ is Π_n (Σ_n), (iii) if ξ_0 is Σ_n (Π_n) and ξ_1 is Π_n (Σ_n), then $\xi_0 \rightarrow \xi_1$ is Π_n (Σ_n), (iv) if ξ is Σ_n (Π_n) and $\delta_f(x_0,\ldots,x_n,y)$ is as in Fact 2, then $\exists z(\delta_f(x_0,\ldots,x_n,z) \wedge \xi)$ and $\forall z(\delta_f(x_0,\ldots,x_n,z) \rightarrow \xi)$ are Σ_n (Π_n), (v) Σ_n is closed under existential quantification, and (vi) Π_n is closed under universal quantification. It follows that $\Sigma_0 = \Pi_0 = \mathrm{PR}$. B_n is the set of Boolean combinations of Σ_n formulas. Let Φ be either Σ_n or Π_n or B_n. Then $\Phi^T = \{\xi: \exists\eta\in\Phi: T\vdash \xi \leftrightarrow \eta\}$. A formula is Δ_n^T if it is Π_n and Σ_n^T or Σ_n and Π_n^T; $\Delta_n = \Delta_n^{PA}$.

In what follows Γ is either Σ_{n+1} or Π_{n+1} and Γ^+ is either Σ_n or Π_n. Γ^d, the *dual* of Γ, is Σ_n, if Γ is Π_n, and Π_n, if Γ is Σ_n. *In writing* Σ_n, Π_n, Δ_n, *or* B_n *we almost always omit the (obvious) assumption that* $n > 0$.

The arithmetical hierarchy generalizes to formulas containing new symbols for primitive recursive functions in the obvious way: if $\xi(x)$ is Γ^+ and $g(x_0,\ldots,x_n)$ is primitive recursive, then $\xi(g(x_0,\ldots,x_n))$ is Γ^+. In particular, Γ^+ is closed under $\forall x{\leq}f(x_0,\ldots,x_n)$ and $\exists x{\leq}f(x_0,\ldots,x_n)$.

From the definition of Σ_n and Π_n and Fact 4 (a), (b), (c) we get:

Fact 4. (d) The sets Σ_n and Π_n are primitive recursive.

Fact 5. (a) For each Σ_{n+1} formula (sentence) σ, we can effectively find a Π_n formula $\pi(x)$ such that $PA\vdash \sigma \leftrightarrow \exists x\pi(x)$.

(b) For each Π_{n+1} formula (sentence) π, we can effectively find a Σ_n formula $\sigma(x)$ such that $PA\vdash \pi \leftrightarrow \forall x\sigma(x)$.

By Fact 5, if we are working in an extension of PA, we can always assume that any Σ_{n+1} formula (sentence) is of the form $\exists x\pi(x)$, where $\pi(x)$ is Π_n and that any Π_{n+1} formula (sentence) is of the form $\forall x\sigma(x)$, where $\sigma(x)$ is Σ_n. Also note that it follows from Fact 5 that for each Σ_{n+1} formula $\sigma(x_0,\ldots,x_{k-1})$ there is a PR formula $\rho(x_0,\ldots,x_{k-1},y_0,\ldots,y_n)$ such that

$$\mathrm{PA} \vdash \sigma(x_0,\ldots,x_{k-1}) \leftrightarrow \exists y_0 \forall y_1 \ldots Q y_n \rho(x_0,\ldots,x_{k-1},y_0,\ldots,y_n),$$

where Q is $\exists$ or $\forall$ according as n is even or odd. Similarly, for each Π_{n+1} formula $\pi(x_0,\ldots,x_{k-1})$, there is a PR formula $\rho(x_0,\ldots,x_{k-1},y_0,\ldots,y_n)$ such that

$$\mathrm{PA} \vdash \pi(x_0,\ldots,x_{k-1}) \leftrightarrow \forall y_0 \exists y_1 \ldots Q y_n \rho(x_0,\ldots,x_{k-1},y_0,\ldots,y_n),$$

where Q is $\forall$ or $\exists$ according as n is even or odd. (This includes the case $k = 0$, in which case σ and π are sentences.)

Fact 3 (a) can be improved as follows:

Fact 3. (b) The formula $\delta_f(x_0,\ldots,x_n,y)$ of Fact 3 (a) can be taken to be Σ_1.

Corollary 1 can now be improved as follows.

Corollary 3. For every recursive set X (relation R) there is a Σ_1 formula and, therefore, a Π_1 formula binumerating X (R) in Q.

(b) For every r.e. set X (relation R) there are a Σ_1 formula and a Π_1 formula numerating X (R) in Q.

A theory T is *Γ–sound* if every Γ sentence provable in T is true.

For every r.e. set X, there is a primitive recursive relation R(k,m) such that X = {k: $\exists$mR(k,m)}. Thus, from Corollary 2 (ii), we get the following:

Corollary 4. Suppose T is Σ_1–sound. Then for every r.e. set X, there is a Σ_1 formula numerating X in T.

In Chapter 3 it will be shown that the assumption that T is Σ_1–sound can be omitted (Theorem 3.1).

A function $f(k_0,\ldots,k_n)$ is *provably recursive* in T if there is a Σ_1 formula $\delta_f(x_0,\ldots,x_n,y)$ such that

(i) $\mathrm{T} \vdash \delta_f(k_0,\ldots,k_n,y) \leftrightarrow y = f(k_0,\ldots,k_n)$.

(ii) $\mathrm{T} \vdash \delta_f(x_0,\ldots,x_n,y) \wedge \delta_f(x_0,\ldots,x_n,z) \rightarrow y = z$,

(iii) $\mathrm{T} \vdash \exists y \delta_f(x_0,\ldots,x_n,y)$.

(In (i) $f(k_0,\ldots,k_n)$ is, of course, a numeral.) Thus, all primitive recursive functions are provably recursive in PA.

Suppose (i), (ii), (iii) are true. Then we may add the function symbol f to the language of T and add

$$f(x_0,\ldots,x_n) = y \leftrightarrow \delta_f(x_0,\ldots,x_n,y)$$

as a new axiom, where $\delta_f(x_0,\ldots,x_n,y)$ is as above. The resulting theory is then a conservative extension of T.

Suppose $\alpha(x)$ and $\beta(y)$ are Σ_{n+1} and $f(k_0,\ldots,k_n)$ is provably recursive in T. Then

(1) $\exists x(\alpha(x) \wedge \forall y \leq f(x_0,\ldots,x_n)\beta(y))$

is not (necessarily) Σ_{n+1}; it is, however, $\Sigma^{\mathrm{T}}_{n+1}$, since it is, provably in T, equivalent to

$$\exists xz(\alpha(x) \wedge \delta_f(x_0,\ldots,x_n,z) \wedge \forall y{\le}z\beta(y)).$$

Similarly, if $\alpha(x)$ is Σ_{n+1} and $\beta(y)$ is Π_{n+1}, then

(2) $\forall x(\alpha(x) \to \exists y{\le}f(x_0,\ldots,x_n)\beta(y))$

is Π^T_{n+1}, since it is, provably in T, equivalent to

$$\forall xz(\alpha(x) \wedge \delta_f(x_0,\ldots,x_n,z) \to \exists y{\le}z\beta(y)).$$

(The reason why we don't extend the sets of Σ_{n+1} and Π_{n+1} to comprise the formulas (1) and (2), respectively, is that Σ_{n+1} and Π_{n+1} would then be nonrecursive (see Fact 4 (d)).

By Fact 4 (b), there is a primitive recursive function $Sbst_1(k,m,n)$ such that if n is a formula, then $Sbst_1(k,m,n)$ is the result of replacing v_k in that formula by the numeral for the number m. Thus, if n is $\xi(v_k)$, then $Sbst_1(k,m,\xi(v_k)) := \xi(m)$. Let

$$Sbst_2(k_0,m_0,k_1,m_1,n) = Sbst_1(k_0,m_0,Sbst_1(k_1,m_1,n)).$$

By Fact 2, there are formulas $Subst_1(x,y,z,u)$ and $Subst_2(x_0,y_0,x_1,y_1,z,u)$ such that

$$Q\vdash Subst_1(k,m,\xi(v_k),u) \leftrightarrow u = \xi(m),$$

$$Q\vdash Subst_2(k_0,m_0,k_1,m_1,\eta(v_{k_0},v_{k_1}),u) \leftrightarrow u = \eta(m_0,m_1).$$

As already mentioned, we may in any extension of PA introduce the corresponding function symbols $Sbst_1$ and $Sbst_2$.

If v_k is the only free variable of ξ, we write $\xi(\dot{x})$ for $Sbst_1(k,x,\xi)$. In writing, for example, $\eta(\dot{x}, \dot{y})$ we assume that there are k and m such that $\eta := \eta(v_k,v_m)$ and that $\eta(\dot{x}, \dot{y}) := Sbst_2(k,x,m,y,\eta)$. Note that, although x is not free in, say, $\xi(\eta(x))$, it is free in $\xi(\eta(\dot{x}))$.

Given a formula $\sigma(z)$, let

$$Prf_\sigma(x,y)$$

be a formula whose intuitive meaning is: "there is a v such that $(y)_v = x$, $(y)_u = 0$ for all $u > v$, and for every $u \le v$, either $(y)_u$ is a logical axiom, satisfies $\sigma(z)$, or is obtained from formulas $(y)_w$ with $w < u$ using one of the (logical) rules of derivation"; in other words "y is a proof of the sentence x from the set of sentences satisfying $\sigma(z)$". (Thus, if there are nonsentences "satisfying $\sigma(z)$", they are simply disregarded.) The fact that there is a formula $Prf_\sigma(x,y)$ with the desired properties (see below) follows from Facts 2 (i) and 4 (and the details of the formalization of predicate logic.) If $\sigma(z)$ is Γ^+, then $Prf_\sigma(x,y)$ is Γ^+.

Let

$$Pr_\sigma(x) := \exists y Prf_\sigma(x,y),$$

$$Con_\sigma := \neg Pr_\sigma(\bot),$$

where $\bot := \neg 0 = 0$. Thus, the intuitive meaning of $Pr_\sigma(x)$ is: "the sentence x is provable from the set of sentences satisfying $\sigma(z)$" and Con_σ intuitively says: "the set of sentences satisfying $\sigma(z)$ is consistent". If $\sigma(z)$ is Σ_{n+1}, then $Pr_\sigma(x)$ is Σ_{n+1}, and Con_σ is Π_{n+1}.

For any formula $\sigma(x)$, let

$$(\sigma \restriction y)(x) := \sigma(x) \wedge x \le y,$$

$$(\sigma + y)(x) := \sigma(x) \vee x = y.$$

In what follows we shall use $Prf_S(x,y)$, $Prf_{S+z}(x,y)$, $Prf_{S\restriction z}(x,y)$, $Pr_S(x)$, Con_S, etc. to denote (ambiguously) any formula $\mathrm{Prf}_\sigma(x,y)$, $\mathrm{Prf}_{\sigma+z}(x,y)$, $\mathrm{Prf}_{\sigma\restriction z}(x,y)$, $\mathrm{Pr}_\sigma(x)$, Con_σ, etc. where $\sigma(x)$ is a PR binumeration of S. If $S = \varnothing$, we assume that $\sigma(x) := \neg x = x$; if S is finite and nonempty, $S = \{\varphi_0,\dots,\varphi_n\}$, then $\sigma(x) := x = \varphi_0 \vee \dots \vee x = \varphi_n$.

Fact 6. $\vdash \forall x(\sigma(x) \to \sigma'(x)) \to \forall y(\mathrm{Pr}_\sigma(y) \to \mathrm{Pr}_{\sigma'}(y))$. Consequently $\vdash \forall x(\sigma(x) \to \sigma'(x)) \to (\mathrm{Con}_{\sigma'} \to \mathrm{Con}_\sigma)$.

Fact 7. Suppose $\sigma(x)$ numerates S in T.

(a) If p is a proof of φ in S, then $T\vdash \mathrm{Prf}_\sigma(\varphi,p)$.

(b) If $S\vdash \varphi$, then $T\vdash \mathrm{Pr}_\sigma(\varphi)$.

(c) Suppose $PA\dashv T$. Let $\alpha(x_0,\dots,x_{n-1})$ be any formula whose free variables are $x_0,\dots, x_{n-1}$. If $S\vdash \alpha(x_0,\dots,x_{n-1})$, then $T\vdash \mathrm{Pr}_\sigma(\alpha(\dot{x}_0,\dots,\dot{x}_{n-1}))$.

(d) If $\sigma(x)$ binumerates S in T and p is not a proof of φ in S, then $T\vdash \neg\mathrm{Prf}_\sigma(\varphi,p)$.

Fact 8. Let $\sigma(x)$ be any formula.

(i) $PA\vdash \mathrm{Pr}_\sigma(x) \wedge \mathrm{Pr}_\sigma(x\to y) \to \mathrm{Pr}_\sigma(y)$,

(ii) $PA\vdash \mathrm{Pr}_{\sigma+y}(z) \leftrightarrow \mathrm{Pr}_\sigma(y\to z)$,

(iii) $PA\vdash \mathrm{Pr}_\sigma(x) \to \exists y\mathrm{Pr}_{\sigma|y}(x)$.

Corollary 5. Let $\sigma(x)$ be any formula.

(i) $PA\vdash \mathrm{Pr}_\sigma(\beta(\dot{x})) \to \mathrm{Pr}_\sigma(\exists x\beta(x))$,

(ii) $PA\vdash \mathrm{Pr}_\sigma(\forall x\beta(x)) \to \mathrm{Pr}_\sigma(\beta(\dot{x}))$,

(iii) $PA\vdash \mathrm{Pr}_\sigma(x) \wedge \mathrm{Pr}_\sigma(\neg x) \to \neg\mathrm{Con}_\sigma$,

(iv) $PA\vdash \mathrm{Pr}_\sigma(\neg x) \leftrightarrow \neg\mathrm{Con}_{\sigma+x}$ and $PA\vdash \mathrm{Pr}_\sigma(x) \leftrightarrow \neg\mathrm{Con}_{\sigma+\neg x}$,

(v) if $PA\dashv T$, $\sigma(x)$ numerates S in T, and $S\vdash \gamma(x) \to \delta(x)$, then $T\vdash \mathrm{Pr}_\sigma(\gamma(\dot{x})) \to \mathrm{Pr}_\sigma(\delta(\dot{x}))$,

(vi) if $PA\dashv T$, $\sigma(x)$ numerates S in T, and $S\vdash \varphi \to \psi$, then $T\vdash \mathrm{Pr}_\sigma(\varphi) \to \mathrm{Pr}_\sigma(\psi)$.

All true Σ_1 sentences are provable in Q; in fact, this is provable in PA; in other words, Q is *Σ_1–complete* provably in PA:

Fact 9. Suppose φ and $\delta(x_0,\dots,x_{n-1})$ are Σ_1.

(a) If φ true, then $Q\vdash \varphi$.

(b) $PA\vdash \delta(x_0,\dots,x_{n-1}) \to \mathrm{Pr}_Q(\delta(\dot{x}_0,\dots,\dot{x}_{n-1}))$;

in particular, $PA\vdash \varphi \to \mathrm{Pr}_Q(\varphi)$.

By Fact 9 (a), if ψ is Π_1 and $T + \psi$ is consistent, then ψ is true.

Corollary 6. Suppose $\sigma(x)$ numerates an extension of Q in PA.

(a) If φ is a Σ_1 sentence, then

$PA\vdash \varphi \to \mathrm{Pr}_\sigma(\varphi)$.

(b) If $\sigma(x)$ is Σ_1 and $\tau(x)$ is a numeration of T (in PA), then

$$\mathrm{PA} \vdash \mathrm{Pr}_\sigma(\varphi) \to \mathrm{Pr}_\tau(\mathrm{Pr}_\sigma(\varphi));$$

in particular, $\mathrm{PA} \vdash \mathrm{Pr}_\sigma(\varphi) \to \mathrm{Pr}_Q(\mathrm{Pr}_\sigma(\varphi))$.

The following conditions (cf. Fact 7 (b), Fact 8 (i), and Corollary 6 (b)) are known as the Bernays–Löb provability conditions (for $\mathrm{Pr}_T(x)$).

(BLi) if $T \vdash \varphi$, then $\mathrm{PA} \vdash \mathrm{Pr}_T(\varphi)$,
(BLii) $\mathrm{PA} \vdash \mathrm{Pr}_T(\varphi) \wedge \mathrm{Pr}_T(\varphi \to \psi) \to \mathrm{Pr}_T(\psi)$,
(BLiii) $\mathrm{PA} \vdash \mathrm{Pr}_T(\varphi) \to \mathrm{Pr}_T(\mathrm{Pr}_T(\varphi))$.

The construction of "self–referential" sentences and formulas will play a decisive role in what follows. Such constructions are possible in virtue of the following result, the fixed point lemma; we list a number of special cases; a completely general formulation would be needlessly complicated. φ is a *fixed point* of $\xi(x)$ in T if $T \vdash \varphi \leftrightarrow \xi(\varphi)$.

Lemma 1. (a) For any Γ^+ formula $\gamma(x)$, we can effectively find a Γ^+ sentence φ such that

$$Q \vdash \varphi \leftrightarrow \gamma(\varphi).$$

(b) For any Γ^+ formula $\gamma(x,y)$, we can effectively find a Γ^+ formula $\xi(x)$ such that

$$Q \vdash \xi(x) \leftrightarrow \gamma(x,\xi).$$

(c) For any Γ^+ formulas $\gamma_0(x,y)$ and $\gamma_1(x,y)$, we can effectively find Γ^+ sentences φ_0 and φ_1 such that

$$Q \vdash \varphi_0 \leftrightarrow \gamma_0(\varphi_0,\varphi_1),$$
$$Q \vdash \varphi_1 \leftrightarrow \gamma_1(\varphi_0,\varphi_1).$$

(d) For every Γ^+ formula $\gamma(x,y)$, we can effectively find a Γ^+ formula $\xi(x)$ such that for every k,

$$Q \vdash \xi(k) \leftrightarrow \gamma(k,\xi(k)).$$

(e) Suppose $\mathrm{PA} \dashv T$. For every Γ^+ formula $\gamma(x,y)$, we can effectively find a Γ^+ formula $\xi(x)$ such that

$$\mathrm{PA} \vdash \xi(x) \leftrightarrow \gamma(x,\xi(\dot{x})).$$

Proof. In what follows x is v_m and y is v_n.

(a) Let

$$\delta(x) := \exists z(\mathrm{Subst}_1(m,x,x,z) \wedge \gamma(z)).$$

We have

$$Q \vdash \mathrm{Subst}_1(m,\delta,\delta,z) \leftrightarrow z = \delta(\delta).$$

It follows that

$$Q \vdash \delta(\delta) \leftrightarrow \gamma(\delta(\delta)).$$

Thus, $\varphi := \delta(\delta)$ is as desired. ♦

(b) Let

$$\eta(x,y) := \exists z(\mathrm{Subst}_1(n,y,y,z) \wedge \gamma(x,z)).$$

We have

$$Q\vdash \mathrm{Subst}_1(n,\eta,\eta,z) \leftrightarrow z = \eta(x,\eta).$$

It follows that

$$Q\vdash \eta(x,\eta) \leftrightarrow \gamma(x,\eta(x,\eta)).$$

Thus, $\xi(x) := \eta(x,\eta)$ is as desired. ♦

(c) For $i = 0, 1$, let

$$\delta_i(x,y) := \exists z_0 z_1(\mathrm{Subst}_2(m,x,n,y,x,z_0) \wedge \mathrm{Subst}_2(m,x,n,y,y,z_1) \wedge \gamma_i(z_0,z_1)).$$

We have

$$Q\vdash \mathrm{Subst}_2(m,\delta_0,n,\delta_1,\delta_0,z_0) \leftrightarrow z_0 = \delta_0(\delta_0,\delta_1),$$
$$Q\vdash \mathrm{Subst}_2(m,\delta_0,n,\delta_1,\delta_1,z_1) \leftrightarrow z_1 = \delta_1(\delta_0,\delta_1).$$

It follows that

$$Q\vdash \delta_0(\delta_0,\delta_1) \leftrightarrow \gamma_0(\delta_0(\delta_0,\delta_1),\delta_1(\delta_0,\delta_1)),$$
$$Q\vdash \delta_1(\delta_0,\delta_1) \leftrightarrow \gamma_1(\delta_0(\delta_0,\delta_1),\delta_1(\delta_0,\delta_1)).$$

Thus, $\varphi_0 := \delta_0(\delta_0,\delta_1)$ and $\varphi_1 := \delta_1(\delta_0,\delta_1)$ are as desired. ♦

(d) Let

$$\eta(x,y) := \exists z(\mathrm{Subst}_2(m,x,n,y,y,z) \wedge \gamma(x,z)).$$

We have

$$Q\vdash \mathrm{Subst}_2(m,k,n,\eta,\eta,z) \leftrightarrow z = \eta(k,\eta).$$

It follows that

$$Q\vdash \eta(k,\eta) \leftrightarrow \gamma(k,\eta(k,\eta)).$$

Thus, $\xi(x) := \eta(x,\eta)$ is as desired. ♦

(e) Let

$$\eta(x,y) := \gamma(x,\mathrm{Sbst}_2(m,x,n,y,y)).$$

We have

$$PA\vdash \mathrm{Sbst}_2(m,x,n,\eta,\eta) = \eta(\dot{x},\eta).$$

It follows that

$$PA\vdash \eta(x,\eta) \leftrightarrow \gamma(x,\eta(\dot{x},\eta)).$$

Thus, $\xi(x) := \eta(x,\eta)$ is as desired. ■

The cases listed in the above formulation of the fixed point lemma do not exhaust the possibilities of self–reference. (This should be clear from the proof.) However, applications of self–reference (in what follows) not covered by these examples can be obtained by straightforward generalization.

For example, let $\gamma(x,y)$ be any formula and suppose we want to construct a sentence θ such that

$$Q\vdash \theta \leftrightarrow \gamma(\theta,\neg\theta).$$

This can be done as follows. There is a (PR) formula $\nu(x,y)$ such that for every φ,

$$Q\vdash \nu(\varphi,y) \leftrightarrow y = \neg\varphi.$$

Let $\delta(x) := \exists y(\nu(x,y) \wedge \gamma(x,y))$. By the fixed point lemma, there is a sentence θ such that $Q\vdash \theta \leftrightarrow \delta(\theta)$. Clearly θ is as desired.

From this point on the fixed point lemma will be used without further mention. The phrase "let φ be such that $Q\vdash \varphi \leftrightarrow \xi(\varphi)$", where $\xi(x)$ is Γ^+, is short for "let φ be a (Γ^+) sentence such that $Q\vdash \varphi \leftrightarrow \xi(\varphi)$" and the same applies *mutatis mutandis* to all similar phrases.

Applying the fixed point lemma we now prove two basic and very important theorems.

A theory S is *decidable* if Th(S) is recursive, otherwise *undecidable*. S is *essentially undecidable* if S and all its consistent extensions are undecidable.

Theorem 2. Q is essentially undecidable.

This follows at once from Corollary 1 (a) and:

Lemma 2. There is no formula binumerating Th(T) in T.

Proof. Suppose $\tau(x)$ binumerates Th(T) in T. Let φ be such that

(1) $Q\vdash \varphi \leftrightarrow \neg\tau(\varphi)$.

If $T\vdash \varphi$, then $T\vdash \tau(\varphi)$ and so, by (1), $T\vdash \neg\varphi$. But then T is inconsistent, contrary to Convention 2. It follows that $T\nvdash \varphi$. Since $\tau(x)$ binumerates Th(T) in T, this implies that $T\vdash \neg\tau(\varphi)$ and so, by (1), $T\vdash \varphi$, a contradiction. ■

Let U be any, not necessarily r.e., consistent extension of Q. By a *truth–definition for* U we understand a formula $\upsilon(x)$ such that for every sentence φ,

(tr) $U\vdash \varphi \leftrightarrow \upsilon(\varphi)$.

The following result is known as the Tarski, or Gödel–Tarski, theorem.

Theorem 3. There is no truth–definition for U.

Proof. The proof is almost the same as that of Lemma 2. Suppose $\upsilon(x)$ is a truth–definition for U. Let φ be such that

$$Q\vdash \varphi \leftrightarrow \neg\upsilon(\varphi).$$

This together with (tr) implies that U is inconsistent, contrary to assumption. ■

The proof of Theorem 3 is a formal version of the so called liar paradox. In the latter one considers a sentence saying of itself that it isn't true:

(*) (*) isn't true.

(*) is both true and not true, a contradiction. Thus, a sentence saying, what (*) seems to say, cannot exist.

Let **M** be any model of Q. The set X of natural numbers is *defined* in **M** by the formula $\xi(x)$ if

$$X = \{k: \xi(k) \text{ is true in } \mathbf{M}\}.$$

X is *definable* in **M** if there is a formula defining X in **M**. Applying Theorem 3 to the set of sentences true in **M** we get the following:

Corollary 7. Suppose **M** is a model of Q. The set of sentences true in **M** is not definable in **M**; in particular, this is true of **N**.

Thus, there is no full truth–definition in arithmetic. We do, however, have the following partial positive fact. A *partial truth–definition* for Γ sentences in T is a formula $Tr_\Gamma(x)$ such that for every Γ sentence φ,

$$T\vdash \varphi \leftrightarrow Tr_\Gamma(\varphi).$$

Let $\Gamma(x)$ be a "natural" PR binumeration of the set of Γ sentences (cf. Fact 4 (d)). We (may) assume that if $\Gamma \subseteq \Gamma'$, then

$$PA\vdash \Gamma(x) \rightarrow \Gamma'(x).$$

Fact 10. (a) There is a Γ formula $Sat_\Gamma(x,y)$ with the following properties:

(i) For every Γ formula $\gamma(x)$,

$$PA\vdash \gamma(x) \leftrightarrow Sat_\Gamma(x,\gamma).$$

(ii) Let $Tr_\Gamma(x) := Sat_\Gamma(0,x)$. Then for every Γ formula $\gamma(x)$,

$$PA\vdash \gamma(x) \leftrightarrow Tr_\Gamma(\gamma(\dot{x}))$$

and so for every Γ sentence φ,

$$PA\vdash \varphi \leftrightarrow Tr_\Gamma(\varphi).$$

(iii) $PA\vdash \Gamma^d(x) \wedge \Gamma(y) \wedge Tr_{\Gamma^d}(x) \wedge Tr_\Gamma(x \rightarrow y) \rightarrow Tr_\Gamma(y).$

(b) There is a Δ_{n+1} formula $Sat_{B_n}(x,y)$ such that for every B_n formula $\beta(x)$,

$$PA\vdash \beta(x) \leftrightarrow Sat_{B_n}(x,\beta).$$

Fact 10 (a) (i), (ii) can be used to justify self–referential constructions such as the following one. Let $\gamma(x,y)$ be any Γ formula. There is then a Γ formula $\xi(x)$ such that

$$PA\vdash \xi(k) \leftrightarrow \xi(k+1) \vee \gamma(k,\xi).$$

Indeed, this is equivalent to

$$PA\vdash \xi(k) \leftrightarrow Tr_\Gamma(\xi(k+1)) \vee \gamma(k,\xi).$$

Applying Fact 10 (a), we can now show that the arithmetical hierarchy, pertaining to formulas of L_A, is proper; for the corresponding result for sentences, see Corollary 2.5.

Theorem 4. Suppose $PA \dashv T$. There is a Γ formula which is not $\Gamma^{d,T}$.

Proof. Let $\gamma(x) := Sat_\Gamma(x,x)$. Suppose $\eta(x)$ is Γ^d and $T\vdash \eta(x) \leftrightarrow \gamma(x)$. $\neg\eta(x)$ is Γ. Thus, $T\vdash \neg\eta(x) \leftrightarrow Sat_\Gamma(x,\neg\eta)$ and so $T\vdash \neg\eta(\neg\eta) \leftrightarrow \gamma(\neg\eta)$. But also $T\vdash \eta(\neg\eta) \leftrightarrow \gamma(\neg\eta)$. It follows that $T\vdash \neg\eta(\neg\eta) \leftrightarrow \eta(\neg\eta)$. But then T is inconsistent, contrary to Convention 2. ■

In terms of the partial truth–definitions we can formulate the following:

Fact 11. For every Γ,

$$PA\vdash \forall x(\Gamma(x) \wedge Pr_\varnothing(x) \rightarrow Tr_\Gamma(x)).$$

Let $X{\restriction}k = \{n\in X\colon n \le k\}$. (Formulas such as $Pr_{X\restriction k}(x)$ are then ambiguous, but the ambiguity is harmless.) A theory T is *reflexive* if $T\vdash Con_{T\restriction k}$ for every k. T is *essentially reflexive* if every extension of T (in the same language) is reflexive.

Corollary 8. PA is essentially reflexive.

Proof. Suppose PA$\dashv$ T. Let k be arbitrary and let Γ be such that $\neg\bigwedge T \restriction k$ is Γ. By Fact 10 (a) (ii) and Fact 11, T$\vdash Pr_{\varnothing}(\neg\bigwedge T \restriction k) \to \neg\bigwedge T \restriction k$ and so T$\vdash \neg Pr_{\varnothing}(\neg\bigwedge T \restriction k)$. But then, by Corollary 5 (iv), T$\vdash Con_{T\restriction k}$, as desired. ■

From Fact 6 and Corollaries 5 (iv) and 8 we get:

Corollary 9. Suppose PA$\dashv$ T.

(a) If $\tau(x)$ binumerates T in T, then T$\vdash Con_{\tau\restriction k}$ for every k.

(b) For all k and φ, T$\vdash Pr_{T\restriction k}(\varphi) \to \varphi$.

Corollary 8 will be of crucial importance, especially in Chapters 6 and 7. But it should be observed that, although many results proved in the following pages for extensions of PA do depend on Corollary 8, others, for example, most of those of Chapter 2 and all results of Chapter 5, do not. The latter results generalize to (possibly finitely) axiomatized, consistent extensions of PA, not (necessarily) formalized in L_A.

The following elementary observations are occasionally useful.

Lemma 3. Let

$\pi := \forall x(\alpha(x) \to \exists y{\le}x\beta(y))$, $\qquad\theta := \forall y(\beta(y) \to \exists x{<}y\alpha(x))$,

$\sigma := \exists x(\alpha(x) \wedge \forall y{\le}x\neg\beta(y))$, $\qquad\chi := \exists y(\beta(y) \wedge \forall x{<}y\neg\alpha(x))$.

Then

(i) PA$\vdash \pi \vee \theta$,

(ii) PA$\vdash \neg(\sigma \wedge \chi)$,

(iii) PA$\vdash (\pi \wedge \theta) \to \forall x\neg\alpha(x)$,

(iv) PA$\vdash \exists x\alpha(x) \to (\sigma \vee \chi)$,

(v) PA$\vdash \sigma \leftrightarrow (\exists x\alpha(x) \wedge \theta)$,

(vi) PA$\vdash \exists x\alpha(x) \to (\chi \leftrightarrow \neg\sigma)$.

Proof. (i) Argue in PA: "Suppose $\neg\pi$ and $\neg\theta$. Let z and u be such that $\alpha(z)$, $\neg\exists y{\le}z\beta(y)$, $\beta(u)$, $\neg\exists y{<}u\alpha(y)$. Then $\neg u \le z$ and $\neg z < u$, impossible. Thus, $\pi \vee \theta$." This proves (i).

(ii) follows from (i).

(iii) Argue in PA: "Suppose π, θ, and $\exists x\alpha(x)$. By the least number principle, there is a smallest z such that $\alpha(z)$. Since π holds, there is a $u \le z$ such that $\beta(u)$. But then, by θ, there is a $v < u$ such that $\alpha(v)$. It follows that $v < z$, a contradiction." This proves (iii).

(iv) follows from (iii).

(v) follows from (i) and (iii).

(vi) follows from (ii) and (iv). ■

Corollary 10. Suppose $\alpha(x)$ and $\beta(y)$ are PR. Let θ be as in Lemma 3.

(a) If $\exists x\alpha(x)$ and $\forall y\neg\beta(y)$ are true, then PA$\vdash \theta$.

(b) PA$\vdash (\exists x\alpha(x) \wedge \forall y\neg\beta(y)) \rightarrow Pr_{PA}(\theta)$.

Proof. Let π be as in Lemma 3.

(a) If $\exists x\alpha(x)$ and $\forall x\neg\beta(x)$ are true, so is $\neg\pi$. Since $\neg\pi$ is Σ_1, it follows that PA$\vdash$ $\neg\pi$. But then, by Lemma 3 (i), PA$\vdash \theta$. ♦

(b) $\vdash (\exists x\alpha(x) \wedge \forall y\neg\beta(y)) \rightarrow \neg\pi$. Since $\neg\pi$ is Σ_1, we have PA$\vdash \neg\pi \rightarrow Pr_{PA}(\neg\pi)$, by Fact 9 (b). By Lemma 3 (i), PA$\vdash \neg\pi \rightarrow \theta$. By (BLi) and (BLii), we get PA$\vdash Pr_{PA}(\neg\pi) \rightarrow Pr_{PA}(\theta)$. Putting these together we get the desired conclusion. ■

For the concepts and results of (elementary) recursion theory used in this book we refer to Soare (1987). A set is Π^0_n (Σ^0_n) if it is defined in **N** by a Π_n (Σ_n) formula. X is a *complete* Π^0_n (Σ^0_n) set if X is Π^0_n (Σ^0_n) and for every Π^0_n (Σ^0_n) set Y, there is a recursive function f(k) such that for every k, $k\in Y$ iff $f(k)\in X$. No complete Π^0_n (Σ^0_n) set is Σ^0_n (Π^0_n). Complete Π^0_n (Σ^0_n) sets exist. If T is true, Theorem 4 follows directly from the fact that for each $n > 0$, there is a Π^0_n (Σ^0_n) set which isn't Σ^0_n (Π^0_n).

The following notions will be needed in Chapter 7. A partially ordered set **L** = (L, $\le$) is a *lattice* if any two members a, b of L have a *least upper bound* (l.u.b.) $a \cup b$ and a *greatest lower bound* (g.l.b.) $a \cap b$. Thus, $a \cup b \le c$ iff $a \le c$ and $b \le c$; similarly, $c \le a \cap b$ iff $c \le a$ and $c \le b$. It follows that $a \le b$ iff $a \cap b = a$ iff $a \cup b = b$. **L** is *distributive* if for all a, b, $c\in L$,

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c);$$

or equivalently,

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c).$$

The inequalities

$$(a \cap b) \cup (a \cap c) \le a \cap (b \cup c),$$
$$a \cup (b \cap c) \le (a \cup b) \cap (a \cup c)$$

hold in all lattices.

Suppose **L** has a minimal (maximal) element 0_L (1_L). If $a \cap b = 0_L$ and $a \cup b = 1_L$, then b is a *complement* of a (and a a complement of b). If **L** is distributive, each a has at most one complement. If $b = \max\{c\colon a \cap c = 0_L\}$, then b is the *pseudocomplement* (*p.c.*) of a.

Exercises for Chapter 1.

1. Improve Lemma 2 by showing that if U is any consistent extension of Q, not necessarily r.e., there is no formula numerating N – Th(U) in U.

2. (a) Let Y be any r.e. set of formulas decidable in T. Show that there is a recursive set X such that no member of Y binumerates X in T.

(b) Suppose PA$\dashv$ T. Show that every Δ_1 formula is decidable in T. Conclude that there is a recursive set which is not binumerated by any Δ_1 formula in T (compare Corollary 3 (a)).

3. (a) Suppose T is Σ_1–sound. Show that not every recursive function is provably recursive in T (compare Exercise 2.28 (b)). [Hint: Let $\delta_0(x,y)$, $\delta_1(x,y)$,... be an effective enumeration of all Σ_1 formulas $\delta(x,y)$ provably in T defining (total) functions, i.e. such that

(tot) $\quad T\vdash \forall x\exists y\forall z(\delta(x,z) \leftrightarrow z = y)$.

For each m, let $f_m(k)$ be the recursive function defined by $\delta_m(x,y)$ in T. The function $g(k) = f_k(k)+1$ is recursive.]

(b) Suppose PA$\dashv$ T. Show that if T is not Σ_1–sound, then every recursive function is provably recursive in T. (Thus, the restriction to Σ_1–sound theories in (a) is essential.)

(c) Show that for each recursive function f(k), there is a formula $\delta(x,y)$ defining f in T and such that (tot) holds. (Thus, the restriction to Σ_1 formulas in the definition of "provably recursive" is essential.)

4. Show that there is no formula $\alpha(x)$ such that for all φ, ψ,

(i) $\quad T\vdash \alpha(\varphi) \to \varphi$,

(ii) $\quad T\vdash \alpha(\alpha(\varphi)\to\varphi)$,

(iii) $\quad$ if $\vdash \varphi$, then $T\vdash \alpha(\varphi)$,

(iv) $\quad T\vdash \alpha(\varphi)$ and $T\vdash \alpha(\varphi\to\psi)$, then $T\vdash \alpha(\psi)$.

(This improves the Gödel–Tarski theorem; see also Exercise 4.4.) [Hint: Let $\chi := \bigwedge Q$. Let φ be such that

$$Q\vdash \varphi \leftrightarrow \neg\alpha(\chi\to\varphi).$$

It follows that

$$\vdash (\alpha(\chi\to\varphi) \to (\chi \to \varphi)) \to (\chi \to \varphi).$$

Show that $T\vdash \varphi$ and $T\vdash \neg\varphi$.]

5. Suppose PA$\dashv$ T.

(a) Show that $Tr_\Gamma(x)$ is not $\Gamma^{d,T}$ (compare Theorem 4).

(b) Show that there is a Δ_{n+1} formula which is not B_n^T (compare Theorem 4).

6. (a) Show that if T is Σ_n–sound, then T is Π_{n+1}–sound.

(b) Suppose PA$\dashv$ T and T is true. Let φ be such that

$$PA\vdash \varphi \leftrightarrow \exists z(\Sigma_n(z) \wedge Pr_{T+\varphi}(z) \wedge \neg Tr_{\Sigma_n}(z)).$$

(These sentences will reappear in Chapter 5.) Thus, φ "says" that T + φ is not Σ_n–sound. φ is Σ_{n+1}. Show that φ is false. Conclude that Σ_n–soundness does not imply Σ_{n+1}–soundness.

Notes for Chapter 1.

The background material on formal arithmetic and Gödel numberings presupposed in this book can be found in several textbooks, for example, Kleene (1952a), Mendelson (1987), Smoryński (1985), Kaye (1991), Boolos (1979), (1993), Hájek and Pudlák (1993). We follow Feferman (1960) in identifying formal expressions with their Gödel numbers. Facts 2 and 4 are due to Gödel (1931). When, somewhat later, the recursive functions were defined, the proof of Fact 3 presented no new difficulties. For proofs of these Facts and Fact 1, see any one of the textbooks just mentioned. The terms "numerate" and "binumerate" are due to Feferman (1960). Theorem 1 is due to Craig (1953). The exact definitions of PR, Σ_n, Π_n, Δ_n vary from one author to another, depending on the intended applications; for example, other authors often use Δ_0 to denote the set of bounded formulas, i.e. formulas all of whose quantifiers are bounded (cf. e.g. Kaye (1991) and Hájek and Pudlák (1993)). The present definitions have the advantage that the concepts are easy to work with (in the present setting) and that the sets PR, Σ_n, Π_n are primitive recursive (Fact 4 (d)) (Δ_n, $n > 0$, is not recursive; see Exercise 2.4 (d)). The formulas $\mathrm{Prf}_\sigma(x,y)$, $\mathrm{Pr}_\sigma(x)$, Con_σ were introduced by Feferman (1960). Fact 9 is due to Feferman (1960). The Bernays–Löb provability conditions are due to Löb (1955), simplifying the original conditions due to Bernays (cf. Hilbert and Bernays (1939)). Part (a) of the fixed point lemma is implicit in Gödel (1931); it was first stated explicitly by Carnap (1934) (see also Gödel (1934)). The more general versions (b) – (e) were subsequently obtained by Ehrenfeucht and Feferman (1960) and Montague (1962). Lemma 2 and Theorem 2 first appeared in Tarski, Mostowski, Robinson (1953); for a stronger result, see Exercise 2.3. Theorem 3 was first published by Tarski (1933) (see also Gödel (1934)). The application of (partial) truth–definitions goes back to Hilbert and Bernays (1934, 1939); a full proof of Fact 10 is given in Kaye (1991). For a slightly different proof of Theorem 4 and a related result, see Exercise 5. Fact 11 is essentially due to Kreisel and Wang (1955) (see also Mostowski (1952a)); for a sketch of a proof of a related result, which can easily be turned into a proof of Fact 11, see Kaye (1991), p. 140. Corollary 8 is due to Mostowski (1952a).

Exercise 4 is due to Montague (1963).